The nature of equilibria under noncollusive product design and collusive pricing

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Abstract

The basic framework is Hotelling's model of product choice with quadratic transportation cost. Duopolists choose locations in the initial period and compete in prices in subsequent infinite periods. The firms share profits on the profit possibility frontier. It is shown that under very general conditions, both the firms locating at the center is an equilibrium. It is not necessarily unique and multiple symmetric equilibria can exist. Thus the products are not necessarily minimally differentiated. How the profits are shared when the firms are located together off the center has a critical bearing on the nature of equilibria. If the firms share profits equally at those locations, then all the equilibria are symmetric. Otherwise, asymmetric equilibria can appear. The equilibria can be classified into three types: a unique equilibrium at the center of the market, multiple symmetric equilibria and multiple asymmetric agglomerated equilibria. The second case entails nonminimal product differentiation. Sufficient conditions for each of these equilibria are given. Necessary conditions for multiple symmetric equilibria off the center are also obtained.

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1 Introduction

The literature on product differentiation originating from Hotelling (1929) is by now vast. d'Aspremont et al. (1979) and Neven (1985) have shown that, in a two stage model with quadratic transportation cost, where duopolists first choose locations and then compete in prices, the equilibrium locations are at the extreme endpoints of the market segment.

This model, with variations, has been extensively used to examine a wide range of theoretical and policy issues. Schmalensee (1978) is an early empirical investigation of product positioning and entry deterrence. A more recent study on firm entry with endogenous product choice is Seim (2006). Mazzeo (2002) examines the effect of product differentiation on oligopoly market structure. An excellent survey of endogenous product choice can be found in Crawford (2012). Sweeting (2013) develops a dynamic model of product positioning to evaluate the effect of taxation on product repositioning. The effect on firm location when the transportation cost is shared between the producers and the consumers is examined in Kyureghian et al. (2013). Draganska et al. (2009) and Sullivan (2017) develop models that integrate product choice, price competition and collusion in differentiated product markets. They also provide empirical analyses of the food industry. Fan (2013) studies the effects of mergers on both prices and product characteristics. Using data on newspaper characteristics and prices, the author finds that a merger would have led to higher prices and lower product quality. Wollmann (2018) distinguishes between entry and exit of products, rather than firms. In stage one firms choose product characteristics for their commercial vehicles and in stage two, firms observe product characteristic choices of competitors and simultaneously choose prices. The author examines the efficacy of the automotive industry bailout policy.

The model has also been examined in a supergame setting, with payoff discounting. The firms choose locations in the initial period and prices in subsequent infinite periods. A justification for this is that redesigning the product is often more difficult than changing the price. The profit possibility frontier (PPF) is the set of Pareto optimal profit allocations between the two firms. Even though locations are chosen independently, the firms can tacitly collude in the price setting stage. Friedman and Thisse (1993) has termed this "partial collusion." In general, the optimal locations chosen in the beginning by the firms will depend on the collusive prices charged subsequently. In Schmalensee (1987), the collusive profit ratio is determined by the market-share ratio of the one-shot Nash equilibrium [see also Friedman and Thisse (1993, p.641)]. In Jehiel (1992), the prices are determined by the Nash bargaining solution at each pair of locations. In Friedman and Thisse (1993), the firms share profits on the PPF in proportion to the Nash equilibrium profits of the one–shot game. In all these cases, central agglomeration is the unique subgame perfect Nash equilibrium (SPNE) outcome.

An often perceived explanation for this minimum differentiation is that, in the one–shot game the firms locate at the market extremes to minimize competition and earn higher profits through higher prices. In a repeated setting with price collusion, competition is softened and each firm finds it advantageous to move inwards, towards the other. So, both firms locate at the market center.

This intuition, however, is not quite correct. Rath and Zhao (2003) has shown that if the prices are determined either by the egalitarian [Kalai (1977)] or the Kalai and Smorodinsky (1975) bargaining solutions then there are multiple symmetric equilibria. If there is cost differential between firms, then minimal product differentiation is not obtained; see Matsumura and Matsushima (2011). An inward move by a firm results in a lower price that does not compensate for the increase in market share. Profit goes down and the firms locate off the center. So, in equilibrium, the products are not necessarily minimally differentiated. The relative magnitude of the reservation price and the transportation cost parameter determines the extent of differentiation. Thus, the equilibrium outcomes obtained in these papers are very detail specific and do not support any general conclusion on the nature of product differentiation.

This paper focuses on these minimal and nonminimal product differentiation results. Friedman and Thisse (1993) provided a set of sufficient conditions on profit sharing which ensures central agglomeration as the unique equilibrium outcome. Unfortunately, it was recently discovered in Rath and Zhao (2021) that those conditions are too strong. In the presence of some mild continuity requirements, there is exactly one profit sharing rule that satisfies those conditions. Under this rule, the firms charge identical prices at each pair of locations. Furthermore, with discounting, this sharing rule cannot be supported as an SPNE outcome at every pair of locations. This brings out an inconsistency in those conditions, and so cannot be used to characterize equilibria.

In light of the preceding discussion, it is futile to consider any strengthening of those conditions for uniqueness, rather one should consider a weakening of the conditions to encompass a wider class of profit sharing rules. However, slightly weakened versions can result in multiple equilibria. For the specific details, see Rath and Zhao (2021). The shortcomings of the conditions in Friedman and Thisse (1993) is a serious setback to the existence claims in this literature. In particular, sufficient conditions for a unique equilibrium is not available at present.

This paper addresses this issue and explores the possible nature of equilibria in this framework. If the firms share profits on the PPF then under very general conditions both the firms locating at the center is an equilibrium. The required conditions are: (a) the profits of the two firms be identical if they are symmetrically located and (b) the profit ratio be bounded either above or below one at distinct asymmetric locations, depending on how asymmetrically the firms are located. These conditions also rule out certain other locations being an equilibrium. However, at this level of generality, one should not expect a unique outcome, and further conditions are required for possible equilibrium predictions.

As is well known, if the firms locate together then the profit of each firm is indeterminate. This is a primary motivation behind Simon and Zame (1990) and it shows that alternative criteria for breaking ties have a critical bearing on possible equilibrium outcomes.

Two conditions on profit sharing between the firms, which arise naturally when the firms are located at the same point, are explored in the sequel. In one, the firms share the profits equally if they are identically located.[1] As a result, in the presence of symmetry, agglomeration off the center does not survive as an equilibrium. All equilibria are symmetric; in some cases, the equilibrium at the center emerges as the unique one but in others there are multiple equilibria inside market quartiles. In the other variant, the profit of each firm is determined by the limit of the profit as one firm approaches the other. Continuity of the profit functions is retained in this case. The profit of a firm might decrease or increase as it moves towards the other firm, and in the latter case, asymmetric equilibria with the firms agglomerating together can appear.

We describe below why these different results obtain and how the equilibrium outcomes are characterized. The primitives of this model are the reservation price, the transportation cost parameter and the discount factors of the firms. Given these entities, it is clear from the existing literature that different profit sharing rules yield different equilibrium outcomes (e.g., the Nash and the Kalai and Smorodinsky bargaining solutions). So, any equilibrium characterization must involve restrictions on the profit sharing rule itself. In this context, the profit ratio and

[1] Typically, this introduces a discontinuity in the profit functions at locations off the center. Since Dasgupta and Maskin (1986a, 1986b), the literature on the existence of Nash equilibrium in discontinuous games has grown considerably. See Reny (2016) and the other papers in the same volume.

the market shares of the firms emerge as natural candidates for further exploration since these are likely to be known in most situations and are easy to interpret.

Suppose that the firms share profits equally when they are identically located. Then the possible equilibrium outcomes are the symmetric locations inside the market quartiles. If a firm moves outwards, then typically its profit declines. On the other hand, an inward move results in a lower price but the market share can increase. Thus the effect on profit is ambiguous. If the profit increases, then the given pair of symmetric locations is not an equilibrium and if this is the case at all symmetric locations off the center, then there is a unique equilibrium at the center of the market. This increase in profit is captured through lower bounds on the derivatives of the ratio of profits (with respect to locations) to yield sufficient conditions for a unique equilibrium. In contrast, if the profit decreases by an inward move, then the pair of symmetric locations emerges as an equilibrium. This phenomenon provides upper bounds on market shares, which give necessary and sufficient conditions for symmetric equilibria off the center. These conditions have easy and interesting geometric interpretations. The other remaining scenario is that the profits are determined as limits when the firms agglomerate together. Again, an increase in profits as a firm moves inwards translates into lower bounds on the derivatives of the ratio of profits. These bounds determine whether asymmetric agglomerated equilibria exist or not.

The paper is organized as follows. The one–shot model is given in the next section. Conditions on profit sharing are given in section 3. Some implications for possible equilibrium locations are noted in section 4. Example 2 in this section demonstrates that under general conditions, every pair of symmetric locations inside the market quartiles can be an equilibrium. This underscores the importance of further conditions to restrict the set of equilibria. Section 5 gives sufficient conditions for a unique equilibrium at the center of the market. These are in the form of lower bounds on the derivatives of the profit ratio. Section 6 gives conditions for existence of multiple symmetric equilibria and interprets them geometrically. Section 7 deals with agglomerated equilibria off the center. Some aspects of sustaining collusion are discussed in section 8. Section 9 concludes. The proofs of the results are given in section 10.

2 The one-shot model

The consumers are uniformly distributed over the unit interval [0, 1]. The two firms are located in the interval at $x_1 \leq x_2$. Typically, a consumer is located distinctly from a firm and so incurs a utility cost in consuming the firm's product. This is formalized as a transportation cost. This cost is quadratic and the associated parameter is t. If the consumer is located at $s \in [0,1]$ and firm i charges a price p_i, $i = 1,2$, then the transportation cost is $t(s - x_i)^2$ and the delivered price is $p_i + t(s - x_i)^2$. Each consumer buys a unit of a good from the firm with the lower delivered price. When $x_1 < x_2$, the (marginal) consumer y who faces the same price from the two firms is given by $p_1 + t(y - x_1)^2 = p_2 + t(y - x_2)^2$. The solution gives

$$y = \frac{p_2 - p_1}{2t(x_2 - x_1)} + \frac{x_1 + x_2}{2}. \tag{1}$$

Each consumer in $[0, y]$ buys a unit from firm 1 and each consumer in $(y, 1]$ buys a unit from firm 2. The production costs are zero and the profit of the two firms are $p_1 y$ and $p_2(1 - y)$ respectively.

For each pair of locations a Nash equilibrium in prices exists. If $x_1 = x_2$ then the Nash equilibrium profit of each firm is zero. When $x_1 < x_2$ the Nash equilibrium profits are $R_{1N} = t(x_2 - x_1)(2 + x_1 + x_2)^2/18$ and $R_{2N} = t(x_2 - x_1)(4 - x_1 - x_2)^2/18$. The profit of a firm increases as it moves away from the other firm. So, in a one-shot, two stage game, where the duopolists choose locations first and prices next, the equilibrium locations are at the extreme endpoints of the market segment. For details, see d'Aspremont et al. (1979, p. 1149), Neven (1985), or Friedman and Thisse (1993, p. 634).

3 Some conditions on profit sharing

Let the time periods be given by $\{0, 1, 2, \ldots\}$. Suppose that the firms choose locations in period 0 and in subsequent infinite periods compete in prices. Future payoffs are discounted. Some general conditions on profit sharing by the two firms and their implications for equilibrium outcomes are examined in this section.

In the one-shot game of the previous section, the profit functions of the firms (in contrast to the Nash equilibrium profits) can be unbounded. Since we wish to address the issue of price collusion, we introduce a reservation price for the consumers which makes the profit functions bounded. It is presumed that the consumers buy a unit of the product per unit time from the cheapest seller, subject to the reservation price π. Given firm locations x_1, x_2 and prices p_1, p_2, the profit of firm i is $F_i(x_1, x_2, p_1, p_2)$, $i = 1,2$. If we let $x = (x_1, x_2)$ denote the pair of locations and if $p_1(x)$ and $p_2(x)$ are given, then the profit of firm i can be abbreviated to

$F_i(x)$ and the pair of profits can be written as $F(x) = (F_1(x), F_2(x))$. We will follow this interpretation. The ratio of profits is denoted by $\Lambda = F_1/F_2$. The PPF is the set of Pareto optimal profit allocations between the two firms. Some conditions on profit sharing between the two firms are given below.

(C1) For any $x_1 \leq x_2$, (F_1, F_2) is on the PPF and the profit of each firm is positive.

(C2) For any $x_1 \leq x_2$, $F_1 = F_2$ if $x_1 + x_2 = 1$.

(C3) For any $x_1 < x_2$, $F_1/F_2 < 1$ if $x_1 + x_2 < 1$ and $F_1/F_2 > 1$ if $x_1 + x_2 > 1$.

Lemma 3 in Friedman and Thisse (1993) shows that if $\pi \geq 3t$ then at any PPF prices the entire market is served. This assumption $\pi \geq 3t$ will be maintained throughout. If $\pi \geq 3t$, then (C1) is equivalent to the two conditions that (i) at a pair of prices the entire market is served, each firm has a positive market share and (ii) some consumer pays the reservation price. So, the PPF prices can be of three types depending on the location of the reservation price consumer: $p_1 = \pi - tx_1^2$, or $p_2 = \pi - t(1 - x_2)^2$, or $p_1 + t(y - x_1)^2 = \pi = p_2 + t(y - x_2)^2$. These correspond respectively to the cases that the reservation price is paid by consumer zero, or one, or the marginal consumer y.

(C2) is the symmetry condition. This requires that the profits be identical if the firms are symmetrically located. (C3) provides an upper or lower bound on the profit ratio depending on the type of asymmetry in locations. If $x_1 < x_2$ and $x_1 + x_2 < 1$ then firm 2 is closer to the market center than firm 1. In this case (C3) requires a higher profit for firm 2.

The profits need to be specified if the firms are located at the same point $x_1 = x_2 \neq 1/2$. Two possible alternative conditions (tie-breaking rules) are examined. Let $\bar{x}_1 = \bar{x}_2$.

(C4) $F_1(\bar{x}_1, \bar{x}_2) = F_2(\bar{x}_1, \bar{x}_2)$.

(C5) For $i = 1, 2$, $F_i(\bar{x}_1, \bar{x}_2) = \lim_{x_1 \to \bar{x}_2} F_i(x_1, \bar{x}_2) = \lim_{x_2 \to \bar{x}_1} F_i(\bar{x}_1, x_2)$.

(C4) is equal division. It stipulates that whenever the firms are located at the same point their profits are identical. (C5), on the other hand, requires that if the firms are located at the same point then the profits are determined by the limits of the profits as one firm gradually moves towards the other. It is presupposed that these limits exist and are identical.

(C4) can always be exogenously imposed by the modeler. (C5), on the other hand, determines profits endogenously by the structure of the profit sharing rule. If $\lim_{x_1 \to x_2} \Lambda \neq 1$ then (C4) introduces a discontinuity in the profit ratio if the firms are located at the same point. If

$\lim_{x_1 \to x_2} \Lambda = 1$ then the choice between (C4) and (C5) is inconsequential.[2] The implications of both these conditions are worth examining. It will be seen in sections 5–7 that the equilibrium outcomes are different under these two conditions.

First we present an example in which, given any arbitrary fixed locations $x^* = (x_1^*, x_2^*)$, there is a profit sharing rule under which x^* is the unique SPNE outcome. The profit sharing rule satisfies (C2)–(C4), but (C1) is violated at some (but not at all) pairs of locations. The example thus underscores the importance of (C1) at all locations.

Example 1 Fix a pair of locations $x^* = (x_1^*, x_2^*)$, $x_1^* \leq x_2^*$. Assume that the firms are on the PPF at x^*. If either $x_1^* = x_2^*$, or $x_1^* + x_2^* = 1$, then $F_1(x^*) = F_2(x^*)$. If $x_1^* < x_2^*$ and $x_1^* + x_2^* \neq 1$ then $F_1(x^*)/F_2(x^*) = 1/2$ when $x_1^* + x_2^* < 1$ and $F_1(x^*)/F_2(x^*) = 2$ when $x_1^* + x_2^* > 1$. (The point on the PPF corresponding to the profit ratio determines the prices of the firms.) At any $x = (x_1, x_2) \neq (x_1^*, x_2^*)$, $F_i(x) = R_{iN}(x)$, $i = 1, 2$. This profit sharing rule satisfies (C2)–(C4). (C1) is violated if $x \neq x^*$.

Consider the following strategy of firm i, $i = 1, 2$. (I) In period 0, choose location x_i^*. (II) Let $t \geq 1$. If the locations are x^*, charge the price corresponding to the profits $(F_1(x^*), F_2(x^*))$. Continue to charge this price until a defection from these prices obtain. In case of a defection, charge the one-shot Nash equilibrium price at x^* for the remainder of the game. If the locations are $x \neq x^*$ to begin with, then charge the one-shot Nash equilibrium price at x in all periods. We will show that these strategies are subgame perfect and yield the equilibrium outcome x^*.

At any pair of locations $R_{iN} \leq t/2$. On the other hand, at x^*, because of (C1) and $\pi \geq 3t$, some consumer pays the reservation price and the entire market is served. So, each price is greater then or equal to $\pi - t$, the total profit is greater then or equal to $\pi - t$ and each firm's profit is greater than or equal to $(\pi - t)/3$. From $\pi \geq 3t$, $(\pi - t)/3 \geq 3t/5$. Therefore, for $i = 1, 2$; $F_i(x^*) > F_i(x)$ for any $x \neq x^*$ and $F_i(x^*) \geq R_{iN}(x^*) + (t/10)$. The defection profit (at x^*) is at most π. This shows that for sufficiently high discount factors, the given strategies are subgame perfect and the unique equilibrium outcome is x^*.[3]

[2] This is true for some specific profit sharing rules, such as the Nash, Kalai–Smorodinsky and egalitarian bargaining solutions. See Rath and Zhao (2003), sections 4 and 6.

[3] Notice that there are two sources of payoff discontinuities in this example. One is caused by (C4), and is inevitable. Let $1/2 < x_1 < x_1^* = x_2^*$. Then $\Lambda(x_1^*, x_2^*) = 1$ and $\Lambda(x_1, x_2^*) = (2 + x_1 + x_2^*)^2/(4 - x_1 - x_2^*)^2 \not\to 1$ as $x_1 \to x_1^*$. The other is, if $1/2 < x_1 < x_1^* \leq x_2^*$ then $\lim_{x_1 \to x_1^*} F_i(x_1, x_2^*) \leq t/2 < 3t/5 \leq F_i(x_1^*, x_2^*)$. The latter should not be a concern. A modified version of this example (which we do not present here) has (C4) as the only source of discontinuity, but the conclusion remains the same.

A further aspect of the example merits attention. A particular pair of locations was fixed *a priori*, and the profit sharing rule was such that (C1) was satisfied at those locations and nowhere else. The unique equilibrium outcome was the starting pair of locations. Alternatively, one could consider profit sharing rules so that (C1) holds at all locations. This is the import of partial collusion. The subsequent sections address this issue. As will be seen, the set of equilibria is (considerably) restricted under (C1) and in some cases the equilibrium is unique.

4 Some implications of (C1)–(C4)

Lemmas 1 and 2 below characterize possible equilibrium outcomes.[4] The conclusions of these two lemmas are combined to yield Proposition 1. It states that under (C1)–(C4), the possible equilibrium outcomes are the symmetric ones inside the market quartiles.

Lemma 1 *Suppose that (C1)–(C3) hold.*

(i) $x_1 = x_2 = 1/2$ *is an equilibrium. Furthermore, any* $x_1 < x_2 = 1/2$*, or* $1/2 = x_1 < x_2$ *is not an equilibrium.*

(ii) Let $x_1 \leq 1/2 \leq x_2$*. If* $x_1 + x_2 < 1$ *then* $F_1(x_1, x_2) < F_1(1 - x_2, x_2)$*. If* $x_1 + x_2 > 1$ *then* $F_2(x_1, x_2) < F_2(x_1, 1 - x_1)$*. So,* $x_1 \leq 1/2 \leq x_2$ *and* $x_1 + x_2 \neq 1$ *is not an equilibrium.*

(iii) Any symmetric pair of locations $(x_1, 1 - x_1)$ *with* $x_1 < 1/4$ *is not an equilibrium.*

If (C1)–(C3) hold then (1/2, 1/2) is an equilibrium. Part (*ii*) rules out asymmetric equilibria on opposite sides of the center of the market. The other candidates for equilibrium are the symmetric locations $(x_1, 1 - x_1)$ with $1/4 \leq x_1 < 1/2$ and locations on the same side of the center of the market, i.e., $x_1 < x_2 < 1/2$, or $1/2 < x_1 < x_2$, or $x_1 = x_2 \neq 1/2$. These asymmetric equilibria can be ruled out if (C4) holds.[5]

Lemma 2 *Let (C1) and (C4) hold.*

(i) If (C2) holds then $x_1 = x_2 \neq 1/2$ *is not an equilibrium.*

(ii) If (C3) holds then neither $x_1 < x_2 < 1/2$ *nor* $1/2 < x_1 < x_2$ *can occur in equilibrium.*

[4] For the time being, if a particular pair of locations is claimed to be an equilibrium, it is made under the caveat that it can be sustained in a repeated setting. This aspect is examined in section 8.

[5] In some cases, asymmetric equilibria can be ruled out without the aid of (C4). Suppose that (C1)–(C3) and (C5) hold. If $\partial F_1/\partial x_1$ is negative for all $1/2 < x_1 < x_2$ then firm 1 will not locate at $x_1 \in (1/2, x_2]$. If a similar condition also holds for the profit function of firm 2 then under (C1)–(C3) and (C5), (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric ones $(x_1, 1 - x_1)$ with $1/4 \leq x_1 < 1/2$.

Proposition 1 *Suppose that (C1)–(C4) hold. Then (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric locations* $(x_1,\, 1 - x_1)$ *with* $1/4 \leq x_1 < 1/2$.

These results have been proved before under different conditions, see for example, Friedman and Thisse (1993) and Rath and Zhao (2003). The conditions (C1)–(C4) above are the weakest. In particular, if the entire market is served at a pair of prices, then (C3) above is weaker than (P4) in Rath and Zhao (2003). This is discussed in subsection 10.3.

Proposition 1 above provides the smallest set of equilibria under general conditions. Without additional assumptions on the profit sharing rule, the set of equilibria cannot be restricted further, as the following example shows.

Example 2 Let $1/4 \leq x_1^* \leq 1/2$. Then there is a profit sharing rule satisfying (C1)–(C4) such that every symmetric pair of locations in $[x_1^*, 1 - x_1^*]$ is an equilibrium.

Let $0 \leq \mu \leq 1$. Suppose that at any pair of locations the firms remain on the PPF. If $x_1 < x_2$, the firms charge prices such that the marginal consumer given in (1) is

$$y \;=\; \frac{1}{2} + \frac{t\mu}{\pi}(x_1 + x_2 - 1) \tag{2}$$

and if $x_1 = x_2$ then the profits are shared equally, i.e., $F_1(x) = F_2(x)$. At distinct locations, if μ increases then the market share of firm 1 increases, which could result in a higher profit. So, a firm might find it advantageous to move inwards.

This example is analyzed in detail in Rath and Zhao (2021). Some pertinent facts are given here. Since $t\mu/\pi \leq 1/3$, $1/6 \leq y \leq 5/6$. So, the profit of each firm is positive and (C1) holds. By assumption, (C4) holds and (C2) is immediate. To verify (C3), assume that $x_1 + x_2 > 1$. Then $y \geq 1/2$. Since $t\mu/\pi \leq 1/3$, $y < (x_1 + x_2)/2$. From (1), $p_2 < p_1$. Therefore, $p_1 y > p_2(1 - y)$, i.e., $F_1 > F_2$.

If $\mu \leq 4\pi/(16\pi - t)$ then any pair of symmetric locations in [1/4, 3/4] is an equilibrium. If $\mu \geq 2\pi/(4\pi - t)$ then (1/2, 1/2) is the unique equilibrium. If $4\pi/(16\pi - t) < \mu < 2\pi/(4\pi - t)$ then there is a unique $x_1^* \in (1/4, 1/2)$ such that any pair of symmetric locations in the interval $[x_1^*,\, 1 - x_1^*]$ is an equilibrium, where

$$x_1^* \;=\; \frac{-\pi + \sqrt{\pi^2 + 4\pi t\mu^2}}{2t\mu}. \tag{3}$$

As μ increases, the RHS increases and the set of equilibria becomes smaller.

The example shows that under (C1)–(C4), there can be a unique equilibrium at the market center, but multiple symmetric equilibria inside the market quartiles cannot be ruled out.

5 Sufficient conditions for a unique equilibrium

In the last section we saw that the only candidate for a unique equilibrium is at the market center (Proposition 1), and that, in general multiple equilibria exist (Example 2). Therefore, further structure on the profit sharing rule is needed to narrow down the set of equilibrium outcomes. There is no specific restriction that one can impose and alternative characterizations are possible. Any such condition should be intuitive and easily verifiable. The profit ratio Λ naturally suggests itself since it is easy to interpret and is likely to be known in many situations.

The next theorem provides sufficient conditions for a unique equilibrium involving the derivatives of Λ. It is worth noting that typically the profit functions are not differentiable at symmetric locations. Nevertheless, Λ might be differentiable. Since only inward moves need to be examined to determine equilibria because of Lemma 1 (*ii*), the derivatives are usually the right (left) hand side derivatives at symmetric locations with respect to changes in x_1 (x_2).

Theorem 1 *Suppose that (C1)–(C4) hold. For* $x_1 \in [1/4, 1/2)$, *if* $\partial\Lambda/\partial x_1 \geq t(1+2x_1)/(\pi - tx_1^2)$ *at* $(x_1, 1-x_1)$ *then the pair of locations is not an equilibrium. For* $x_2 \in (1/2, 3/4]$, *if* $\partial\Lambda/\partial x_2 \geq t(3-2x_2)/[\pi - t(1-x_2)^2]$ *at* $(1-x_2, x_2)$ *then the pair of locations is not an equilibrium. So, (1/2, 1/2) is the unique equilibrium if either* $\partial\Lambda/\partial x_1 \geq t(1+2x_1)/(\pi - tx_1^2)$ *at all symmetric locations with* $x_1 \in [1/4, 1/2)$, *or* $\partial\Lambda/\partial x_2 \geq t(3-2x_2)/[\pi - t(1-x_2)^2]$ *at all symmetric locations with* $x_2 \in (1/2, 3/4]$.

By Proposition 1, the possible equilibria are the symmetric ones in [1/4, 3/4]. The lower bounds on the derivatives of Λ ensure that the profit of a firm increases by an inward move from symmetric locations off the center. So, (1/2, 1/2) emerges as the unique equilibrium.

Corollary 1 *Let (C1)–(C4) hold. Suppose that* $\partial\Lambda/\partial x_1 \geq 8/11$ *at all symmetric locations* $(x_1, 1-x_1)$, $x_1 \in [1/4, 1/2)$, *or* $\partial\Lambda/\partial x_2 \geq 8/11$ *at all symmetric locations* $(1-x_2, x_2)$, $x_2 \in (1/2, 3/4]$. *Then (1/2, 1/2) is the unique equilibrium.*

This follows readily from the theorem. $t(1+2x_1)/(\pi - tx_1^2)$ is an increasing function of x_1 and equals $8t/(4\pi - t)$ when $x_1 = 1/2$. Since $\pi \geq 3t$, $8/11 \geq 8t/(4\pi - t)$. Therefore, if $\partial\Lambda/\partial x_1 \geq 8/11$ at all symmetric locations $(x_1, 1 - x_1)$ for $x_1 \in [1/4, 1/2)$ then central agglomeration is the unique equilibrium. A similar argument applies if $\partial\Lambda/\partial x_2 \geq 8/11$.

Below we consider three profit sharing rules to further elucidate Theorem 1. The same profit sharing rules are also examined after Theorems 2 and 3.

Illustrations (1) If the firms share profits on the PPF in proportion to the one–shot Nash equilibrium profits, as in Friedman and Thisse (1993), then $\Lambda = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$. If $x_2 \neq 1/2$ then Λ does not tend to 1 as x_1 tends to x_2, so one needs to assume (C4). Clearly, (C1)–(C3) are satisfied. $\partial\Lambda/\partial x_1 = 12(2 + x_1 + x_2)/(4 - x_1 - x_2)^3$ and equals 4/3 (> 8/11) at symmetric locations. So, central agglomeration is the unique equilibrium.

(2) If the firms share profits on the PPF in proportion to the Nash equilibrium market shares, see Schmalensee (1987) or Friedman and Thisse (1993), $\Lambda = (2 + x_1 + x_2)/(4 - x_1 - x_2)$. (C1)–(C3) are satisfied, and assume (C4). $\partial\Lambda/\partial x_1 = 6/(4 - x_1 - x_2)^2$ and equals 2/3 at symmetric locations. If $\pi \geq 4t$ then $\partial\Lambda/\partial x_1 > 8t/(4\pi - t)$ and central agglomeration is the unique equilibrium.

(3) Consider Example 2 in section 4. Unlike the preceding two illustrations, here the profit ratio Λ is not explicitly given. So, a bit more work is involved to determine a unique equilibrium. Let $\bar{x}_1 + \bar{x}_2 > 1$ and $\bar{x}_2 \leq 3/4$. Then $y \geq 1/2$. If $p_1 + t(y - \bar{x}_1)^2 = \pi = p_2 + t(y - \bar{x}_2)^2$, then either consumer zero is not served (if $y = 1/2$), or consumer 1 is not served (if $y > 1/2$), a contradiction. Since $y < (\bar{x}_1+\bar{x}_2)/2$, $p_2 < p_1$ and from $\pi - t(1-\bar{x}_2)^2 > \pi - t\bar{x}_1^2$, $p_2 \neq \pi - t(1-\bar{x}_2)^2$. Thus $p_1 = \pi - t\bar{x}_1^2$.

In this case, the two derivatives $\partial y/\partial x_1$ and $\partial\Lambda/\partial x_1$ are related by (5) in section 10. At symmetric locations, the equation simplifies to $p_1(\partial\Lambda/\partial x_1) = [4p_1 - 2t(1-2x_1)](\partial y/\partial x_1) + t(1-2x_1)$. So, $p_1(\partial\Lambda/\partial x_1) \geq t(1+2x_1) \Leftrightarrow [4p_1 - 2t(1-2x_1)](\partial y/\partial x_1) \geq 4tx_1$. Using $\partial y/\partial x_1 = t\mu/\pi$, we get $\mu \geq 2\pi x_1/[2p_1 - t(1 - 2x_1)]$. The RHS is increasing in x_1. Letting $x_1 = 1/2$ yields $\mu \geq 2\pi/(4\pi - t)$. This is exactly the bound obtained in the example for a unique equilibrium at the center of the market. In particular, since $3/5 > 2\pi/(4\pi - t)$, if $\mu \geq 3/5$ then there is a unique equilibrium at the market center.

6 Necessary and sufficient conditions for symmetric equilibria off the center

Consider a pair of distinct symmetric locations inside the market quartiles. It is an equilibrium if neither an outward nor an inward move by a firm is profitable. Outward moves are never profitable by part (*ii*) of Lemma 1. So, only inward moves need to be examined to determine whether the pair of locations is an equilibrium or not. In general, such a move by a firm results in a lower price. So, if the market share does not increase significantly, then profit decreases and the given pair of symmetric locations emerges as an equilibrium. Along these lines, consider the following entities.

For $x_1 < x_2$, let $\sigma_1 = [\pi - t(1-x_2)^2]/[2(\pi - tx_1^2)]$, $\sigma_2 = (\pi - tx_1^2)/[2(\pi - t(1-x_2)^2)]$,

$$\begin{aligned} Q_1 &= \frac{\pi - t(1-x_2)^2}{2[\pi - tx_1^2 + t(x_2 - x_1)(2\sigma_1 - x_1 - x_2)](1-\sigma_1)}, \\ Q_2 &= \frac{2[\pi - t(1-x_2)^2 - t(x_2 - x_1)(2 - 2\sigma_2 - x_1 - x_2)](1-\sigma_2)}{\pi - tx_1^2}. \end{aligned}$$

These expressions play a central role in characterization of symmetric equilibrium locations off the center in the next theorem. An inward move and nonincreasing profit automatically put restrictions on the reservation price consumer and the market shares. These determine the bounds σ_1 and σ_2 on the market shares. Q_1 and Q_2, respectively, are the profit ratios when the market shares equal these bounds.

Theorem 2 *Suppose that (C1)–(C4) hold. Let (x_1^*, x_2^*) be a pair of symmetric locations, $1/2 < x_2^* \leq 3/4$. Then the following conditions are equivalent.*

(i) The pair of symmetric locations (x_1^*, x_2^*) is an equilibrium.

(ii) $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$ and $1 - y(x_1^*, x_2) \leq \sigma_2(x_1^*, x_2)$ for $x_1^* < x_2 < x_2^*$.

(iii) $\Lambda(x_1, x_2^*) \leq Q_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$ and $\Lambda(x_1^*, x_2) \geq Q_2(x_1^*, x_2)$ for $x_1^* < x_2 < x_2^*$.

Theorem 2 admits the following corollary, which is applicable in many situations.

Corollary 2 *Let (C1)–(C4) hold. Suppose that $\Lambda(1 - x_2, 1 - x_1) = 1/\Lambda(x_1, x_2)$ if $x_1 < x_2$. Let (x_1^*, x_2^*) be symmetric locations, $1/2 < x_2^* \leq 3/4$. Then the following conditions are equivalent.*

(i) The pair of symmetric locations (x_1^*, x_2^*) is an equilibrium.

(ii) $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$.

(iii) $\Lambda(x_1, x_2^*) \leq Q_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$.

The condition $\Lambda(1-x_2, 1-x_1) = 1/\Lambda(x_1, x_2)$ means that if the locations are flipped symmetrically, then the profit ratio reverses.[6] In this scenario, it is enough to check for inward moves of only one firm.

Illustrations (1) Let $\Lambda = (2+x_1+x_2)^2/(4-x_1-x_2)^2$. Then $\Lambda(1-x_2, 1-x_1) = 1/\Lambda(x_1, x_2)$ if $x_1 < x_2$. It was shown in the preceding section that central agglomeration is the unique equilibrium under this profit sharing rule. We will show that *(iii)* of Corollary 2 is violated.

Consider symmetric locations (x_1, x_2), $1/2 < x_2 \leq 3/4$. If $\Lambda(x_2, x_2) > Q_1(x_2, x_2)$ then by continuity, $\Lambda(\bar{x}_1, x_2) > Q_1(\bar{x}_1, x_2)$ for some $x_1 < \bar{x}_1 < x_2$. In this case, $\Lambda(x_2, x_2) = (1+x_2)^2/(2-x_2)^2$ and $Q_1(x_2, x_2) = [\pi - t(1-x_2)^2]/[\pi - tx_2^2 - t(2x_2-1)] = [\pi - tx_2^2 + t(2x_2 - 1)]/[\pi - tx_2^2 - t(2x_2-1)]$. So, $\Lambda(x_2, x_2) > Q_1(x_2, x_2)$ iff $3\pi - 3tx_2^2 - 5t + 2tx_2(1-x_2) > 0$. Since $\pi \geq 3t$, the inequality holds. Thus, the unique equilibrium is at the center of the market.

(2) If $\Lambda = (2+x_1+x_2)/(4-x_1-x_2)$ then $\Lambda(1-x_2, 1-x_1) = 1/\Lambda(x_1, x_2)$ if $x_1 < x_2$. Consider symmetric locations (x_1, x_2), $1/2 < x_2 \leq 3/4$. $\Lambda(x_2, x_2) = (1+x_2)/(2-x_2)$ and $\Lambda(x_2, x_2) > Q_1(x_2, x_2)$ iff $\pi - tx_2^2 - 3t > 0$. If $\pi \geq 4t$ then the inequality holds and central agglomeration is the unique equilibrium.

(3) If $y = (1/2) + (t\mu/\pi)(x_1 + x_2 - 1)$, as in Example 2, then $y(1-x_2, 1-x_1) = 1 - y(x_1, x_2)$ [which is equivalent to $\Lambda(1-x_2, 1-x_1) = 1/\Lambda(x_1, x_2)$, see Lemma 6]. In this case, $y \leq \sigma_1 \Leftrightarrow \pi(1+x_1-x_2) - 2\mu(\pi - tx_1^2) \geq 0$.[7] The LHS is increasing in x_1. As an equality at symmetric locations this becomes $t\mu x_1^2 + \pi x_1 - \pi\mu = 0$. Its positive root determines the equilibria in (3).

The conditions in Theorem 2 characterizing the equilibrium locations have nice geometric interpretations which we describe below. The shift in the PPF as a firm changes its location is instrumental in obtaining these results. Consider a pair of symmetric locations (x_1, x_2) with $1/2 < x_2 \leq 3/4$. In Figure 1, DSB is the PPF for this pair of locations. The line OS is the $45°$ line. Since all our profit sharing rules satisfy symmetry (C2), the solution of any profit sharing rule at the pair of symmetric locations (x_1, x_2) is at S and the profit of each firm is $[\pi - t(1-x_2)^2]/2 = (\pi - tx_1^2)/2$.

Suppose that firm 1 moves to the right from x_1 to $\bar{x}_1$, $x_1 < \bar{x}_1 < x_2$. The PPF for the locations $(\bar{x}_1, x_2)$ is not drawn but two of its important qualitative properties are: *(i)* on the

[6] Lemma 6 in section 10 examines some implications of this condition.

[7] If μ is close to 1 this inequality never holds and central agglomeration is the unique equilibrium.

horizontal axis the PPF expands and (*ii*) the point S no longer belongs to the profit possibility set (PPS) at locations $(\bar{x}_1, x_2)$.[8,9]

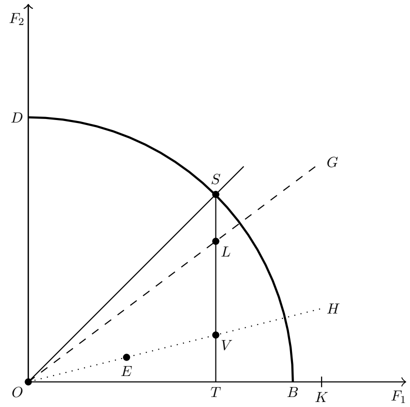

Figure 1

Along the line ST the profit of firm 1 is always $(\pi - tx_1^2)/2$. To determine which portion of the line ST belongs to the PPS at $(\bar{x}_1, x_2)$, one can maximize the profit of firm 2 subject to the constraint that the profit of firm 1 is $(\pi - tx_1^2)/2$. This determines the point L, its coordinates are given below.

Let $\bar{p}_1 = \pi - t\bar{x}_1^2$, $\bar{y} = (\pi - tx_1^2)/[2(\pi - t\bar{x}_1^2)]$, $\bar{p}_2 = \bar{p}_1 + t(x_2 - \bar{x}_1)(2\bar{y} - \bar{x}_1 - x_2)$ and define $\bar{F}_1 = \bar{p}_1\bar{y} = (\pi - tx_1^2)/2$, $\bar{F}_2 = \bar{p}_2(1 - \bar{y})$. The coordinates of point L are $(\bar{F}_1, \bar{F}_2)$. Notice two important facts here: $\bar{y}$ above is precisely $\sigma_1(\bar{x}_1, x_2)$ and $\bar{F}_1/\bar{F}_2 = Q_1(\bar{x}_1, x_2)$. Also observe that, since the PPS is convex and both L and K belong to it, so is the entire line segment LK. At any interior point of this line segment the profit of firm 1 is strictly greater than $(\pi - tx_1^2)/2$.

We can now examine the effects on profits and consequent equilibrium locations because of relocation by firm 1. At $(\bar{x}_1, x_2)$ symmetry no longer holds. So, in general, the profit allocations will be different under different profit sharing rules. Consider the line OG passing through L. The condition $Q_1(\bar{x}_1, x_2) < \Lambda(\bar{x}_1, x_2)$ means the line passing through the profit

[8] At locations (x_1, x_2), B corresponds to the maximum profit of firm 1 when firm 2 has zero profit. This happens when firm 1 serves the entire market, i.e., the price as well as the profit of firm 1 is $\pi - t(1 - x_1)^2$. By similar reasoning, at locations $(\bar{x}_1, x_2)$, firm 1 can earn $\pi - t(1 - \bar{x}_1)^2 > \pi - t(1 - x_1)^2$ if $\bar{x}_1 \leq 1/2$, or $\pi - t\bar{x}_1^2 > \pi - t(1 - x_1)^2$ if $\bar{x}_1 > 1/2$. So, on the horizontal axis the PPF at $(\bar{x}_1, x_2)$ expands, say to K, where the profit of firm 1 is either $\pi - t(1 - \bar{x}_1)^2$, or $\pi - t\bar{x}_1^2$.

[9] Point S has coordinates $\big([\pi - tx_1^2]/2, [\pi - tx_1^2]/2\big)$. What is *extremely important* is that point S no longer belongs to the PPS at locations $(\bar{x}_1, x_2)$. If S belongs to the PPS, then without loss of generality we can suppose that there is (p_1, p_2, y) such that $p_1 y \geq (\pi - tx_1^2)/2$, $p_2(1 - y) \geq (\pi - tx_1^2)/2$ and $(p_1 y, p_2[1 - y])$ is on the PPF. $p_1 y \geq (\pi - tx_1^2)/2$ and $p_1 \leq \pi - t\bar{x}_1^2 < \pi - tx_1^2$ imply that $y > 1/2$. On the other hand, $p_2(1 - y) \geq (\pi - tx_1^2)/2$ and $p_2 \leq \pi - t(1 - x_2)^2 = \pi - tx_1^2$ imply that $1 - y \geq 1/2$, i.e., $y \leq 1/2$. This is a contradiction. So, at the locations $(\bar{x}_1, x_2)$, the point S as well as points near S lie outside the PPS.

allocation $(F_1(\bar{x}_1, x_2), F_2(\bar{x}_1, x_2))$ [under the specific sharing rule given by $\Lambda(\cdot,\cdot)$] lies to the right of the line OG, say OH in the figure. This implies that $F_1(\bar{x}_1, x_2) > (\pi - tx_1^2)/2$ and the pair of symmetric locations (x_1, x_2) is not an equilibrium. This phenomenon is captured in condition (iii) of Theorem 2.

On the other hand (and unlike the figure), if the line OH were to the left of the line OG, then $\Lambda(\bar{x}_1, x_2) < Q_1(\bar{x}_1, x_2)$, the profit of firm 1 decreases as it moves inwards $[F_1(\bar{x}_1, x_2) < (\pi - tx_1^2)/2]$, and the pair of symmetric locations (x_1, x_2) is a candidate for an equilibrium as depicted by Theorem 2.

It is worth exploring how these general insights translate into specific profit sharing rules. We examine further the three illustrations given after Theorem 2. We know that if $\Lambda = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$, or $\Lambda = (2 + x_1 + x_2)/(4 - x_1 - x_2)$ then there is a unique equilibrium at the market center.

Suppose that $\Lambda = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$. Let point E in the figure correspond to the one-shot Nash equilibrium profits at $(\bar{x}_1, x_2)$. Its coordinates are $(R_{1N}, R_{2N}) = \big(t(x_2 - \bar{x}_1)[2 + \bar{x}_1 + x_2]^2/18, t(x_2 - \bar{x}_1)[4 - \bar{x}_1 - x_2]^2/18\big)$. Given Λ, the solution will be where the line OH intersects the PPF at $(\bar{x}_1, x_2)$.

The geometric interpretation of $Q_1(\bar{x}_1, x_2) < \Lambda(\bar{x}_1, x_2)$ is that the line OH lies below the line OG. This is equivalent to point V lies below point L. Since V lies on ST and OH, its coordinates are $\big(\bar{F}_1, \bar{F}_1[R_{2N}/R_{1N}]\big)$. Point L lies above point V iff $\bar{p}_2(1 - \bar{y}) - \bar{F}_1(R_{2N}/R_{1N}) > 0$. This inequality hods for $\bar{x}_1 \in (x_1, x_2)$, i.e., the profit of firm 1 increases because of the inward move.

An analogous explanation applies if $\Lambda = (2+x_1+x_2)/(4-x_1-x_2)$. Now assume that $\pi \geq 4t$ and point E has coordinates $([2 + \bar{x}_1 + x_2]/6, [4 - \bar{x}_1 - x_2]/6)$, given by the one-shot Nash equilibrium market shares. The coordinates of point V are $\big(\bar{F}_1, \bar{F}_1[(4 - \bar{x}_1 - x_2)/(2 + \bar{x}_1 + x_2)]\big)$. Point L lies above point V iff $\bar{p}_2(1 - \bar{y}) - \bar{F}_1[(4 - \bar{x}_1 - x_2)/(2 + \bar{x}_1 + x_2)] > 0$. As in the earlier case, this inequality hods for $\bar{x}_1 \in (x_1, x_2)$.

Lastly, consider the profit sharing rule of Example 2. In this case, $y = (1/2) + (t\mu/\pi)(x_1 + x_2 - 1)$ and the set of equilibria is determined by μ. Recall that $\Lambda(\bar{x}_1, x_2) \leq Q_1(\bar{x}_1, x_2) \Leftrightarrow y(\bar{x}_1, x_2) \leq \sigma_1(\bar{x}_1, x_2)$. Consider the line OH, point E can be ignored now and V is relevant only if OH lies to the right of OG. If the line OH is indeed to the right of line OG as in the figure, then $\Lambda(\bar{x}_1, x_2) > Q_1(\bar{x}_1, x_2)$ and those locations are not equilibrium locations. On the other hand, if the line OH is to the left of the line OG, or coincides with it, then $\Lambda(\bar{x}_1, x_2) \leq Q_1(\bar{x}_1, x_2)$

and equilibrium locations are obtained.

The exact placement of the line OH vis-a-vis the line OG is determined by the symmetric equilibrium locations (x_1, x_2) given in (3). In particular, if $\mu = 0$ then $y(\bar{x}_1, x_2) = 1/2 < (\pi - tx_1^2)/[2(\pi - t\bar{x}_1^2)] = \sigma_1(\bar{x}_1, x_2)$. So, any pair of symmetric locations (x_1, x_2) with $1/2 \leq x_2 \leq 3/4$ is an equilibrium. At the other extreme, if $\mu = 1$ then $y(\bar{x}_1, x_2) = (1/2) + (t/\pi)(\bar{x}_1 + x_2 - 1) > (\pi - tx_1^2)/[2(\pi - t\bar{x}_1^2)] = \sigma_1(\bar{x}_1, x_2)$ is equivalent to $\pi(2 - x_1 - \bar{x}_1) - 2t\bar{x}_1^2 > 0$. Since $x_1 < 1/2$ and $\bar{x}_1 \leq 3/4$, the inequality holds and central agglomeration is the unique equilibrium.

7 Existence of asymmetric equilibria

For many profit sharing rules, $\lim_{x_1 \to x_2} \Lambda \neq 1$ for some $x_2 \neq 1/2$. To preserve the continuity of Λ, (C5) can be imposed instead of (C4). As Lemma 2 suggests, (C4) primarily rules out asymmetric equilibria. Without (C4) it may not be possible to rule those out. So, asymmetric equilibria can exist and the firms may agglomerate at a point off the center.

Theorem 3 *Suppose that (C1)–(C3) and (C5) hold.*

(i) $(1/2, 1/2)$ *is an equilibrium.*

(ii) Let $x_1^* = x_2^* \neq 1/2$ *and* $x^* = (x_1^*, x_2^*)$*. If* $(\partial\Lambda/\partial x_1)/\Lambda^2 \geq t(1 + 2x_1)/\left(\pi - tx_2^{*2}\right)$ *at all* $x_1 < x_2^*$ *and* $\partial\Lambda/\partial x_2 \geq t(3 - 2x_2)/\left[\pi - t(1 - x_1^*)^2\right]$ *at all* $x_2 > x_1^*$ *then* x^* *is an equilibrium.*

Like Theorem 1, the lower bounds on the derivatives of Λ ensure that the profit of a firm increases by an inward move. So, x^* is an equilibrium.

Corollary 3 *Let (C1)–(C3) and (C5) hold. Suppose that* $\Lambda(1 - x_2, 1 - x_1) = 1/\Lambda(x_1, x_2)$ *if* $x_1 < x_2$*. Let* $x_1^* = x_2^* \neq 1/2$ *and* $x^* = (x_1^*, x_2^*)$*.*

If $x_2^* > 1/2$, $[\partial\Lambda(x_1, x_2^*)/\partial x_1]/\Lambda^2 \geq t(1+2x_1)/\left(\pi - tx_2^{*2}\right)$ *at all* $x_1 < x_2^*$ *and* $[\partial\Lambda(x_1, x_2)/\partial x_1]/\Lambda^2 \geq t/(\pi - t)$ *at all* $x_1 < x_2 < 1/2$*, then* x^* *is an equilibrium.*

If $x_1^* < 1/2$, $\partial\Lambda(x_1^*, x_2)/\partial x_2 \geq t(3-2x_2)/\left[\pi - t(1 - x_1^*)^2\right]$ *at all* $x_2 > x_1^*$ *and* $\partial\Lambda(x_1, x_2)/\partial x_2 \geq t/(\pi - t)$ *at all* $x_2 > x_1 > 1/2$ *then* x^* *is an equilibrium.*

The profit ratio reversal condition in the corollary ensures that it is enough to examine changes in location of only one firm to determine equilibria.

Illustrations (1) If $\Lambda = (2 + x_1 + x_2)^2/(4 - x_1 - x_2)^2$ then $\partial\Lambda/\partial x_1 = \partial\Lambda/\partial x_2 = 12(2 + x_1 +$

$x_2)/(4-x_1-x_2)^3$ and $(\partial\Lambda/\partial x_1)/\Lambda^2 = 12(4-x_1-x_2)/(2+x_1+x_2)^3$. If $x_1 < x_2 < 1/2$ then $(\partial\Lambda/\partial x_1)/\Lambda^2 \geq 4/3 > t/(\pi - t)$ and if $1/2 < x_1 < x_2$ then $\partial\Lambda/\partial x_2 \geq 4/3 > t/(\pi - t)$.

The function $[(\partial\Lambda/\partial x_1)/\Lambda^2] - [t(1+2x_1)/(\pi - tx_2^2)]$ is decreasing in both x_1 and x_2. To identify equilibria with $x_1^* = x_2^* > 1/2$, one can take $x_1 = x_2$ and solve $[3(2-x_2)/(1+x_2)^3] - [t(1+2x_2)/(\pi - tx_2^2)] \geq 0$. This is decreasing in x_2. The function $(\partial\Lambda/\partial x_2) - (t(3-2x_2)/[\pi - t(1-x_1)^2])$ is increasing in both x_1 and x_2. To identify equilibria with $x_1^* = x_2^* < 1/2$, one can take $x_2 = x_1$ and solve $[3(1+x_1)/(2-x_1)^3] - (t(3-2x_1)/[\pi - t(1-x_1)^2]) \geq 0$. This is increasing in x_1. If $\pi \geq 3t$ then any point in the interval $[0.35, 0.65]$ is an equilibrium. If $\pi \geq 9t$ then any point in the interval $[0,1]$ is an equilibrium.

This observation also underscores the role of (C4) in section 5, that the profits be identical if the firms are located together, to obtain central agglomeration as the unique equilibrium.

(2) If $\Lambda = (2+x_1+x_2)/(4-x_1-x_2)$ then $\partial\Lambda/\partial x_1 = \partial\Lambda/\partial x_2 = 6/(4-x_1-x_2)^2$ and $(\partial\Lambda/\partial x_1)/\Lambda^2 = 6/(2+x_1+x_2)^2$. If $x_1 < x_2 < 1/2$ then $(\partial\Lambda/\partial x_1)/\Lambda^2 \geq 2/3 > t/(\pi - t)$ and if $1/2 < x_1 < x_2$ then $\partial\Lambda/\partial x_2 \geq 2/3 > t/(\pi - t)$ when $\pi \geq 4t$.

The function $[(\partial\Lambda/\partial x_1)/\Lambda^2] - [t(1+2x_1)/(\pi - tx_2^2)]$ is decreasing in both x_1 and x_2. To identify equilibria with $x_1^* = x_2^* > 1/2$, one can take $x_1 = x_2$ and solve $[(3/2)/(1+x_2)^2] - [t(1+2x_2)/(\pi - tx_2^2)] \geq 0$. This is decreasing in x_2. The function $(\partial\Lambda/\partial x_2) - (t(3-2x_2)/[\pi - t(1-x_1)^2])$ is increasing in both x_1 and x_2. To identify equilibria with $x_1^* = x_2^* < 1/2$, one can take $x_2 = x_1$ and solve $[(3/2)/(2-x_1)^2] - (t(3-2x_1)/[\pi - t(1-x_1)^2]) \geq 0$. This is increasing in x_1. If $\pi \geq 4t$ then any point in the interval $[0.45, 0.55]$ is an equilibrium. If $\pi \geq 9t$ then any point in the interval $[0,1]$ is an equilibrium.

(3) Let $\mu = 1$ in Example 2. Fix $1/2 < x_2 < 3/4$. Lemma 1 shows that if $x_1 + x_2 < 1$ then $F_1(x_1, x_2) < F_1(1-x_2, x_2)$. Assume that $x_1 + x_2 > 1$. Then $p_1 = \pi - tx_1^2$. Since $\partial y/\partial x_1 = t/\pi$, $\partial F_1/\partial x_1 = (\pi - tx_1^2)(t/\pi) - 2tyx_1$. This is decreasing in x_1 and is positive when $x_1 = 1/2$.

Now let $1/2 < x_1 < x_2$. Then $p_1 = \pi - tx_1^2$. It is easily verified that $y < (x_1+x_2)/2$, i.e., $p_2 < p_1$. Note that $1 - y = (1/2) - (t/\pi)(x_1 + x_2 - 1) < (1/2) - (t/\pi)(2x_1 - 1)$. By (C5), $F_2(x_1, x_1) = p_1[(1/2) - (t/\pi)(2x_1 - 1)] > p_2(1-y) = F_2(x_1, x_2)$.

Analogously, it can be shown that (i) if $1/4 < x_1 < 1/2$, then $F_2(x_1, x_2) < F_2(x_1, 1-x_1)$ when $x_1 + x_2 > 1$ and $\partial F_2/\partial x_2$ is negative when $x_2 = 1/2$ and (ii) if $x_1 < x_2 < 1/2$ then $F_1(x_2, x_2) > F_1(x_1, x_2)$.

This means, there is an interval containing $1/2$ such that any point in it is an equilibrium.

Therefore, if μ is large then the firms may locate together off the center.

We conclude this section with some observations on the similarities and dissimilarities between Theorems 1 and 3. Under each, (1/2, 1/2) is an equilibrium. More generally, in these two results, the firms always agglomerate. This happens because each firm finds it advantageous to move towards the other. However, in Theorem 1 the equilibrium at the center is unique, but Theorem 3 exhibits multiple equilibria off the center. This difference highlights the distinction between (C4) and (C5).

8 Sustaining collusion

In our setting, the firms choose locations in the initial period and prices in subsequent infinite periods. If a specific pair of locations is an SPNE outcome of the supergame, then the collusive prices at any given pair of locations is an SPNE outcome of the repeated game. In this context, one can examine optimal punishment paths as in Abreu (1988). However, for existence purposes reversion to Nash equilibrium prices of the one-shot game is sufficient.

If at each pair of locations, the firm profits are (uniformly) bounded away from the Nash equilibrium profits, then for sufficiently high discount factors, the prices are subgame perfect in the infinite horizon location choice game. Formally, for $i = 1$, 2 and $x_1 \leq x_2$, let $s_i = \inf_{(x_1,x_2)}\{F_i(x_1,x_2) - R_{iN}(x_1,x_2)\}$ and $s = \min\{s_1, s_2\}$. Since $F_i(x_1,x_2) < \pi$ for $i = 1$, 2, $s < \pi$. That s be positive is essential. Denote by δ_i the discount factor of firm i, $i = 1, 2$.

Lemma 3 *Let (C1)–(C3) and one of (C4) and (C5) hold for the profit allocation* (F_1, F_2). *If* $s > 0$ *and* $\delta_i \in [(\pi - s)/\pi, 1)$ *for* $i = 1$, 2, *then* $(F_1(x_1,x_2), F_2(x_1,x_2))$ *is an SPNE outcome at each* (x_1, x_2).

The lemma shows the existence of SPNE under very general conditions. In all the illustrations considered in the preceding sections, $s > 0$. On occasions, however, it may not be possible to directly compute s. More critically, s might be nonpositive. A positive lower bound on s can be obtained with some other structure on the profit ratio Λ.

Lemma 4 *Let (C1)–(C3) and one of (C4) and (C5) hold. Suppose that for all* $x_1 < x_2$: *(i)* $1/r \leq \Lambda \leq r$ *for some* $1 < r < \infty$, *(ii)* $(x_1 + x_2)/(2 - x_1 - x_2) \leq \Lambda$ *if* $x_1 + x_2 < 1$ *and* $\Lambda \leq (x_1 + x_2)/(2 - x_1 - x_2)$ *if* $x_1 + x_2 > 1$. *Then* $s \geq t/(1 + r) > 0$.

Recall that (C3) provided a bound for the profit ratio in one direction, depending on location asymmetry. Condition (ii) above supplements this by providing a bound in the other direction. This lemma gives sufficient conditions for s to be positive, and in turn Lemma 3 ensures the existence of positive discount factors less than one.

9 Conclusion

The general framework is that symmetric duopolists choose locations once and then repeatedly choose prices to remain on the PPF. Schmalensee (1987), Jehiel (1992), Friedman and Thisse (1993) and Rath and Zhao (2003) have identified the equilibria in this setting when the firms share profits in proportion to the one-shot Nash equilibrium profits and market shares and under different bargaining solutions. In some cases, there is a unique equilibrium at the market center (minimal product differentiation), but in others multiple symmetric equilibria can exist (nonminimal product differentiation).

Friedman and Thisse (1993) did propose a set of sufficient conditions for a unique equilibrium at the market center. However, it was shown recently in Rath and Zhao (2021) that those conditions are inconsistent in some sense and cannot be used to characterize equilibria, unique or not. The discovery of this flaw is a serious setback to the existence claims in this literature. The consequence is that the works cited above address some special cases only but no general characterization result is available. In particular, sufficient conditions for a unique equilibrium is yet unknown.

This paper has systematically explored the nature of equilibria in this framework. It has been shown that if at symmetric locations inside market quartiles the derivatives of the profit ratio exceed certain magnitudes then there is a unique equilibrium. In our opinion, this is the first general result asserting a unique outcome at the market center and minimal product differentiation. Depending on the profit sharing rule, other symmetric equilibria might exist. Necessary and sufficient conditions for this have been identified in terms of upper bounds on the market shares and the profit ratio. That these conditions have nice geometric interpretations is appealing. Furthermore, asymmetric equilibria might exist if equal division of profits when the firms agglomerate together is not assumed. These results, thus characterize the equilibria in this framework under fairly general conditions.

Throughout, we have assumed that the costs of the two firms are identical. In Matsumura and Matsushima (2011), there is cost differential between the firms. They have shown that the agglomeration results in Jehiel (1992) and Friedman and Thisse (1993) do not hold under firm asymmetry, but have not derived the equilibrium locations. This is an interesting topic for further research.

10 Proofs of the results

In the next two subsections, we present some preliminary results and the derivatives of profit functions. The main results of the paper are proved subsequently.

10.1 Some preliminary results

Claim 1 *For fixed locations $x_1 < x_2$, suppose that $\min\{p_1, p_2\} \geq 2t(x_2 - x_1)$, the entire market is served at these prices and the profit of each firm is positive. Then the profit function of a firm is decreasing in its own price and is increasing in the other price. So, Λ is decreasing in p_1 and is increasing in p_2.*

This is immediate from $\partial y/\partial p_i = (-1)^i/[2t(x_2 - x_1)]$, $i = 1, 2$.

The next claim shows that if at fixed locations, the profit ratio or the market shares are held fixed, then the corresponding prices are unique.

Claim 2 *For fixed locations $x_1 < x_2$, let $p = (p_1, p_2)$ and $w = (w_1, w_2)$ be two sets of prices such that: (a) the corresponding profit allocations are on the PPF and (b) the profit of each firm is positive. Then $p = w$, if either (i) $F_1(p)/F_2(p) = F_1(w)/F_2(w)$, or if (ii) $p_2 - p_1 = w_2 - w_1$.*

Proof (i) Suppose that $w_1 > p_1$. If $w_2 > p_2$ then the entire market is not served at w, a contradiction. So, $w_2 \leq p_2$. Observe that the entire market is served and the profit of each firm is positive at the prices (p_1, w_2). By Claim 1, $\Lambda(p) \geq \Lambda(p_1, w_2) > \Lambda(w)$, a contradiction.

Suppose that $w_1 < p_1$. If $w_2 < p_2$ then no consumer pays the reservation price at the prices (w_1, w_2), a contradiction. So, $w_2 \geq p_2$. Observe that the entire market is served and the profit of each firm is positive at the prices (w_1, p_2). By Claim 1, $\Lambda(p) < \Lambda(w_1, p_2) \leq \Lambda(w)$, a contradiction. So, $w_1 = p_1$. If $w_2 > p_2$ then $\Lambda(p) < \Lambda(w)$ and if $w_2 < p_2$ then $\Lambda(p) > \Lambda(w)$. So, $w_2 = p_2$ and $w = p$.

(ii) If $w_2 > p_2$ then $w_1 > p_1$ and the entire market is not served at the prices (w_1, w_2), a contradiction. If $w_2 < p_2$ then $w_1 < p_1$ and no consumer pays the reservation price at the prices (w_1, w_2), a contradiction. So, $w = p$. ■

The following lemma provides a partial characterization of the prices and also determines the effect of particular prices on profits.

Lemma 5 *Let (C1) and (C2) hold and $1/4 \leq x_1 < x_2 \leq 3/4$.*

(i) If $x_1 + x_2 \neq 1$, then $p_1 \neq \pi - t(y - x_1)^2$ and $p_2 \neq \pi - t(y - x_2)^2$.

(ii) If $x_1+x_2 > 1$ and $p_2 = \pi - t(1-x_2)^2$, then $F_1(x_1, x_2)/F_2(x_1, x_2) > (x_1+x_2)/(2-x_1-x_2)$ and $F_1(x_1, x_2) > F_1(1 - x_2, x_2) = p_2/2$.

(iii) If $x_1 + x_2 < 1$ and $p_1 = \pi - tx_1^2$, then $F_1(x_1, x_2)/F_2(x_1, x_2) < (x_1 + x_2)/(2 - x_1 - x_2)$ and $F_2(x_1, x_2) > F_2(x_1, 1 - x_1) = p_1/2$.

Proof (i) Suppose that $x_1 + x_2 \neq 1$. Specifically, let $x_1 + x_2 > 1$. If $p_1 = \pi - t(y - x_1)^2$ then $p_2 = \pi - t(y - x_2)^2$. Either one is not served (if $y > 1/2$), or zero is not served (if $y \leq 1/2$), a contradiction. A similar argument applies if $x_1 + x_2 < 1$.

(ii) Let $x_1 + x_2 > 1$ and $p_2 = \pi - t(1 - x_2)^2$. Let $x = (x_1, x_2)$ and $p = (p_1, p_2)$. Since (C1) holds, $F_i(x, p) > 0$ for $i = 1, 2$. Let $\bar{p}_1 = \pi - tx_1^2$. Then $p_1 \leq \bar{p}_1 < p_2$.

Since the entire market is served at the price vector p and $p_1 < p_2$, the entire market is served as well at the price vector (p_1, p_1). $F_2(x, p) > 0$ implies that $F_2(x, p_1, p_1) > 0$. Notice that $p_1 + tx_1^2 < p_1 + tx_2^2$, i.e., zero is not served by firm 2 at the prices (p_1, p_1). So, $F_1(x, p_1, p_1) > 0$. By Claim 1, $F_1(x, p)/F_2(x, p) > F_1(x, p_1, p_1)/F_2(x, p_1, p_1) = (x_1 + x_2)/(2 - x_1 - x_2)$.

It is easy to show that the entire market is served at the prices $(\bar{p}_1, p_2)$. $F_2(x, p) > 0$ implies that $F_2(x, \bar{p}_1, p_2) > 0$. From $x_1 + x_2 > 1$, $x_2 > 1/2$. So, $p_2 + tx_2^2 = \pi + t(2x_2 - 1) > \pi$, which means that firm 2 does not serve zero. This gives $F_1(x, \bar{p}_1, p_2) > 0$.

By Claim 1, $F_1(x, p) \geq F_1(x, \bar{p}_1, p_2)$. When the prices are $(\bar{p}_1, p_2)$, the market share of firm 1 is greater than $(x_1 + x_2)/2$ from (1), i.e., $F_1(x, \bar{p}_1, p_2) \geq (\pi - tx_1^2)(x_1 + x_2)/2$. Therefore, to show that $F_1(x, p) > p_2/2$, it suffices to show that $(\pi - tx_1^2)(x_1 + x_2) > \pi - t(1 - x_2)^2$. This is true since the LHS is increasing in x_1 and equals the RHS when $x_1 = 1 - x_2$.

(iii) The proof is analogous. ■

10.2 The derivatives of the profit functions

The derivatives of the profit functions with respect to locations are needed in the proofs of Theorems 1 and 3. These are derived at the outset.

Since $p_1 y = p_2(1-y)\Lambda$,

$$\frac{\partial p_1}{\partial x_1} y + p_1 \frac{\partial y}{\partial x_1} = p_2(1-y)\frac{\partial \Lambda}{\partial x_1} + \Lambda\left(\frac{\partial p_2}{\partial x_1}(1-y) - p_2\frac{\partial y}{\partial x_1}\right). \tag{4}$$

As noted earlier, depending on the reservation price consumer, the prices can be of three types: (i) $p_1 = \pi - tx_1^2$, or (ii) $p_2 = \pi - t(1-x_2)^2$, or (iii) $p_1 = \pi - t(y-x_1)^2$. The prices of the two firms are related by $p_2 = p_1 + t(x_2 - x_1)(2y - x_1 - x_2)$. Given one of the prices and its derivative (which may involve the derivative of the market share), the derivative of the other can be determined. These in turn determine the derivatives of the profit functions.

If $p_1 = \pi - tx_1^2$ then $\partial p_1/\partial x_1 = -2tx_1$. From (4),

$$\frac{\partial y}{\partial x_1} = \frac{(p_1 y/\Lambda)(\partial\Lambda/\partial x_1) - 2ty(1-y)\Lambda + 2tyx_1}{p_1 + [p_2 - 2t(x_2-x_1)(1-y)]\Lambda} \tag{5}$$

$$\frac{1}{\Lambda y} \times \frac{\partial F_1}{\partial x_1} = \frac{(p_1^2/\Lambda^2)(\partial\Lambda/\partial x_1) - 2tp_1(1 + x_1 - y) + 2t^2(x_2-x_1)(x_1+x_2+2-4y)x_1}{p_1 + [p_2 - 2t(x_2-x_1)(1-y)]\Lambda}. \tag{6}$$

If $p_2 = \pi - t(1-x_2)^2$, then $\partial p_2/\partial x_1 = 0$.

$$\frac{1}{\Lambda y} \times \frac{\partial F_1}{\partial x_1} = \frac{(p_1/\Lambda^2)(\partial\Lambda/\partial x_1)[p_1 - 2t(x_2-x_1)y] + 2tp_2(y-x_1)}{p_1 - 2t(x_2-x_1)y + p_2\Lambda}. \tag{7}$$

If $p_1 = \pi - t(y-x_1)^2$, then $\partial p_1/\partial x_1 = 2t(y-x_1) - 2t(y-x_1)(\partial y/\partial x_1)$.

$$\frac{1}{\Lambda y} \times \frac{\partial F_1}{\partial x_1} = \frac{(p_1/\Lambda^2)(\partial\Lambda/\partial x_1)[p_1 - 2t(y-x_1)y] + 2t(y-x_1)[p_2 + 2t(y-x_2)(1-y)]}{p_1 - 2t(y-x_1)y + [p_2 + 2t(y-x_2)(1-y)]\Lambda}. \tag{8}$$

Similarly, from $p_1 y = p_2(1-y)\Lambda$,

$$\frac{\partial p_1}{\partial x_2} y + p_1 \frac{\partial y}{\partial x_2} = p_2(1-y)\frac{\partial \Lambda}{\partial x_2} + \Lambda\left(\frac{\partial p_2}{\partial x_2}(1-y) - p_2\frac{\partial y}{\partial x_2}\right). \tag{9}$$

If $p_1 = \pi - tx_1^2$, then $\partial p_1/\partial x_2 = 0$. From (9),

$$\frac{1}{1-y} \times \frac{\partial F_2}{\partial x_2} = \frac{2tp_1(y-x_2) - p_2[p_2 - 2t(x_2-x_1)(1-y)](\partial\Lambda/\partial x_2)}{p_1 + [p_2 - 2t(x_2-x_1)(1-y)]\Lambda}.$$

If $p_2 = \pi - t(1-x_2)^2$ then $\partial p_2/\partial x_2 = 2t(1-x_2)$.

$$\frac{1}{1-y} \times \frac{\partial F_2}{\partial x_2} = \frac{2tp_2(1-x_2+y) + 2t^2(x_2-x_1)(x_1+x_2-4y)(1-x_2) - p_2^2(\partial\Lambda/\partial x_2)}{p_1 - 2t(x_2-x_1)y + p_2\Lambda}.$$

If $p_1 = \pi - t(y-x_1)^2$, then $\partial p_1/\partial x_2 = -2t(y-x_1)(\partial y/\partial x_2)$.

$$\frac{1}{1-y} \times \frac{\partial F_2}{\partial x_2} = \frac{2t(y-x_2)[p_1 - 2t(y-x_1)y] - p_2[p_2 + 2t(y-x_2)(1-y)](\partial\Lambda/\partial x_2)}{p_1 - 2t(y-x_1)y + [p_2 + 2t(y-x_2)(1-y)]\Lambda}.$$

10.3 Relationship between two conditions on profit sharing

It is shown below that if the entire market is served at a pair of prices then (C3) in section 3 is weaker than (P4) in Rath and Zhao (2003). In the notations of the present paper, (P4) there can be stated as follows.

(P4) Let $x_1 < x_2$.

(i) If $x_1 + x_2 > 1$ then $1/2 \leq y < 1$ and $p_1 \in \{\pi - tx_1^2, \pi - t(y - x_1)^2\}$.

(ii) If $x_1 + x_2 < 1$ then $0 < y \leq 1/2$ and $p_2 \in \{\pi - t(1 - x_2)^2, \pi - t(y - x_2)^2\}$.

First we show that (P4) implies (C3). Without loss of generality, let $x_1 + x_2 > 1$. Using (1),

$$\begin{aligned} F_1 - F_2 &= p_1 y - [p_1 + t(x_2 - x_1)(2y - x_1 - x_2)](1 - y) \\ &= [p_1 - t(x_2 - x_1)(1 - y)](2y - 1) + t(x_2 - x_1)(x_1 + x_2 - 1)(1 - y). \end{aligned}$$

Since $p_1 - t(x_2 - x_1)(1 - y) > 0$, $2y - 1 \geq 0$ and $1 - y > 0$, $F_1 > F_2$, i.e., (C3) holds.

The converse is not true, (C3) does not imply (P4). Suppose the profit sharing rule is such that, when $x_1 < x_2$, the market share of firm 1 is

$$y = \frac{1}{2} + (x_2 - x_1)(1 - x_1 - x_2)\epsilon$$

where $0 < \epsilon \leq t/(8\pi)$.

If $x_1 + x_2 > 1$ then $y < 1/2$. So, (P4) is violated. That (C3) holds can be seen as follows. Note that $2y - 1 = 2(x_2 - x_1)(1 - x_1 - x_2)\epsilon$ and $2y - x_1 - x_2 = (1 - x_1 - x_2)[1 + 2(x_2 - x_1)\epsilon]$. Moreover, from $\pi \geq 3t$, $\epsilon \leq 1/24$ and $y \in [11/24, 13/24]$. Since the entire market is served, the profit of each firm is positive. In particular, $p_1 \leq \pi$.

$$\begin{aligned} F_1 - F_2 &= p_1 y - [p_1 + t(x_2 - x_1)(2y - x_1 - x_2)](1 - y) \\ &= p_1(2y - 1) - t(x_2 - x_1)(2y - x_1 - x_2)(1 - y) \\ &= (x_2 - x_1)(1 - x_1 - x_2)\left[2p_1\epsilon - t[1 + 2(x_2 - x_1)\epsilon](1 - y)\right]. \end{aligned}$$

It is shown below that $2p_1\epsilon - t[1 + 2(x_2 - x_1)\epsilon](1 - y)$ is negative. Therefore, $F_1 - F_2$ has the same sign as $x_1 + x_2 - 1$, i.e., (C3) holds. Notice that $2p_1\epsilon \leq 2p_1 t/(8\pi) \leq t/4$. On the other hand, $1 - y \geq 11/24 > 1/3$.

10.4 Proofs of Lemmas 1 and 2

Proof of Lemma 1 Let $x_1 < x_2$. First we show that if $x_1 + x_2 < 1$ then the profit of firm 1 is less than $p_2/2$ and if $x_1 + x_2 > 1$ then the profit of firm 2 is less than $p_1/2$.

Assume that $x_1 + x_2 < 1$. If $y \geq 1/2$, then by (C3), $p_1 y < p_2(1-y) \leq p_2/2$. Let $y < 1/2$. From (1), $p_1 = p_2 - t(x_2 - x_1)(2y - x_1 - x_2)$. Therefore, $p_1 y < p_2/2$ iff $2t(x_2-x_1)(x_1+x_2-2y)y < p_2(1-2y)$. Since $p_2 \geq \pi - t \geq 2t$ and $y < 1/2$, it is enough to show that $x_1 + x_2 - 2y < 1 - 2y$, i.e., $x_1 + x_2 < 1$, which holds. We have shown that the profit of firm 1 is less than $p_2/2$.

Analogously it can be shown that if $x_1 + x_2 > 1$ then the profit of firm 2 is less than $p_1/2$.

(i) Fix any $x_1 < x_2 = 1/2$. Then $x_1 + x_2 < 1$. We have shown that the profit of firm 1 is less than $p_2/2$. Since $x_2 = 1/2$ and some consumer pays the reservation price, $p_2 = \pi - t(1-x_2)^2$. If firm 1 relocates at x_2, by (C2), its profit is $p_2/2$ since p_2 remains unchanged.

The preceding arguments show that firm 1 is better off at $x_1 = x_2 = 1/2$ than at $x_1 < x_2 = 1/2$. So, $x_1 < x_2 = 1/2$ is not an equilibrium. By similar arguments, firm 2 is better off at $1/2 = x_1 = x_2$ than at $1/2 = x_1 < x_2$. So, $1/2 = x_1 < x_2$ is not an equilibrium. Moreover, $(1/2,\, 1/2)$ is an equilibrium.

(ii) Let $x_1 \leq 1/2 \leq x_2$. Without loss of generality suppose that $x_1 + x_2 < 1$. Then $x_1 < 1/2$ and $x_1 < x_2$.

We have shown that $F_1(x_1, x_2) < p_2/2$. Moreover, $p_2 \leq \pi - t(1-x_2)^2$ and $p_2 \leq \pi - t(y-x_2)^2$. Suppose that $x_2 \leq 3/4$. Then $F_1(x_1, x_2) < \left[\pi - t(1-x_2)^2\right]/2 = F_1(1-x_2, x_2)$.

Now suppose that $x_2 > 3/4$. Assume that $y \leq 1/2$. Note that $p_1 \leq \pi - t(y-x_1)^2$. Therefore, $F_1(x_1, x_2) = p_1 y \leq \left[\pi - t(y-x_1)^2\right] y \leq \left(\pi - t[(1/2) - x_1]^2\right)/2 < \left(\pi - t[(1/2) - x_2]^2\right)/2 = F_1(1-x_2, x_2)$. Let $y > 1/2$. By (C3), $F_1(x_1, x_2) < F_2(x_1, x_2) \leq \left[\pi - t(y-x_2)^2\right](1-y) < \left(\pi - t[(1/2) - x_2]^2\right)/2 = F_1(1-x_2, x_2)$.

(iii) Fix any (x_1, x_2) with $x_2 = 1 - x_1$ and $x_1 < 1/4$. Then the profit of each firm is $\left[\pi - t((1/2) - x_2)^2\right]/2$. Let $x_1 < \bar{x}_1 < 1/4$. Then $\bar{x}_1 + x_2 > 1$. We will consider the three possibilities for the prices.

First suppose that $p_1 = \pi - t\bar{x}_1^2$. We will show that $F_2 > F_1$, which violates (C3). So, this possibility cannot hold. Using (1),

$$\begin{aligned} F_2 - F_1 &= \left[p_1 + t(x_2 - \bar{x}_1)(2y - \bar{x}_1 - x_2)\right](1-y) - p_1 y \\ &= \left[p_1 - t(x_2 - \bar{x}_1)(1-y)\right](1-2y) - t(x_2 - \bar{x}_1)(\bar{x}_1 + x_2 - 1)(1-y). \end{aligned}$$

This is a decreasing function of y. Since $p_1 = \pi - t\bar{x}_1^2$, the market share of firm 1 is at most $2\bar{x}_1$, i.e., $y \leq 2\bar{x}_1 < 1/2$. Therefore,

$$F_2 - F_1 \geq \left[p_1 - t(x_2 - \bar{x}_1)(1 - 2\bar{x}_1)\right](1 - 4\bar{x}_1) - t(x_2 - \bar{x}_1)(\bar{x}_1 + x_2 - 1)(1 - 2\bar{x}_1).$$

Since $p_1 > 2t$, the RHS is positive when $\bar{x}_1 = x_1$. Hence, $F_2 - F_1 > 0$, if $\bar{x}_1 - x_1$ is small.

Next suppose that $p_2 = \pi - t(1-x_2)^2$. We will show that $F_1 > \left[\pi - t((1/2)-x_2)^2\right]/2$, i.e., the profit of firm 1 increases because of relocation.

Using (1), $F_1 = p_1 y = \left[p_2 - t(x_2 - \bar{x}_1)(2y - \bar{x}_1 - x_2)\right] y$. This is increasing in y. Since $p_2 = \pi - t(1-x_2)^2$, the market share of firm 2 is at most $2 - 2x_2$, i.e., $y \geq 2x_2 - 1$. So, $F_1 \geq \left[p_2 - t(x_2 - \bar{x}_1)(4x_2 - 2 - \bar{x}_1 - x_2)\right](2x_2 - 1)$. To show that $F_1 > \left[\pi - t((1/2)-x_2)^2\right]/2$, it suffices to show that $2\left[p_2 - t(x_2 - \bar{x}_1)(4x_2 - 2 - \bar{x}_1 - x_2)\right](2x_2-1) > \left[\pi - t((1/2)-x_2)^2\right]$. It is enough to establish this when $\bar{x}_1 = x_1$. In that case, the inequality can be written as

$$\pi(4x_2 - 3) - 2t(2x_2-1)^2(4x_2-3) + \frac{t}{4}(2x_2-1)^2 - 2t(1-x_2)^2(2x_2-1) \ > \ 0.$$

Since $\pi > 2t$, $2x_2 - 1 \leq 1$ and $(2x_2-1)/4 > 1/8 > 2(1-x_2)^2$, the inequality holds.

The remaining possibility is $p_1 + t(y - \bar{x}_1)^2 = \pi = p_2 + t(y - x_2)^2$ at $(\bar{x}_1, x_2)$. If $y < 1/2$, then $F_2(\bar{x}_1, x_2) = \left[\pi - t(y-x_2)^2\right](1-y) > \left[\pi - t((1/2)-x_2)^2\right]/2$. Since $F_1(\bar{x}_1, x_2) > F_2(\bar{x}_1, x_2)$ by (C3), $F_1(\bar{x}_1, x_2) > \left[\pi - t((1/2)-x_2)^2\right]/2$. If $y \geq 1/2$ then $F_1(\bar{x}_1, x_2) = \left[\pi - t(y-\bar{x}_1)^2\right] y > \left[\pi - t(y-x_1)^2\right] y \geq \left[\pi - t((1/2)-x_1)^2\right]/2$. Thus, F_1 increases because of relocation.

We have shown that any symmetric location pair $(x_1, 1-x_1)$ with $x_1 < 1/4$ is not an equilibrium. ∎

Proof of Lemma 2 (*i*) Let $x_1 = x_2 > 1/2$. Then $p_2 = \pi - t x_2^2$ and by (C4), $F_1 = (\pi - t x_2^2)/2$. If firm 1 locates symmetrically instead, its profit is $\left[\pi - t(1-x_2)^2\right]/2$, or $\left[\pi - t((1/2)-x_2)^2\right]/2$ depending on whether $x_2 \leq 3/4$, or $x_2 > 3/4$. Clearly, the profit increases because of relocation. So, $x_1 = x_2 \neq 1/2$ cannot be an equilibrium.

(*ii*) Let $x_1 < x_2 < 1/2$. Then $x_1 + x_2 < 1$ and $p_2 = \pi - t(1-x_2)^2$. By (C3), as argued in the proof of Lemma 1, $F_1 < p_2/2 = \left[\pi - t(1-x_2)^2\right]/2$. If firm 1 relocates at x_2, by (C4), $F_1 = \left[\pi - t(1-x_2)^2\right]/2$. So, $x_1 < x_2 < 1/2$ cannot be an equilibrium. ∎

10.5 Proof of Theorem 1

From Proposition 1, (1/2, 1/2) is an equilibrium and the other candidates for equilibria are the symmetric ones with $1/4 \leq x_1 < 1/2$. To eliminate these, consider a pair of symmetric locations (x_1, x_2). Let $x_1 < \bar{x}_1 < x_2$. Then $\bar{x}_1 + x_2 > 1$. If $p_2 = \pi - t(1-x_2)^2$ at locations $(\bar{x}_1, x_2)$, then Lemma 5 implies that $F_1(\bar{x}_1, x_2) > F_1(x_1, x_2)$, i.e., (x_1, x_2) is not an equilibrium.

Therefore, suppose that $p_1 = \pi - t\bar{x}_1^2$ for all $x_1 < \bar{x}_1 < x_2$. At symmetric locations $(x_1, 1-x_1)$, $y = 1/2$ and $\Lambda = 1$. From (6), $\partial F_1/\partial x_1 > 0$ if

$$p_1^2 \frac{\partial \Lambda}{\partial x_1} - tp_1(1+2x_1) + 2t^2(1-2x_1)x_1 \ > \ 0,$$

which follows from $\partial\Lambda/\partial x_1 \geq t(1+2x_1)/(\pi - tx_1^2)$ and $p_1 = \pi - tx_1^2$. Since F_1 increases at symmetric locations, (x_1, x_2) is not an equilibrium. Thus, (1/2, 1/2) is the unique equilibrium.

The proof when firm 2 changes its location is similar, one uses $\partial F_2/\partial x_2$ given earlier. ∎

10.6 Proofs of Theorem 2 and Corollary 2

Proof of Theorem 2 $(i) \Leftrightarrow (ii)$. Suppose that (x_1^*, x_2^*) is an equilibrium. This implies that inward moves by neither firm is profitable, i.e., $F_1(x_1, x_2^*) \leq F_1(x_1^*, x_2^*) = [\pi - t(1-x_2^*)^2]/2$ for $x_1^* < x_1 < x_2^*$ and $F_2(x_1^*, x_2) \leq F_2(x_1^*, x_2^*) = (\pi - tx_1^{*2})/2$ for $x_1^* < x_2 < x_2^*$.

Consider $F_1(x_1, x_2^*) \leq F_1(x_1^*, x_2^*)$ for $x_1^* < x_1 < x_2^*$. If $p_2 = \pi - t(1-x_2^*)^2$ at locations (x_1, x_2^*), then by (ii) of Lemma 5, $F_1(x_1, x_2^*) > F_1(x_1^*, x_2^*)$, a contradiction. So, $p_1 = \pi - tx_1^2$ for all $x_1^* < x_1 < x_2^*$. Hence, $(\pi - tx_1^2)y = F_1(x_1, x_2^*) \leq F_1(x_1^*, x_2^*) = [\pi - t(1-x_2^*)^2]/2$, which implies that $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$.

Let $x_1^* < x_1 < x_2^*$ and $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$. It is easy to check that $\sigma_1 < (x_1 + x_2^*)/2$ when $x_1 + x_2^* > 1$, which implies that $y < (x_1 + x_2^*)/2$ and $p_2 < p_1$. Since $\pi - t(1-x_2^*)^2 > \pi - tx_1^2$, $p_2 \neq \pi - t(1-x_2^*)^2$ and $p_1 = \pi - tx_1^2$. So, $F_1(x_1, x_2^*) = (\pi - tx_1^2)y \leq (\pi - tx_1^2)\sigma_1 = [\pi - t(1 - x_2^*)^2]/2 = F_1(x_1^*, x_2^*)$. We have shown that an inward move by firm 1 is not profitable. An outward move is not profitable by part (ii) of Lemma 1.

The proof $F_2(x_1^*, x_2) \leq F_2(x_1^*, x_2^*) \Leftrightarrow 1 - y(x_1^*, x_2) \leq \sigma_2(x_1^*, x_2)$ for $x_1^* < x_2 < x_2^*$ is similar.

$(ii) \Leftrightarrow (iii)$. Suppose that $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$ and $x_1^* < x_1 < x_2^*$. Since $x_1 + x_2^* > 1$, $\sigma_1 < (x_1 + x_2^*)/2$. So, $y < (x_1 + x_2^*)/2$, which implies that $p_1 > p_2$. Since $\pi - t(1-x_2^*)^2 > \pi - tx_1^2$, $p_2 \neq \pi - t(1-x_2^*)^2$ and $p_1 = \pi - tx_1^2$. This gives,

$$\Lambda(x_1, x_2^*) = \frac{(\pi - tx_1^2)y}{[\pi - tx_1^2 + t(x_2^* - x_1)(2y - x_1 - x_2^*)](1-y)}.$$

This is an increasing function of y and since $y \leq \sigma_1$, $\Lambda(x_1, x_2^*) \leq Q_1(x_1, x_2^*)$.

Conversely, suppose that $\Lambda(x_1, x_2^*) \leq Q_1(x_1, x_2^*)$. It can be shown that Q_1 is an increasing function of σ_1. Since $x_1 + x_2^* > 1$, $\sigma_1 < (x_1 + x_2^*)/2$ which implies that $Q_1 < (x_1 + x_2^*)/(2 - x_1 - x_2^*)$. Therefore, $\Lambda < (x_1 + x_2^*)/(2 - x_1 - x_2^*)$.

If $p_2 = \pi - t(1-x_2^*)^2$ at (x_1, x_2^*), then Lemma 5 implies that $\Lambda > (x_1 + x_2^*)/(2 - x_1 - x_2^*)$, a contradiction. So, $p_1 = \pi - tx_1^2$ at (x_1, x_2^*) and $\Lambda(x_1, x_2^*)$ is given by the expression above. This is increasing in y. Since $\Lambda \leq Q_1$, $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$.

The proof of $1 - y(x_1^*, x_2) \leq \sigma_2(x_1^*, x_2) \Leftrightarrow \Lambda(x_1^*, x_2) \geq Q_2(x_1^*, x_2)$ for $x_1^* < x_2 < x_2^*$ is similar. ∎

Before proving Corollary 2, we examine some consequences of flipping firm locations.

Lemma 6 *Suppose that (C1) holds. Let $x_1 < x_2$. Then the following conditions are equivalent.*

(i) $\Lambda(1 - x_2, 1 - x_1) = 1/\Lambda(x_1, x_2)$.

(ii) $y(1 - x_2, 1 - x_1) = 1 - y(x_1, x_2)$.

(iii) $F_1(1 - x_2, 1 - x_1) = F_2(x_1, x_2)$ *and* $F_2(1 - x_2, 1 - x_1) = F_1(x_1, x_2)$.

Proof Assume that (C1) holds and $x_1 < x_2$. Given $\Lambda(x_1, x_2) = p_1 y/[p_2(1-y)]$, let $\bar{x}_1 = 1 - x_2$, $\bar{x}_2 = 1 - x_1$, $\bar{p}_1 = p_2$, $\bar{p}_2 = p_1$ and $\bar{y}$ the market share of firm 1 when the prices are $(\bar{p}_1, \bar{p}_2)$ at $(\bar{x}_1, \bar{x}_2)$.

It is easy to check that $\bar{y} = 1 - y$, at the prices$(\bar{p}_1, \bar{p}_2)$ at locations $(\bar{x}_1, \bar{x}_2)$ the entire market is served and some consumer pays the reservation price. Moreover, $\bar{p}_1 \bar{y} = p_2(1 - y)$ and $\bar{p}_2(1 - \bar{y}) = p_1 y$. So, the profit of each firm is positive at $(\bar{x}_1, \bar{x}_2)$.

Clearly, (iii) implies (i). Suppose that $\Lambda(1 - x_2, 1 - x_1) = 1/\Lambda(x_1, x_2)$. At the prices $(\bar{p}_1, \bar{p}_2)$, $\bar{p}_1 \bar{y}/[\bar{p}_2(1 - \bar{y})] = 1/\Lambda(x_1, x_2)$. By Claim 2, these prices are unique. At these prices, $y(1-x_2, 1-x_1) = 1-y(x_1, x_2)$. So, $(i) \Rightarrow (ii)$. Now suppose that $y(1-x_2, 1-x_1) = 1-y(x_1, x_2)$. This is true when the prices are $(\bar{p}_1, \bar{p}_2)$ at $(\bar{x}_1, \bar{x}_2)$. By Claim 2, these prices are unique. At these prices, $F_1(1 - x_2, 1 - x_1) = F_2(x_1, x_2)$ and $F_2(1 - x_2, 1 - x_1) = F_1(x_1, x_2)$. So, $(ii) \Rightarrow (iii)$. ∎

Proof of Corollary 2 It is easy to check that if $\Lambda(1 - x_2, 1 - x_1) = 1/\Lambda(x_1, x_2)$ then $\sigma_2(1 - x_2, 1 - x_1) = \sigma_1(x_1, x_2)$ and $Q_2(1 - x_2, 1 - x_1) = 1/Q_1(x_1, x_2)$.

Suppose that $y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$. Consider $x_1^* < x_2 < x_2^*$. Lemma 6 implies that $1 - y(x_1^*, x_2) = y(x_1, x_2^*)$. So, $1 - y(x_1^*, x_2) = y(x_1, x_2^*) \leq \sigma_1(x_1, x_2^*) = \sigma_2(x_1^*, x_2)$.

Suppose that $\Lambda(x_1, x_2^*) \leq Q_1(x_1, x_2^*)$ for $x_1^* < x_1 < x_2^*$. Consider $x_1^* < x_2 < x_2^*$. Then $\Lambda(x_1^*, x_2) = 1/\Lambda(x_1, x_2^*) \geq 1/Q_1(x_1, x_2^*) = Q_2(x_1^*, x_2)$.

The corollary now follows from the theorem. ∎

10.7 Proofs of Theorem 3 and Corollary 3

Proof of Theorem 3 (i) Lemma 1 implies that $(1/2, 1/2)$ is an equilibrium.

(ii) In order to show that $x^* = (x_1^*, x_2^*)$ is an equilibrium it needs to be shown that $F_1(x_1, x_2^*) < F_1(x_1^*, x_2^*)$ if $x_1 < x_2^*$ and $F_2(x_1^*, x_2) < F_2(x_1^*, x_2^*)$ if $x_2 > x_1^*$.

If $x_2^* \leq 1/2$ then $p_2 = \pi - t(1 - x_2^*)^2$. Since $\partial\Lambda/\partial x_1 > 0$, if $y \geq x_1$ then $\partial F_1/\partial x_1 > 0$ from (7). If $y < x_1$ then $p_1 > p_2$. $\partial F_1/\partial x_1 > 0$ if $[(\partial\Lambda/\partial x_1)/\Lambda^2][p_1 - 2t(x_2^* - x_1)y] > 2t(x_1 - y)$. Since $x_1 < 1/2$, $1 + 2x_1 > 4(x_1 - y)$ and the inequality holds if $2[p_1 - 2t(x_2^* - x_1)y] > \pi - tx_2^{*2}$. This follows from $p_1 > p_2 = \pi - t(1 - x_2^*)^2$, $\pi \geq 3t$ and $(x_2^* - x_1)y < 1/4$.

Now let $x_2^* > 1/2$. Suppose that $x_1 + x_2^* < 1$. Then $x_1 < 1/2$. Lemma 1 shows that $F_1(x_1, x_2^*) < F_1(1 - x_2^*, x_2^*)$. Therefore, suppose that $x_1 + x_2^* \geq 1$. Consider the three cases for the prices.

If $p_2 = \pi - t(1 - x_2^*)^2$ then $p_1 \leq \pi - tx_1^2 \leq p_2$ and $y - x_1 > 0$. If $p_1 = \pi - t(y - x_1)^2$ then $y - x_1 > 0$. Since $\partial\Lambda/\partial x_1 > 0$, from (7) and (8), $\partial F_1/\partial x_1 > 0$ in either of these cases. Therefore, suppose that $p_1 = \pi - tx_1^2$. By (6), $\partial F_1/\partial x_1 > 0$ if

$$\frac{1}{\Lambda^2} \times \frac{\partial\Lambda}{\partial x_1} > \frac{2t}{p_1^2}[p_1(1 + x_1 - y) - t(x_2^* - x_1)(x_1 + x_2^* + 2 - 4y)x_1].$$

The RHS is a decreasing function of y.

Since $p_1 = \pi - tx_1^2$, $p_2 > \pi - tx_2^{*2}$ (otherwise the profit of firm 1 is zero). So, $p_1 y + p_2(1 - y) > \pi - tx_2^{*2}$. (C3) ensures that $p_1 y > (\pi - tx_2^{*2})/2$, i.e., $y > (\pi - tx_2^{*2})/[2(\pi - tx_1^2)] = y^*$, say. Then $y^* < 1/2$. The RHS is less than $2t(1 + x_1 - y^*)/p_1$. It is enough to show that $t(1+2x_1)/(\pi - tx_2^{*2}) > 2t(1 + x_1 - y^*)/(\pi - tx_1^2)$. This is equivalent to $1 + 2x_1 > 4y^*(1 + x_1 - y^*)$. Since $1 > 4y^*(1 - y^*)$ and $2x_1 \geq 4y^* x_1$, the inequality holds. So, $\partial F_1/\partial x_1 > 0$ if $x_2^* > 1/2$.

Analogous arguments show that $\partial F_2/\partial x_2 < 0$ if $x_2 > x_1^*$ and $F_2(x_1^*, x_2) < F_2(x_1^*, x_2^*)$. ∎

Proof of Corollary 3 Let $x_2^* > 1/2$. If $[\partial\Lambda(x_1, x_2^*)/\partial x_1]/\Lambda^2 \geq t(1 + 2x_1)/(\pi - tx_2^{*2})$ at all $x_1 < x_2^*$, then from the theorem, $F_1(x_1, x_2^*) < F_1(x_1^*, x_2^*)$ for all $x_1 < x_2^*$. It needs to be shown that $F_2(x_1^*, x_2) < F_2(x_1^*, x_2^*)$ for all $x_2 > x_1^*$.

Let $\bar{x}_1 = 1 - x_2$ and $\bar{x}_2 = 1 - x_1^*$. Then $\bar{x}_1 < \bar{x}_2 < 1/2$. If $\bar{x}_1 \leq x_1 < \bar{x}_2$, then at locations $(x_1, \bar{x}_2)$, $p_2 = \pi - t(1 - \bar{x}_2)^2$. Since $\partial\Lambda(x_1, \bar{x}_2)/\partial x_1 > 0$, if $y \geq x_1$ then $\partial F_1/\partial x_1 > 0$ from (7).

If $y < x_1$ then $p_1 > p_2$. $\partial F_1/\partial x_1 > 0$ if $[(\partial\Lambda/\partial x_1)/\Lambda^2][p_1 - 2t(\bar{x}_2 - x_1)y] > 2t(x_1 - y)$. Since $p_1 > p_2$, $p_1 - 2t(\bar{x}_2 - x_1)y > \pi - t(1 - \bar{x}_2)^2 - 2t(\bar{x}_2 - x_1)y > \pi - t$. Notice that $x_1 - y < 1/2$. So, $\partial F_1/\partial x_1 > 0$. This means $F_1(\bar{x}_1, \bar{x}_2) < F_1(\bar{x}_2, \bar{x}_2)$.

From Lemma 6, $F_1(\bar{x}_1, \bar{x}_2) = F_2(x_1^*, x_2)$ and $F_2(\bar{x}_1, \bar{x}_2) = F_1(x_1^*, x_2)$. By (C5), $F_1(\bar{x}_2, \bar{x}_2) = F_2(x_1^*, x_1^*)$. So, $F_2(x_1^*, x_2) < F_2(x_1^*, x_1^*)$ and x^* is an equilibrium.

The proof with $\partial\Lambda/\partial x_2$ is analogous. ■

10.8 Proofs of Lemmas 3 and 4

Proof of Lemma 3 Consider the following strategies by the two firms. At any pair of locations, the firms continue to charge the prices corresponding to (F_1, F_2) until a single deviation occurs, in which event they switch to the Nash equilibrium prices for the remainder of the game. Let D_i denote the defection profit of firm i ($i = 1, 2$) from the profits (F_1, F_2). Subgame perfection requires,

$$\frac{1}{1-\delta_i}F_i \ \geq \ D_i + \frac{\delta_i}{1-\delta_i}R_{iN}, \qquad\qquad \delta_i \ \geq \ \frac{D_i - F_i}{D_i - R_{iN}}.$$

We will show that $(\pi - s)/\pi \geq (D_i - F_i)/(D_i - R_{iN})$. This is equivalent to $(F_i - R_{iN})/(D_i - R_{iN}) \geq s/\pi$. Since $F_i - R_{iN} \geq s$ and $D_i - R_{iN} \leq \pi$, the inequality holds. ■

Proof of Lemma 4 By (C1), $F_1 + F_2 \geq \pi - t \geq 2t$.

First suppose that $x_1 = x_2$. Let $i = 1, 2$. If (C4) holds then $F_i = (F_1 + F_2)/2$. If (C5) holds then $1/r \leq F_1/F_2 \leq r$, which yields $F_i \geq (F_1+F_2)/(1+r)$. Since $R_{iN} = 0$, $F_i - R_{iN} \geq 2t/(1+r)$. Henceforth, assume that $x_1 < x_2$.

Consider firm 2. If $x_1 + x_2 \leq 1$, then by (C2) and (C3), $F_2 \geq (F_1 + F_2)/2 \geq t$ and $R_{2N} = t(x_2 - x_1)(4 - x_1 - x_2)^2/18 \leq t/2$. So, $F_2 - R_{2N} \geq t/2 \geq t/(1 + r)$.

If $x_1 + x_2 > 1$ then $R_{2N} \leq t(x_2 - x_1)/2$. There are two possibilities to consider: either $x_2 - x_1 \leq 2/(1 + r)$, or $x_2 - x_1 > 2/(1 + r)$. Suppose that $x_2 - x_1 \leq 2/(1 + r)$. Then $R_{2N} \leq t/(1+r)$. Since $F_1/F_2 \leq r$, $F_2 \geq (F_1+F_2)/(1+r) \geq 2t/(1+r)$ and $F_2 - R_{2N} \geq t/(1+r)$. Now suppose that $x_2 - x_1 > 2/(1 + r)$. From $F_1/F_2 \leq (x_1 + x_2)/(2 - x_1 - x_2)$, $F_2 \geq (F_1 + F_2)(2 - x_1 - x_2)/2 \geq (F_1 + F_2)(x_2 - x_1)/2 \geq t(x_2 - x_1)$. So, $F_2 - R_{2N} \geq t(x_2 - x_1)/2 \geq t/(1+r)$.

Similar arguments show that $F_1 - R_{1N} \geq t/(1 + r)$. So, $s \geq t/(1 + r)$. ■

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