Adiabatic Invariants of Second Order Korteweg-de Vries Type Equation

Piotr Rozmej and Anna Karczewska

Abstract In this chapter we analyze the existence and forms of invariants of the extended Korteweg-de Vries equation (KdV2). This equation appears when the Euler equations for shallow water are extended to the second order, beyond Korteweg-de Vries (KdV). We show that contrary to KdV for which there is an infinite number of invariants, for KdV2 there exists only one, connected to mass (volume) conservation of the fluid. For KdV2 we found only so-called *adiabatic invariants*, that is, functions of the solutions which are constants neglecting terms of higher order than the order of the equation. In this chapter we present two methods for construction of such invariants. The first method, a direct one, consists in using constructions of higher KdV invariants and eliminating non-integrable terms in an approximate way. The second method introduces a near-identity transformation (NIT) which transforms KdV2 into equation (asymptotically equivalent) which is integrable. For the equation obtained by NIT, exact invariants exist, but they become approximate (adiabatic) when the inverse NIT transformation is applied and original variables are restored. Numerical tests of the exactness of adiabatic invariants for KdV2 in several cases of initial conditions are presented. These tests confirm that relative changes in these approximate invariants are small indeed. The relations of KdV invariants and KdV2 adiabatic invariants to conservation laws are discussed, as well.

Keywords Shallow water waves · Nonlinear equations · Invariants of KdV2 equation · Adiabatic invariants

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1 Introduction

The celebrated Korteweg-de Vries equation (KdV) [31], whose origin is the set of Euler's shallow water and long wavelength equations, now enjoys a paradigmatic status in the field of nonlinear partial differential equations. There is a huge number of research papers concerned with weakly nonlinear, dispersive and long wavelength problems in which KdV is shown as the lowest approximation of wave motion in a number of fields of physics, see, e.g., monographs [8, 22, 36, 39, 41] and references therein.

It is accepted fact that KdV gives an infinite number of invariants or conservation laws also referred to as integrals of motion [4, 8, 35, 37]. The two first KdV invariants concern the preservation of mass (volume) of the fluid and conservation of its total momentum. The next one is related to energy conservation. The higher KdV invariants have no simple interpretation. KdV is, however, the result of an approximation of the set of the Euler equations within the perturbation approach, limited to the first order in expansion with respect to parameters assumed to be small. KdV has been extended to the second order (KdV2) by a number of authors, e.g., [6, 21, 25, 33, 34, 42]. In [23, 24, 26, 28] the authors have derived the KdV2 equation for an uneven bottom, introducing an additional small parameter related to bottom variation. Here the term *second order* is defined as the order of perturbation expansion with respect to small parameters. However, this advanced form is lacking in exactly conserved quantities except for the ubiquitous mass law.

Many papers, e.g., [4, 7, 9–11, 13, 16–19, 29, 30, 44] assert integrability of second order KdV type equations and existence of higher invariants. Specifically Benjamin and Olver [4] have discussed Hamiltonian structure, symmetries and conservation laws in respect of water waves. A near-identity transformation (NIT), first published by Kodama [29, 30] and since used by many other authors, e.g., [7, 9–11, 13, 16–19, 44], makes it possible to transform the second order KdV type equations into Hamiltonian forms which are asymptotically equivalent. The existence of the Hamiltonian form for the transformed equation supplies the full hierarchy of invariants, which appear to be adiabatic invariants in respect of the original equation.

The lack of exact invariants in the system forces one to look for adiabatic (approximate) ones, as in [5]. Recently we developed a simple method to calculate such adiabatic invariants, allowing us to derive them directly using the original 'physical' equation (equally applicable to equations expressed in dimensional variables) [21]. Our method is as follows: one constructs the KdV2 in a similar fashion as one does for KdV invariants and then applies the addition of KdV, multiplied by a small parameter, to cancel the non-integrable terms. In [21] we focused on this direct method mentioning NIT-based derivation of adiabatic invariants rather briefly. In this chapter the NIT method is discussed more broadly with particular attention paid to energy conservation law.

It is shown in [40] that KdV2 for uneven bottom [23, 26] is not symmetry-integrable since it admits no genuinely generalized symmetries.

The chapter substantially extends results published recently in [25]. In order to introduce the reader to higher order nonlinear equations beyond KdV several earlier achievements [23, 24, 26, 28] are recalled in Sect. 2. The set of Euler's equations for the inviscid and incompressible fluid and irrotational motion is introduced and the perturbation technique leading to KdV and KdV2 equations is described. Then analytic solutions for KdV and the recently obtained ones for KdV2, solitonic [23] and periodic [21], are presented and their properties compared.

In Sect. 3 we recall derivations of lowest invariants of KdV and their relations to conservation laws. In Sect. 4 a direct extension of the methods used in Sect. 3 for the KdV2 equation is presented. Particular forms of second and third adiabatic invariants for KdV2 are obtained.

In Sect. 5 near-identity transformation is introduced and applied to find general forms of lowest adiabatic invariants for KdV2. Relations of adiabatic invariants of KdV2 to formulas for the momentum and energy of the system are discussed, as well. The quality of adiabatic invariants is tested in numerics in Sect. 6. The main results are summarised in Sect. 7.

2 KdV and KdV2 Equations

First, we will recall briefly the derivation of KdV and KdV2 equations.

The natural assumptions in the shallow water wave problem are the following. Since water viscosity and compressibility are very small the fluid is assumed to be inviscid and incompressible. For gravity waves velocities of fluid particles are small, as well, therefore the motion can be considered as irrotational. This property allows us to introduce velocity potential ϕ. The velocity potential fulfils the Laplace equation for entire fluid volume. The set of Euler's equations contains also the kinematic and dynamic boundary conditions at the free surface and the kinematic boundary condition at the impenetrable bottom. The full set of equations for the velocity potential $\phi(x, y, z, t)$, as well as its derivation, is published in many textbooks, for instance, see [39, Chap. 5]. A typical procedure consists in introducing two small parameters $\alpha = a/h$ and $\beta = (h/l)^2$ and in application of perturbation approach with respect to these parameters. Here a is the amplitude of a surface wave η, h is water depth and l is a typical surface waves wavelength.

An approximation in deriving KdV and higher order nonlinear wave equations is correct when two small parameters α and β are of the same order of magnitude. The definitions of small parameters α and β and the geometry of the problem are shown in Fig. 1. The parameters α, β have the same meaning as the parameters ε, δ^2 in [39], respectively. These notations follow those in the paper [6], in which a systematic method for the derivation of wave equations of different orders is given. In [23, 26] we have introduced a third parameter $\delta = a_h/h$, where a_h denotes the amplitude of bottom changes. This new parameter allowed us to derive second order equation for surface waves over a non-flat bottom using the same perturbative approach as for derivation of KdV or higher-order KdV-like equations.

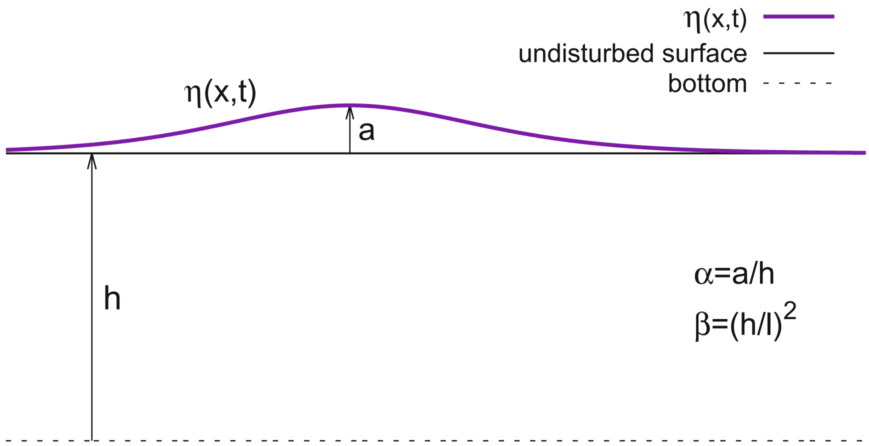

Fig. 1 Schematic view of the geometry of the problem

In what follows we restrict discussion to the 2-dimensional flow, $\phi(x, z, t)$ (which means translational symmetry with respect to y axis). Here, the horizontal coordinate is denoted by x and the vertical one is denoted by z.

A convenient way of studying the problem is introducing non-dimensional variables. They are defined as follows

$$\begin{aligned} \tilde{\eta} &= \eta/a, \quad \tilde{\phi} = \phi/(l\frac{a}{h}\sqrt{gh}), \\ \tilde{x} &= x/l, \quad \tilde{z} = z/h, \quad \tilde{t} = t/(l/\sqrt{gh}). \end{aligned} \tag{1}$$

The set of hydrodynamic equations for 2-dimensional flow in the non-dimensional variables takes the simpler form (henceforth all tildes have been omitted)

$$\beta\phi_{xx} + \phi_{zz} = 0, \tag{2}$$

$$\eta_t + \alpha\phi_x\eta_x - \frac{1}{\beta}\phi_z = 0, \quad \text{for} \quad z = 1 + \alpha\eta \tag{3}$$

$$\phi_t + \frac{1}{2}\alpha\phi_x^2 + \frac{1}{2}\frac{\alpha}{\beta}\phi_z^2 + \eta = 0, \quad \text{for} \quad z = 1 + \alpha\eta \tag{4}$$

$$\phi_z = 0, \quad \text{for} \quad z = 0. \tag{5}$$

The Laplace equation (2) is valid for the entire volume of the fluid. The Eq. (3) represents so called kinematic boundary condition at the (unknown) surface whereas the Eq. (4) is so called dynamic boundary condition at the surface. The Eq. (5) expresses the boundary condition at the impenetrable flat bottom. For abbreviation the partial derivatives with respect to corresponding variables are denoted by subscripts, i.e. $\phi_{nx} \equiv \frac{\partial^n \phi}{\partial x^n}$ and so on.

Next, the velocity potential is postulated in the form of power series

$$\phi(x, z, t) = \sum_{m=0}^{\infty} z^m \phi^{(m)}(x, t). \tag{6}$$

The Laplace equation (2) permits the expression of all $\phi^{(2m)}$ functions by the derivatives $\phi^{(0)}_{2mx}$, and all $\phi^{(2m+1)}$ functions by the derivatives $\phi^{(1)}_{2mx}$. Since the boundary condition at the bottom (5) sets $\phi^{(1)} = 0$, all $\phi^{(2m+1)}$ vanish and one obtains the following velocity potential

$$\phi = \phi^{(0)} - \frac{1}{2}\beta z^2 \phi^{(0)}_{2x} + \frac{1}{24}\beta^2 z^4 \phi^{(0)}_{4x} + \frac{1}{720}\beta^3 z^6 \phi^{(0)}_{6x} + \dots . \tag{7}$$

The presence of small parameters α, β in the set of hydrodynamic equations (2)–(5) and (7) allows us to apply the perturbation technique and to derive equations in first and second order with respect to these parameters. Next we insert $\phi(x, z, t)$ given by (7) into (3) and (4) retaining only terms up to second order in small parameters α, β. The Eq. (4) is then differentiated with respect to x and finally $w(x, t)$ is substituted in place of $\phi^{(0)}_x(x, t)$ in both equations. By this procedure one obtains a set of two coupled nonlinear differential equations which, in general, can be studied at different orders of approximation. This is a second order Boussinesq system

$$\eta_t + w_x + \alpha(\eta w)_x - \frac{1}{6}\beta w_{3x} - \frac{1}{2}\alpha\beta(\eta w_{2x})_x + \frac{1}{120}\beta^2 w_{5x} = 0, \tag{8}$$

$$w_t + \eta_x + \alpha w w_x - \frac{1}{2}\beta w_{2xt} + \frac{1}{24}\beta^2 w_{4xt} + \frac{1}{2}\alpha\beta[-2(\eta w_{xt})_x + w_x w_{2x} - w w_{3x}] = 0. \tag{9}$$

Burde and Sergyeyev [6] show a way of eliminating sequentially the $w(x, t)$ variable and deriving a single equation for $\eta(x, t)$ using the higher order perturbative approach. In their method the properties of solutions to lower order equations for w and η are used in derivations of corrections to equations in the next order. In theory this method can be used up to an arbitrary order. For the reader's convenience we will present it briefly below.

2.1 KdV Equation

Limitation of the Boussinesq system (8), (9) to first order in α, β

$$\eta_t + w_x + \alpha(\eta w)_x - \frac{1}{6}\beta w_{3x} = 0, \tag{10}$$

$$w_t + \eta_x + \alpha w w_x - \frac{1}{2}\beta w_{2xt} = 0 \tag{11}$$

results in the derivation of a KdV equation. First, notice that in zeroth order the above equations

$$\eta_t + w_x = 0, \qquad w_t + \eta_x = 0 \tag{12}$$

hold when $\eta = w$ and $\eta_x = w_x$. It follows that $\eta_t = -\eta_x$ and $w_t = -w_x$.

Next, one seeks solutions of the first order set (10), (11) requiring that w, η fulfil (12) and introducing first order corrections $C^{(\alpha)}$, $C^{(\beta)}$

$$w = \eta + \alpha C^{(\alpha)} + \beta C^{(\beta)}. \tag{13}$$

Insertion of (13) into (10), (11) and neglection of higher order terms gives

$$\alpha\left(C_x^{(\alpha)} + 2\eta\eta_x\right) + \beta\left(C_x^{(\beta)} - \frac{1}{6}\eta_{3x}\right) = 0 \tag{14}$$

$$\alpha\left(C_t^{(\alpha)} + \eta\eta_x\right) + \beta\left(C_t^{(\beta)} - \frac{1}{2}\eta_{2xt}\right) = 0. \tag{15}$$

Because of the correction functions appearing already in first order, it is enough to use a zeroth order formula relating their space and time derivatives. Therefore we use $C_t^{(\alpha)} = -C_x^{(\alpha)}$, $C_t^{(\beta)} = -C_x^{(\beta)}$ (like $\eta_t = -\eta_x$, $w_t = -w_x$) in (14), (15). (Otherwise, if one takes, for instance, $C_t^{(\alpha)} = -C_x^{(\alpha)} + \alpha C_1 + \beta C_2$, then terms with C_1, C_2 appear in second order and consequently are neglected). Inserting these relations, subtracting (14) form (15) and equating separately to zero terms with coefficients α and β one obtains

$$C_x^{(\alpha)} = -\frac{1}{2}\eta\eta_x \qquad \text{and} \qquad C_x^{(\beta)} = \frac{1}{3}\eta_{3x}. \tag{16}$$

Integration gives

$$C^{(\alpha)} = -\frac{1}{4}\eta^2 \qquad \text{and} \qquad C_x^{(\beta)} = \frac{1}{3}\eta_{2x}. \tag{17}$$

Then Eqs. (11) and (10) take the final form

$$w = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{3x}, \tag{18}$$

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} = 0. \tag{19}$$

Equation (19) is the famous Korteweg-de Vries (KdV) equation in fixed reference frame (and scaled dimensionless variables). There are several forms of KdV equation in the literature. Transformation $\bar{x} = x - t, \quad \bar{t} = t$ converts (19) into

$$\eta_{\bar{t}} + \frac{3}{2}\alpha\eta\eta_{\bar{x}} + \frac{1}{6}\beta\eta_{3\bar{x}} = 0. \tag{20}$$

Additional scaling by $\tilde{x} = \sqrt{\frac{3}{2}}\bar{x}, \quad \tilde{t} = \frac{1}{4}\sqrt{\frac{3}{2}}\alpha\bar{t}$ transforms (20) into

$$\eta_{\tilde{t}} + 6\eta\eta_{\tilde{x}} + \frac{\beta}{\alpha}\eta_{3\tilde{x}} = 0 \qquad \text{or} \qquad \eta_{\tilde{t}} + 6\eta\eta_{\tilde{x}} + \eta_{3\tilde{x}} = 0, \quad \text{when} \quad \beta = \alpha. \tag{21}$$

Equation (20) gives the form of KdV in the reference frame moving with a characteristic velocity (equal to 1 in dimensionless variables, which corresponds to $\sqrt{gH}$ in original variables). Forms like (21) or similar are preferred in mathematical papers. Sometimes the inverse transform to dimensional variables is applied yielding

$$\eta_t + c\eta_x + \frac{3}{2}\frac{c}{H}\eta\eta_x + \frac{cH^2}{6}\eta_{3x} = 0, \tag{22}$$

where $c = \sqrt{gH}$ is the limiting long wave speed [3]. Then, solutions of (22) can be directly compared to experimental data.

2.2 *Extended KdV (KdV2)*

Extending considerations of the Boussinesq equations (8), (9) to second order we make use of first order solutions (18) and (19). So, applying the perturbation technique described by [6], we postulate w in the form (18) plus higher order corrections, that is

$$w = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{3x} + \alpha^2 C^{(\alpha^2)} + \alpha\beta C^{(\alpha\beta)} + \beta^2 C^{(\beta^2)}, \tag{23}$$

where $C^{(\alpha^2)}$, $C^{(\alpha\beta)}$, $C^{(\beta^2)}$ are yet unknown functions of η and its derivatives. Proceeding similarly as in first order equations and using the same properties of relations for time and space derivatives of correction functions, that is, $C_t^{(\cdot)} = -C_x^{(\cdot)}$ one obtains them in the form

$$C^{(\alpha^2)} = \frac{1}{8}\eta^3, \qquad C^{(\alpha\beta)} = \frac{3}{16}\eta_x^2 + \frac{1}{2}\eta\eta_{2x}, \qquad C^{(\beta^2)} = \frac{1}{10}\eta_{4x}. \tag{24}$$

Then the final form of second order equations is

$$w = \eta - \frac{1}{4}\alpha\eta^2 + \frac{1}{3}\beta\eta_{3x} + \frac{1}{8}\alpha^2\eta^3 + \alpha\beta\left(\frac{3}{16}\eta_x^2 + \frac{1}{2}\eta\eta_{2x}\right) + \frac{1}{10}\beta^2\eta_{4x}, \tag{25}$$

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{3x} - \frac{3}{8}\alpha^2\alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} = 0. \tag{26}$$

Equation (26) was derived by Marchant and Smyth [33] (directly from the set of Euler equations and alternatively from Luke's Lagrangian [32]) and called by the authors the *extended KdV*. This is a second order extension of KdV in dimensionless variables and fixed reference frame. We call it in short **KdV2**. In principle KdV2 solutions should be a better approximation of the solutions to the Boussinesq set than

KdV solutions. They should be, as well, reasonable approximations in a wider range of small parameters $\alpha,\ \beta$.

2.3 Analytic Solutions of KdV and KdV2

KdV gained enormous success as an approximation common for many problems in nonlinear physics. KdV is integrable and has solutions exhibiting a rich variety of properties. The standard derivation of analytic solutions in the form of single solitonic functions (in terms of hyperbolic functions) and periodic functions (Jacobi elliptic functions) is presented in many textbooks or monographs, see, e.g., [1, 8, 20, 36, 39, 41]. It consists in the introduction of the new variable $\xi = x - vt$. Then KdV is transformed to a nonlinear ordinary differential equation (ODE) which can be integrated two times leading to the equation

$$\frac{\beta}{3\alpha}\left(\eta_\xi\right)^2 = -\eta^3 + 2c_1\eta^2 + r\eta + s, \tag{27}$$

where $c_1 = \frac{v-1}{\alpha}$, r and s are integration constants. The particular case $r = s = 0$ leads to the soliton solution

$$\eta(x,t) = A\,\mathrm{Sech}^2\left(\sqrt{\frac{3A}{4}\frac{\alpha}{\beta}}\left[x - t\left(1 + \frac{\alpha}{2}\right)\right]\right). \tag{28}$$

When one is interested only in mathematical properties of KdV solutions A can be an arbitrary positive constant. However, if physical properties are considered A should be close to one, otherwise the resulting solution contradicts the basic assumption for the derivation ($\frac{A}{H} = \alpha \ll 1$).

When integration constants are nonzero a thorough analysis shows the existence of periodic solutions in terms of Jacobi elliptic functions cn^2 (or equivalently dn^2). The solutions have the form (cnoidal wave)

$$\eta(x,t) = A\,\mathsf{cn}^2\left[B\left(x - vt\right), m\right] + D, \tag{29}$$

where A, B, D, v are constants and $m \in [0,1]$ is the elliptic parameter. Constant $D < 0$ is necessary in order to ensure that the volumes of water elevations and depressions with respect to the undisturbed water level are the same (volume conservation condition). When the elliptic parameter $m \to 1$ the distance between crests of cnoidal wave increases to infinity resulting in a soliton solution as the limit. When $m \to 1$ the limiting profile is the usual cosine wave.

KdV possesses one more important property. There exist exact n-soliton solutions which can be derived from the inverse scattering theory, see, e.g., [1, 2, 12, 14, 38].

Not much was known about analytic solutions to KdV2 till recently. In [23] we showed that KdV2 has an exact single soliton solution of the same form as KdV (28) but with different coefficients. The derivation is following. Proceeding similarly as in the KdV case, that is, introducing $\xi = x - vt$ one transforms (26) into ODE. Postulating the solution in the form $\eta(\xi) = A\,\mathrm{Sech}^2(B\,\xi)$ results in an equation of the form

$$C_2\,\mathrm{Sech}^2(B\,\xi) + C_4\,\mathrm{Sech}^4(B\,\xi) + C_6\,\mathrm{Sech}^6(B\,\xi) = 0, \tag{30}$$

where $C_2,\ C_4,\ C_6$ are functions of unknowns A, B, v and coefficients of the KdV2 equation. Equation (30) holds when all C_i vanish simultaneously. Then, solving the set $C_2 = 0,\ C_4 = 0$ and $C_6 = 0$ one obtains formulas for the coefficients A, B, v which determine the solution. Condition $C_6 = 0$ implies a quadratic equation for $z = \dfrac{B^2\beta}{A\,\alpha}$ with solutions

$$z_1 = \frac{43 - \sqrt{2305}}{152} \approx -0.033, \qquad z_2 = \frac{43 + \sqrt{2305}}{152} \approx 0.599. \tag{31}$$

Then the final formulas are

$$A = \frac{z - \frac{3}{4}}{\alpha\, z(\frac{11}{4} - \frac{19}{3} z)}, \quad B = \sqrt{\frac{z - \frac{3}{4}}{\beta(\frac{11}{4} - \frac{19}{3} z)}}, \quad v = 1 + \frac{z - \frac{3}{4}}{(\frac{11}{4} - \frac{19}{3} z)} \left(\frac{2}{3} + \frac{38}{45}\, \frac{z - \frac{3}{4}}{(\frac{11}{4} - \frac{19}{3} z)} \right). \tag{32}$$

Solutions obtained with $z = z_1$ have to be rejected. In this case B is imaginary, $B = i\bar{B}$. Then $\mathrm{Sech}^2[B(x - vt)] = \left(\cos^2[\bar{B}(x - vt)]\right)^{-1}$. The solution has poles for some arguments, so it has no physical sense.

There is an important difference between solitonic solutions to KdV2 and KdV. There is no freedom for the former ones, and for a given α, β three equations $C_i = 0$ completely determine the coefficients A, B, v of the solutions. For derivation of KdV coefficients the equation analogous to (30) contains only two lower order terms. Then there are only two equations for three coefficients A, B, v. This means that there is one parameter family of solutions. Usually B, v are expressed as functions of positive A which can be arbitrary within some interval (as long as it does not contradict the basic assumption $\frac{A}{h} \ll 1$). Moreover, for KdV2 solitons the ratio $\frac{B^2}{A} = \frac{\alpha}{\beta} z \approx 0.6\frac{\alpha}{\beta}$ and $v \approx 1.1145$ whereas for KdV $\frac{B^2}{A} = 0.75\frac{\alpha}{\beta}$ and $v = 1 + \frac{\alpha}{2}$.

The exact periodic solutions of KdV2 obtained by us in [21] are very fresh. Encouraged by the success of the method used in [23] to derive the soliton solution to KdV2 we postulated periodic solutions to KdV2 in the same form as periodic solutions to KdV (29). Then with a similar procedure as that described above for the solitonic case one arrives at an equation analogous to (30)

$$F_0 + F_2\,\mathrm{cn}^2(B\,\xi) + F_4\,\mathrm{cn}^4(B\,\xi) = 0, \tag{33}$$

where $F_i = F_i(A, B, D, v)$. Then the set of equations $F_i = 0$ supplemented by the volume conservation condition allows us to determine all four unknown coefficients A, B, D, v of the solution as functions of the elliptic parameter m. The condition $F_4 = 0$ gives the same quadratic equation for $z' = \frac{B^2\beta}{A\alpha}\frac{1}{m}$ as the equation $C_6 = 0$ for the solitonic case. Therefore roots z_1', z_2' are the same as z_1, z_2 in (31).

The periodicity of cn function ensures $\lambda/2 = 2K(m)/B$, where λ is the wavelength and $K(m)$ is the complete elliptic integral of the first kind. Then the volume conservation condition

$$\int_0^{\lambda/2} [A\,\mathsf{cn}^2(B\xi, m)]d\xi = 0 \tag{34}$$

yields relations between A, D and m

$$D = -\frac{A}{m}\left(\frac{E(m)}{K(m)} + m - 1\right), \tag{35}$$

where $E(m)$ is the complete elliptic integral. In explicit formulas for coefficients A, B^2, D the factor $\mathsf{EK}(m) = 3\frac{E(m)}{K(m)} + m - 2$ appears. The function $\mathsf{EK}(m),\ m \in [0, 1]$ has the root $m_s \approx 0.96115$ and is positive when $m < m_s$ and negative when $m > m_s$. Then because two z' roots have different signs there are two branches of KdV2 solutions.

1. The branch with $z' = z_2$. $B^2 > 0$ and then the real B is obtained only when $\mathsf{EK}(m) < 0$, that is when $m > m_s$. Therefore $A > 0, D < 0$, and the solution is a 'normal' cnoidal wave with amplitude of crests larger than depressions.
2. Branch with $z' = z_1$. B is real-valued when $m < m_s$. This implies $A < 0, D > 0$, and the solution is an inverted cnoidal profile. Such solutions do not exist for KdV.

For both branches there exist such intervals of m that $B^2 < 0$. However, these solutions (after transforming them to functions of real arguments) exhibit singularities for some arguments and therefore have no physical sense.

For detailed derivation of the analytic periodic solutions of KdV2 and discussion of their properties, see [21].

3 KdV Invariants

It is widely known, see, e.g., [8, Chap. 5], that an equation with a form analogous to the form of the continuity equation

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{36}$$

corresponds to some *conservation law*. In (36) T and X are analogs to density and flux, respectively. Functions T and X may depend on $x, t, \eta, \eta_x, \eta_{2x}, \ldots,$ but not on η_t. The Eq. (36) can be applied, in particular, to KdV and to the equations of the KdV type, such as (47). If functions T and X_x are integrable on $x \in (-\infty, \infty)$ and $\lim_{x\to\pm\infty} X = \text{const}$ (this case corresponds to soliton solutions), then integration of Eq. (36) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{\infty} T\,dx\right) = 0 \quad \text{or} \quad \int_{-\infty}^{\infty} T\,dx = \text{const.} \tag{37}$$

$$\text{since} \qquad \int_{-\infty}^{\infty} X_x\,dx = X(\infty,t) - X(-\infty,t) = 0. \tag{38}$$

The same conclusion can be drawn for periodic solutions (cnoidal waves). In this case limits in the integrals (37), (38) have to be replaced by $(a, a+\Lambda)$, where Λ is the wave length of the cnoidal wave and a is arbitrary.

For the KdV equation (20) the first two invariants are easily obtainable. When (20) is presented in the form

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left(\eta + \frac{3}{4}\alpha\eta^2 + \frac{1}{6}\beta\eta_{xx}\right) = 0, \tag{39}$$

the conservation of mass (volume) law is immediately obtained

$$I^{(1)} = \int_{-\infty}^{\infty} \eta\,dx = \text{const.} \tag{40}$$

Multiplication of (20) by η leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\eta^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{1}{2}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 + \frac{1}{6}\beta\eta\eta_{xx}\right) = 0, \tag{41}$$

which results in the following form of the second invariant

$$I^{(2)} = \int_{-\infty}^{\infty} \eta^2\,dx = \text{const.} \tag{42}$$

Invariants $I^{(1)}$ (40) and $I^{(2)}$ (42) have the same form both in fixed and moving reference frames.

Denoting by $\mathrm{KDV}(x,t)$ the left hand side of (20) and taking

$$3\eta^2 \times \mathrm{KDV}(x,t) - \frac{2}{3}\frac{\beta}{\alpha}\eta_x \times \frac{\partial}{\partial x}\mathrm{KDV}(x,t) = 0 \tag{43}$$

one obtains

$$\frac{\partial}{\partial t}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) + \\ \frac{\partial}{\partial x}\left(\frac{9}{8}\alpha\eta^4 + \frac{1}{2}\beta\eta_{2x}\eta^2 - \beta\eta_x^2\eta + \eta^3 + \frac{1}{18}\frac{\beta^2}{\alpha}\eta_{2x}^2 - \frac{1}{9}\frac{\beta^2}{\alpha}\eta_x\eta_{3x} - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right) = 0. \tag{44}$$

This gives the third invariant for KdV (20) in the fixed reference frame

$$I^{(3)} = \int_{-\infty}^{\infty}\left(\eta^3 - \frac{1}{3}\frac{\beta}{\alpha}\eta_x^2\right)dx = \text{const.} \tag{45}$$

The same formula is obtained for the third KdV invariant in the moving frame [24].

In the subject literature, see, e.g., [3, 8], $I^{(2)}$ is attributed to conservation of momentum and $I^{(3)}$ to conservation of energy. However, as pointed out in [24] they are not exact momentum and energy, respectively.

For any solutions of KdV preserving their shapes during the motion, that is, for single soliton solutions and cnoidal solutions, integrals of any power of the solution $\eta(x,t)$ and any power of its arbitrary derivative with respect to x are invariants. That is,

$$I^{(p,k)} = \int_{-\infty}^{\infty}(\eta_{kx})^p\,dx = \text{const,} \tag{46}$$

where $p \in \mathbb{R}$ is an arbitrary real number, and $k = 0, 1, 2, \ldots$. An arbitrary linear combination of $I^{(p,k)}$ is an invariant, as well.

4 KdV2 Adiabatic Invariants—Direct Method

We now consider the KdV2 equation [24, Eq. (1)]

$$\eta_t + \eta_x + \frac{3}{2}\alpha\,\eta\eta_x + \frac{1}{6}\beta\,\eta_{3x} \\ -\frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} = 0, \tag{47}$$

named as the *extended KdV* by Marchant and Smyth [33, Eq. (2.8)]. They derived (47) both from Euler's hydrodynamic equations and Luke's Lagrangian [32]. The equation has been considered by several authors, see, e.g., [6, 21, 23–26, 28, 33, 34]. As stated above, we call it KdV2.

In [24], we note that $I^{(1)} = \int_{-\infty}^{\infty}\eta\,dx$ is the exact invariant of (47) representing the conservation of mass as it does for KdV.

4.1 Second Invariant

The second invariant of KdV, $I^{(2)} = \int_{-\infty}^{\infty} \eta^2\, dx$ is **not** an invariant of KdV2, because, see [24, Sec. III B], after multiplication of Eq. (47) by η one obtains

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\eta^2\right) + \frac{\partial}{\partial x}\left[\frac{1}{2}\eta^2 + \frac{1}{2}\alpha\eta^3 + \frac{1}{6}\beta\left(-\frac{1}{2}\eta_x^2 + \eta\eta_{2x}\right) - \frac{3}{32}\alpha^2\eta^4 \right. \tag{48}$$
$$\left. + \frac{19}{360}\beta^2\left(\frac{1}{2}\eta_{xx}^2 - \eta_x\eta_{3x} + \eta\eta_{4x}\right) + \frac{5}{12}\alpha\beta\,\eta^2\eta_{2x}\right] + \frac{1}{8}\alpha\beta\,\eta\eta_x\eta_{2x} = 0\,.$$

It is not possible to express the final term in (48) as $\frac{\partial}{\partial x}X(\eta, \eta_x, \ldots)$. Then contrary to KdV case the quantity $\int_{-\infty}^{+\infty} \eta^2 dx$ is not conserved. There are no exact higher order invariants of (47) as well.

It is possible, though, to determine approximate invariants of (47), whose terms which violate the invariance are of the third order in α, β. Our simple method allows us to determine such approximate invariants without big effort. It works by forming an equation which contains functions T and X by means of some manipulations with KdV2. In this equation there are terms in X which are non-integrable with respect to x similarly as the last term in (48). By adding a linear combination of the form $(c_1\alpha + c_2\beta) \times KdV2(x, t)$ to that equation, dropping the third order terms we can determine c_1 and/or c_2 such that the non-integrable terms cancel. (Equivalently, we add a linear combination of the form $(c_1\alpha + c_2\beta) \times KdV(x, t)$ without dropping any term.) This action yields a new T' function and an approximate conservation law for $\int_{-\infty}^{\infty} T' dx$.

The first approximate invariant can be obtained by adding to (48) Eq. (47) multiplied by $c_1\alpha\eta^2$, neglecting terms of third-order and selecting a proper value of c_1 in order to cancel the term $\frac{1}{8}\alpha\beta\,\eta\eta_x\eta_{2x}$. When this is done we are left with the expression

$$c_1\alpha\eta_t\eta^2 + c_1\alpha\eta^2\eta_x + c_1\frac{3}{2}\alpha^2\eta^3\eta_x + c_1\frac{1}{6}\alpha\beta\eta^2\eta_{3x}. \tag{49}$$

In integration over x of (49), terms $c_1\alpha\eta^2\eta_x$ and $c_1\frac{1}{6}\alpha\beta\eta^2\eta_{3x}$ are integrable with respect to x and then can be included into the flux function X.

The last term in (49) can be transformed to $-\frac{1}{3}c_1\alpha\beta\eta\eta_x\eta_{2x}$. It cancels with $\frac{1}{8}\alpha\beta\,\eta\eta_x\eta_{2x}$ when $c_1 = \frac{3}{8}$. Then the first term in (49) yields

$$c_1\alpha\eta_t\eta^2 = \frac{\partial}{\partial t}\left(\frac{1}{8}\alpha\eta^3\right). \tag{50}$$

Due to integrability of the other terms the approximate invariant of KdV2 is obtained as ($\frac{1}{2}$ is omitted)

$$I_{\mathrm{ad}}^{(2\alpha)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{4}\alpha\, \eta^3\right) dx \approx \text{const.} \tag{51}$$

However, there is another way to remove the last term in (48) and get an alternative form of the second approximate invariant. This goal can be achieved by adding to (48) Eq. (47) multiplied by $c_2\beta\eta_{2x}$, dropping again third-order terms and selecting a proper value of c_2 to remove the term $\frac{1}{8}\alpha\beta\, \eta\eta_x\eta_{2x}$. Then new terms are

$$c_2\beta\eta_t\eta_{2x} + c_2\beta\eta_x\eta_{2x} + c_2\frac{3}{2}\alpha\beta\eta\eta_x\eta_{2x} + c_2\frac{1}{6}\beta^2\eta_{2x}\eta_{3x}. \tag{52}$$

In integration over x of (52), the terms $c_2\beta\eta_x\eta_{2x}$ and $c_2\frac{1}{6}\beta^2\eta_{2x}\eta_{3x}$ are integrable with respect to x and then can be included into X. The cancellation of non-integrable terms

$$c_2\frac{3}{2}\alpha\beta\eta\eta_x\eta_{2x} + \frac{1}{8}\alpha\beta\, \eta\eta_x\eta_{2x} = 0$$

implies $c_2 = -\frac{1}{12}$.

Integration of the first term in (52) over x gives

$$\int_{-\infty}^{\infty} c_2\beta\eta_t\eta_{2x}dx = c_2\beta\left(\eta_t\eta_x|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\eta_{tx}\eta_x\right) = -c_2\beta\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\frac{1}{2}\eta_x^2\right). \tag{53}$$

Since terms with $\eta_x\eta_{2x}$ and $\eta_{2x}\eta_{3x}$ can be expressed as $\left(-\frac{1}{2}\eta_x^2\right)_x$ and $\left(-\frac{1}{2}\eta_{2x}^2\right)_x$, respectively, one gets finally

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}\left(\eta^2 + \frac{1}{12}\beta\eta_x^{\,2}\right)dx + F(\eta,\eta_x,\eta_{2x})|_{-\infty}^{\infty} = O(\alpha^3), \tag{54}$$

where $F(\eta,\eta_x,\eta_{2x})$ results from the integration of the flux term. Since solutions to KdV2 are either solitonic or periodic then this term vanishes.

This gives an adiabatic (approximate) invariant of KdV2 (47) in the form

$$I_{\mathrm{ad}}^{(2\beta)} = \int_{-\infty}^{\infty}\left(\eta^2 + \frac{1}{12}\beta\,\eta_x^{\,2}\right)dx \approx \text{const.} \tag{55}$$

The existence of two independent adiabatic invariants $I_{\mathrm{ad}}^{(2\alpha)}$ and $I_{\mathrm{ad}}^{(2\beta)}$ means also that

$$I_{\mathrm{ad}}^{(2)} = \varepsilon\, I_{\mathrm{ad}}^{(2\alpha)} + (1-\varepsilon)I_{\mathrm{ad}}^{(2\beta)} = \int_{-\infty}^{\infty}\left(\eta^2 + \varepsilon\frac{1}{12}\alpha\eta^3 + (1-\varepsilon)\frac{1}{12}\beta\,\eta_x^{\,2}\right)dx \tag{56}$$

is an adiabatic invariant for any ε, that is, there exists one parameter family of adiabatic second invariant of KdV2.

4.2 Third Invariant

In order to find the third invariant for KdV2 one can follow the procedure described in Sect. 3, Eqs. (43)–(45), but for KdV2 equation. Take

$$3\eta^2 \times \mathrm{KDV2}(x,t) - \frac{2}{3}\frac{\beta}{\alpha}\eta_x \times \frac{\partial}{\partial x}\mathrm{KDV2}(x,t) = 0 \tag{57}$$

and consider a simpler case, when $\beta = \alpha$. The result is

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\eta^3 - \frac{1}{3}\eta_x^2\right) + \frac{\partial}{\partial x}\left(\eta^3 - \frac{1}{3}\eta_x^2 + \alpha\frac{9}{8}\eta^4 - \alpha^2\frac{9}{40}\eta^5\right) \\ &+ \alpha\left(-\eta_x^3 - \eta\eta_x\eta_{2x} + \frac{1}{2}\eta^2\eta_{3x}\right) + \alpha^2\left(\frac{1}{2}\eta\eta_x^3 + \frac{25}{8}\eta^2\eta_x\eta_{2x} - \frac{23}{36}\eta_x\eta_{2x}^2\right. \\ &\left. + \frac{5}{4}\eta^3\eta_{3x} - \frac{11}{12}\eta_x^2\eta_{3x} - \frac{5}{18}\eta\eta_x\eta_{4x} + \frac{19}{120}\eta^2\eta_{5x}\right). \end{aligned} \tag{58}$$

In (58) we omitted terms which vanish under integration over x. All terms in the second and third rows of (58) are non-integrable. However, taking an integral of the form $\int_{-\infty}^{\infty} \ldots dx$ and integrating by parts they can be reduced to two types of non-integrable terms. All terms in the bracket with α become proportional to $\eta\eta_x\eta_{2x}$. All terms in the bracket with α^2 reduce to $\eta\eta_x\eta_{2x}$ and $\eta_x\eta_{2x}^2$. Then using procedures described above for second adiabatic invariant, that is, by adding to (58) the KdV multiplied by proper factors one can cancel these non-integrable terms. The added terms supply additional terms in the T function. As in the case of second invariant this action is not unique and there is some freedom in the form of final adiabatic invariant. One of admissible forms is

$$I_{\mathrm{ad}}^{(3)} = \int_{-\infty}^{\infty}\left(\eta^3 - \frac{1}{3}\eta_x^2 - \alpha\eta^4 + \frac{7}{12}\alpha\,\eta\eta_x^2\right)dx. \tag{59}$$

It should be noted that the first two terms in (59) are the same as the third KdV invariant.

The method presented enables us to derive higher order adiabatic invariants, as well.

5 Near-Identity Transformation for KdV2 in Fixed Frame

Our research was performed in the fixed reference frame. It was motivated by two facts. First, already pointed out in [24, Eq. (39)], even for KdV energy has **noninvariant form** (Ali and Kalisch [3] showed this fact in dimension variables). Second, our purpose is to study invariants, and approximate invariants not only for KdV and KdV2, but also for the KdV2 equation with non-flat bottom, derived in [23, 26]. For this equation it is only the fixed reference frame that makes sense.

Second order versions of KdV type equations are not unique since there exist transformations which transform the given equation into an equation of the same form but with some coefficients altered. These equations are asymptotically equivalent, that is, their solutions converge to the same form when small parameters tend to zero. Therefore such transformation, called *near-identity transformation* (NIT), is often used to convert higher order nonlinear differential equations to their asymptotically equivalent forms which can be integrable. Such NIT was first introduced by Kodama [29, 30] and then used and generalized by many authors, see, e.g., [9–11, 13, 15, 16, 19, 34]. Below we apply NIT in the form suitable for the KdV2 equation.

Introduced below is the near-identity transformation in the form used by the authors of [9]

$$\eta = \eta' + \alpha a \eta'^2 + \beta b \eta'_{xx} + \dots, \tag{60}$$

where a, b are some constants. (Here, we choose $+$ sign. The inverse transformation, up to terms of second order, is $\eta' = \eta - \alpha a \eta^2 - \beta b \eta_{xx} + \dots$).

NIT should preserve the form of the KdV2 (47), at most altering some coefficients. Then it is possible to choose coefficients a, b of NIT such that the transformed equation possesses a Hamiltonian (see the consequences in the Sect. 5.2).

Insertion (60) into (47) yields (terms of order higher than the second in α, β are neglected)

$$\begin{aligned}
&\eta'_t + \eta'_x + \alpha\left[\left(\frac{3}{2} + 2a\right)\eta'\eta'_x + 2a\eta'\eta'_t\right] + \beta\left[\left(\frac{1}{6} + b\right)\eta'_{3x} + b\eta'_{xxt}\right] \\
&\quad + \alpha\beta\left\{\left[\left(\frac{23}{24} + a + \frac{3}{2}b\right)\eta'_x\eta'_{2x}\right] + \left[\left(\frac{5}{12} + \frac{1}{3}a + \frac{3}{2}b\right)\eta'\eta'_{3x}\right]\right\} \\
&\quad + \alpha^2\left(-\frac{3}{8} + \frac{9}{2}a\right)\eta'^2\eta'_x + \beta^2\left[\left(\frac{19}{360} + \frac{1}{6}b\right)\eta'_{5x}\right] = 0.
\end{aligned} \tag{61}$$

Since terms with time derivatives (η'_t, η'_{xxt}) appear in first order with respect to small parameters we can replace them by appropriate expressions obtained from KdV2 (47) limited to first order, that is from KdV (20)

$$\eta'_t = -\eta'_x - \frac{3}{2}\alpha\eta'\eta'_x - \frac{1}{6}\beta\eta'_{3x} \tag{62}$$

and

$$\eta'_{xxt} = \partial_{xx}\left(-\eta'_x - \frac{3}{2}\alpha\eta'\eta'_x - \frac{1}{6}\beta\eta'_{3x}\right) = -\eta'_{3x} - \frac{3}{2}\alpha(3\eta'_x\eta'_{2x} + \eta'\eta'_{3x}) - \frac{1}{6}\beta\eta'_{5x}. \tag{63}$$

Inserting (62) and (63) into (61) one obtains

$$\begin{aligned}\eta'_t + \eta'_x + \frac{3}{2}\alpha\eta'\eta'_x + \frac{1}{6}\beta\eta'_{3x} + \alpha^2\left(-\frac{3}{8} + \frac{3}{2}a\right)\eta'^2\eta'_x \\ + \alpha\beta\left[\left(\frac{23}{24} + a - 3b\right)\eta'_x\eta'_{2x} + \frac{5}{12}\eta'\eta'_{3x}\right] + \frac{19}{360}\beta^2\eta'_{5x} = 0.\end{aligned} \tag{64}$$

Equation (64) for η' has the same form as KdV2 (47) with only two coefficients altered. The coefficient in front of the term with $\alpha^2\eta^2\eta_x$ is changed from $-\frac{3}{8}$ to $-\frac{3}{8} + \frac{3}{2}a$ and the coefficient in front of the term with $\alpha\beta\eta_x\eta_{2x}$ is changed from $\frac{23}{24}$ to $\frac{23}{24} + a - 3b$.

5.1 NIT—Second Adiabatic Invariant

For the NIT-transformed KdV2 equation (64) one can find the second invariant in the same way as previously, that is multiplying (64) by η' and requiring that the coefficient in front of the non-integrable term vanishes. This gives

$$\int_{-\infty}^{\infty} \eta'\left[\frac{5}{12}\eta'\eta'_{3x} + \left(\frac{23}{24} + a - 3b\right)\eta'_x\eta'_{2x}\right]dx = 0. \tag{65}$$

Since

$$\int_{-\infty}^{\infty} \eta'^2\eta'_{3x}\,dx = -2\int_{-\infty}^{\infty} \eta'\eta'_x\eta'_{2x}\,dx \tag{66}$$

one obtains

$$\left(-2\frac{5}{12} + \frac{23}{24} + a - 3b\right)\int_{-\infty}^{\infty} \eta'\eta'_x\eta'_{xx}\,dx = 0 \quad\Longrightarrow\quad a - 3b + \frac{1}{8} = 0. \tag{67}$$

Then under the condition

$$a - 3b = -\frac{1}{8} \tag{68}$$

the integral $\int_{-\infty}^{\infty} \eta'^2 dx$ is the exact invariant of the Eq. (64).

Using inverse NIT

$$\eta' = \eta - \alpha a\eta^2 - \beta b\eta_{xx} + \dots, \tag{69}$$

and neglecting higher order terms, one gets

$$\int_{-\infty}^{\infty} \eta'^2 dx \approx \int_{-\infty}^{\infty} [\eta^2 - 2\alpha a\eta^3 - 2\beta b\eta\eta_{xx}] = \int_{-\infty}^{\infty} [\eta^2 - 2\alpha a\eta^3 + 2\beta b\eta_x^2]\, dx, \quad (70)$$

where the last term was obtained through integration by parts. The r.h.s. of (70) is the most general form of the second adiabatic invariant of KdV2 under the condition (68), that is, one parameter family of adiabatic invariants

$$I_{\mathrm{ad}}^{(2)} = \int_{-\infty}^{\infty} [\eta^2 - 2\alpha a\eta^3 + 2\beta b\eta_x^2]\, dx \approx \text{const.} \quad (71)$$

In particular, with $a = 0$, $b = \frac{1}{24}$

$$I_{\mathrm{ad}}^{(2)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{12}\beta\eta_x^2\right) dx = I_{\mathrm{ad}}^{(2\beta)} \quad (72)$$

and with $b = 0,\ a = -\frac{1}{8}$

$$I_{\mathrm{ad}}^{(2)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{4}\alpha\eta^3\right) dx = I_{\mathrm{ad}}^{(2\alpha)}. \quad (73)$$

These adiabatic invariants are the same as those obtained in the direct way in (51) and (55).

The above formulas come from NIT (60) in which the sign + was used. However, if in (60) the sign − is chosen then the condition (68) is replaced by $a - 3b = \frac{1}{8}$. The signs of the inverse NIT become opposite and then the final forms of adiabatic invariants remains the same as in (71)–(73).

5.2 NIT—Third Adiabatic Invariant

NIT-transformed KdV2 (64) describes waves in the fixed frame. In order to determine its Hamiltonian form let us convert (64) to a moving frame by transformation

$$\bar{x} = x - t, \quad \bar{t} = t, \quad \partial_x = \partial_{\bar{x}}, \quad \partial_t = -\partial_{\bar{x}} + \partial_{\bar{t}}. \quad (74)$$

Then (64) can be written in more general form as

$$\eta_{\bar{t}} + \alpha A\eta\eta_{\bar{x}} + \beta B\eta_{3\bar{x}} + \alpha^2 A_1\eta^2\eta_{\bar{x}} + \beta^2 B_1\eta_{5\bar{x}} + \alpha\beta\,(G_1\eta\eta_{3\bar{x}} + G_2\eta_{\bar{x}}\eta_{2\bar{x}}) = 0, \quad (75)$$

where

$$A=\frac{3}{2},\quad B=\frac{1}{6},\quad A_1=-\frac{3}{8}+\frac{3}{2}a,\quad B_1=\frac{19}{360},\quad G_1=\frac{5}{12}\quad G_2=\frac{23}{24}+a-3b. \tag{76}$$

In the following we drop bars over t and x, remembering that now we work in the moving reference frame.

In particular, the parameters a, b of NIT can be chosen such that

$$G_2=2G_1. \tag{77}$$

In this case the Hamiltonian for the Eq. (75) exists. The condition (77) with (76) gives

$$\frac{23}{24}+a-3b=2\frac{5}{12} \qquad \Longrightarrow \qquad a-3b=-\frac{1}{8}.$$

This is the same condition as (68). This condition supplies one parameter family of NIT, assuring Hamiltonian form of the NIT-transformed KdV2 (75) in the moving frame.

This Hamiltonian form is

$$\eta'_t=\frac{\partial}{\partial x}\left(\frac{\delta\mathscr{H}}{\delta\eta'}\right), \tag{78}$$

where the Hamiltonian $H=\int_{-\infty}^{\infty}\mathscr{H}\,dx$ has density

$$\mathscr{H}=-\frac{1}{6}\alpha A\eta'^3+\frac{1}{2}\beta B\eta_x'^2-\frac{1}{12}\alpha^2A_1\eta'^4-\frac{1}{2}\beta^2B_1\eta_{xx}'^2+\frac{1}{2}\alpha\beta G_1\eta'\eta_x'^2. \tag{79}$$

Since $\mathscr{H}=\mathscr{H}(\eta',\eta'_x,\eta'_{xx})$, then the *functional derivative* in (78) is

$$\begin{aligned}\frac{\delta\mathscr{H}}{\delta\eta'}&=\frac{\partial\mathscr{H}}{\partial\eta'}-\frac{\partial}{\partial x}\frac{\partial\mathscr{H}}{\partial\eta'_x}+\frac{\partial^2}{\partial x^2}\frac{\partial\mathscr{H}}{\partial\eta'_{xx}}\\&=-\frac{1}{2}\alpha A\eta'^2-\beta B\eta'_{xx}-\frac{1}{3}\alpha^2A_1\eta'^3-\alpha\beta G_1\left(\frac{1}{2}\eta_x'^2+\eta'\eta'_{xx}\right)-\beta^2B_1\eta'_{4x}.\end{aligned} \tag{80}$$

Insertion (80) into (78) yields

$$\eta'_t=-\alpha A\eta'\eta'_x-\beta B\eta'_{3x}-\alpha^2A_1\eta'^2\eta'_x+\beta^2B_1\eta'_{5x}-\alpha\beta G_1(2\eta'_x\eta'_{xx}+\eta'\eta'_{xx}). \tag{81}$$

We see that the Hamiltonian form of KdV2 in the moving frame exists under the condition that the coefficient at the term $\eta'_x\eta'_{xx}$ is two times larger that the coefficient at the term $\eta'\eta'_{xxx}$. This is obtained by a proper choice of a, b parameters of NIT, which is the condition (68).

Now, the Hamiltonian is the exact constant of motion for the NIT-transformed equation (75) under the condition (68)

$$\int_{-\infty}^{\infty}\left[-\frac{1}{6}\alpha A\eta'^3+\frac{1}{2}\beta B\eta_x'^2-\frac{1}{12}\alpha^2A_1\eta'^4-\frac{1}{2}\beta^2B_1\eta_{xx}'^2+\frac{1}{2}\alpha\beta G_1\eta'\eta_x'^2\right]dx=\text{const.} \tag{82}$$

In order to obtain the adiabatic invariant of the original Eq. (47) it is necessary to perform the inverse NIT, that is

$$\eta' = \eta - \alpha a \eta^2 - \beta b \eta_{xx} \tag{83}$$

and then to neglect in the Hamiltonian density all higher order terms. This yields

$$\begin{aligned}\mathscr{H} = & -\frac{1}{6}\alpha A \eta^3 + \frac{1}{2}\beta B \eta_x^2 + \alpha^2 \left(\frac{1}{2}aA - \frac{1}{12}A_1\right)\eta^4 \\ & + \beta^2\left(-\frac{1}{2}B_1\eta_{2x}^2 - bB\eta_x\eta_{3x}\right) + \alpha\beta\left[\left(\frac{1}{2}G_1 - 2aB\right)\eta\eta_x^2 + \frac{1}{2}bA\eta^2\eta_{2x}\right],\end{aligned} \tag{84}$$

with the condition (68).

Now, we restore the original notation $A = \eta$ and numerical values of coefficients (76). Using relations which come from integration by parts

$$\int_{-\infty}^{\infty} \eta_x \eta_{3x}\, dx = -\int_{-\infty}^{\infty} \eta_{2x}^2\, dx, \quad \int_{-\infty}^{\infty} \eta^2 \eta_{2x}\, dx = -2\int_{-\infty}^{\infty} \eta\eta_x^2\, dx$$

and changing irrelevant sign one obtains finally

$$\begin{aligned}I_{\mathrm{ad}}^{(3)} = \int_{-\infty}^{\infty} & \left[\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 - \alpha^2\left(\frac{1}{32} + \frac{5}{8}a\right)\eta^4 + \beta^2\left(\frac{19}{720} - \frac{1}{6}b\right)\eta_{2x}^2\right. \\ & \left. + \alpha\beta\left(\frac{1}{3}a + \frac{3}{2}b - \frac{5}{24}\right)\eta\eta_x^2\right].\end{aligned} \tag{85}$$

We obtain one parameter family (68) of adiabatic invariants related to energy.

In a particular case, when in (68), we set $a = 0, \quad b = \frac{1}{24}$ and then

$$I_{\mathrm{ad}}^{(3)} = \int_{-\infty}^{\infty} \left[\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 - \frac{1}{32}\alpha^2\eta^4 + \frac{7}{720}\beta^2\eta_{2x}^2 - \frac{7}{48}\alpha\beta\eta\eta_x^2\right] dx. \tag{86}$$

When, in (68), we set $a = -\frac{1}{8}, \quad b = 0$, then we obtain

$$I_{\mathrm{ad}}^{(3)} = \int_{-\infty}^{\infty} \left[\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 + \frac{3}{64}\alpha^2\eta^4 + \frac{19}{720}\beta^2\eta_{2x}^2 - \frac{1}{4}\alpha\beta\eta\eta_x^2\right] dx. \tag{87}$$

Another particular form of (85) can be obtained when one sets

$$\frac{19}{720} - \frac{1}{6}b = 0 \quad \Longrightarrow \quad b = \frac{19}{120}, \quad a = \frac{7}{20}.$$

Then, (85) reduces to

$$I_{\rm ad}^{(3)} = \int_{-\infty}^{\infty} \left[\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 - \frac{1}{4}\alpha^2\eta^4 + \frac{7}{240}\alpha\beta\eta\eta_x^2\right] dx. \tag{88}$$

In a similar way one can set

$$\frac{1}{3}a + \frac{3}{2}b - \frac{5}{24} = 0 \quad \Longrightarrow \quad b = \frac{1}{10}, \quad a = \frac{7}{40}.$$

In this case the adiabatic invariant has the form

$$I_{\rm ad}^{(3)} = \int_{-\infty}^{\infty} \left[\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 - \frac{9}{64}\alpha^2\eta^4 + \frac{7}{720}\beta^2\eta_{2x}^2\right] dx. \tag{89}$$

5.3 *Momentum and Energy for KdV2*

Relations between invariants and conservation laws are not as simple as might be expected, even for KdV. In this subsection we present these relations for motion in a fixed reference frame. Expressions of energy for KdV and KdV2 in the moving frame can be found in [24, 28].

5.3.1 KdV Case

The first KdV invariant, $\int_{-\infty}^{\infty} \eta\, dx = \text{const}$, represents volume (mass) conservation of the incompressible fluid.

When components of momentum are calculated as integrals over the fluid volume from momentum density the results are as follows.

$$p_x = p_0 \int_{-\infty}^{\infty}\left[\eta + \frac{3}{4}\alpha\eta^2\right] dx = p_0\left[I_1 + \frac{3}{4}\alpha\, I_2\right] \quad \text{and} \quad p_y = 0, \tag{90}$$

where p_0 is a constant in units of momentum. Since the vertical component of the momentum is zero and the horizontal component is expressed by the two lowest invariants we have the conservation of momentum law.

The total energy in the fixed frame is, see, e.g., [24, Eq. (39)] (E_0 is a constant in energy units)

$$\begin{aligned} E_{\rm tot} &= E_0 \int_{-\infty}^{\infty} \left(\alpha\eta + (\alpha\eta)^2 + \frac{1}{4}(\alpha\eta)^3\right) dx \\ &= E_0 \left(\alpha I^{(1)} + \alpha^2 I^{(2)} + \frac{1}{4}\alpha^2 I^{(3)} + \frac{1}{12}\alpha^2\beta \int_{-\infty}^{\infty} \eta_x^2\, dx\right). \end{aligned} \tag{91}$$

The energy (91) in the fixed reference frame **has noninvariant form**. The last term in (91) generates tiny deviations from energy conservation only when η_x changes in time in the soliton frame of reference, which occurs during soliton collisions only.

5.3.2 KdV2 Case

Volume conservation, $I_1 = \int_{-\infty}^{\infty} \eta\, dx = \text{const}$, is fulfilled for KdV2, too.

Calculation of momentum components within second order approximation of Euler's equations gives also a vanishing vertical component $p_y = 0$. For a horizontal component one gets

$$\begin{aligned} p_x &= p_0 \int_{-\infty}^{\infty} \left(\eta + \frac{3}{4}\alpha\eta^2 - \frac{1}{8}\alpha^2\eta^3 - \frac{7}{48}\alpha\beta\eta_x^2 \right) dx \\ &= p_0 \left[I_1 + \frac{3}{4}\alpha \int_{-\infty}^{\infty} \left(\eta^2 - \frac{1}{6}\alpha\eta^3 - \frac{7}{36}\beta\eta_x^2 \right) dx \right]. \end{aligned} \tag{92}$$

The total momentum of the fluid is composed of two terms. The first is proportional to the volume. The second, an integral in the lower row of (92), contains the same functional terms η^2, η^3, η_x^2 as the expressions for the second adiabatic invariants (56) and (71) but with slightly different coefficients. Analogously to the KdV case (90) one can write

$$p_{x(\mathrm{ad})} \approx p_0 \left[I_1 + \frac{3}{4}\alpha I_{\mathrm{ad}}^{(2)} \right]. \tag{93}$$

We will see in Sect. 6 that $p_{x(\mathrm{ad})}$, given by (93) has much smaller deviations from a constant value than p_x given by (92).

Energy, $E_{\mathrm{tot}} = T + V$, for the system governed by KdV2 is, see, e.g., [24, Eq. (91)],

$$E_{\mathrm{tot}} = E_0 \int_{-\infty}^{\infty} \left(\alpha\eta + (\alpha\eta)^2 + \frac{1}{4}(\alpha\eta)^3 - \frac{3}{32}(\alpha\eta)^4 - \frac{7}{48}\alpha^3\beta\eta\eta_x^2 \right) dx. \tag{94}$$

This expression can be written as

$$\begin{aligned} E_{\mathrm{tot}} &= E_0 \left[\alpha I_1 + \alpha^2 I_{\mathrm{ad}}^{2\beta} + \alpha^2 \int_{-\infty}^{\infty} \left(\frac{1}{4}\alpha\eta^3 - \frac{1}{12}\beta\eta_x^2 - \frac{3}{32}\alpha^2\eta^4 - \frac{7}{48}\alpha\beta\eta\eta_x^2 \right) dx \right] \\ &\approx E_0\, \alpha \left[I_1 + \alpha \left(I_{\mathrm{ad}}^{2\beta} + \alpha I_{\mathrm{ad}}^{3} \right) \right], \end{aligned} \tag{95}$$

where $I_{\mathrm{ad}}^{2\beta}$ is given by (55) and I_{ad}^{3} was chosen in the form (88). Equation (95) shows that the energy of the system described by KdV2 in a fixed frame is approximately given by the sum of exact first invariant and combination of second and third adiabatic invariants. Since there is one parameter freedom in these adiabatic invariants other particular approximate formulas for the energy are admissible, as well. Because of the approximate character of adiabatic invariants the energy of the system is not a conserved quantity. When motion of several solitons is considered the largest changes in the energy occur when solitons change their shapes during collisions, see, e.g., [24, Fig. 4].

6 Numerical Tests

One might question how good these invariants are. The calculations given below offer some insight.

To start with we calculated the time evolution, governed by Eq. (47), for three particular waves. The finite difference method (FDM) of Zabusky [43], generalized for precise calculation of higher derivatives [23, 26] was used. The finite element method (FEM) used for the same problems in [27] give the same results for soliton's motion. 1-, 2- and 3-soliton solutions of KdV were chosen as initial conditions. For the 3-soliton solution the amplitudes were chosen to be 1.0, 0.6 and 0.3, for the 2-soliton solution the amplitudes were chosen as 1.0 and 0.3 and for the single soliton the chosen amplitude was 1.0. The profiles of these waves evolving according to (47) at some instants are presented in Fig. 2. Vertical shifts by 0.2 and horizontal shifts by 30 were used on the figure to avoid overlaps. All these results were obtained for small parameters $\alpha = \beta = 0.1$.

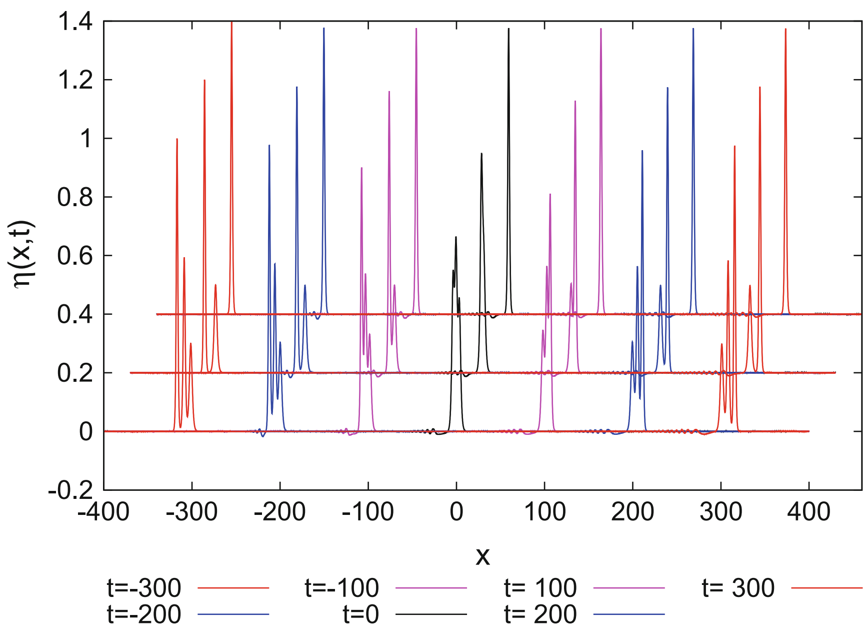

Fig. 2 Time evolution of initially 1-, 2- and 3-soliton KdV solution according to KdV2 (47). Reproduced with permission from [21]. Copyright (2017) by Elsevier

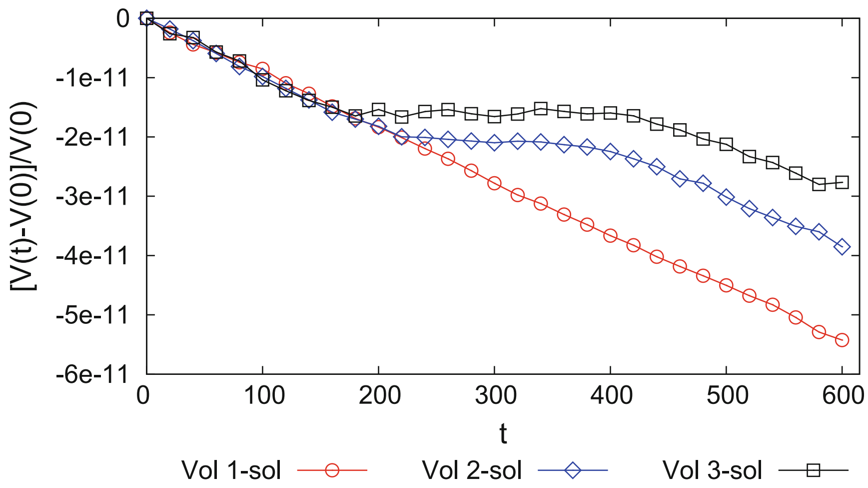

Fig. 3 Numerical precision of the volume conservation law for the three waves displayed in Fig. 2. Reproduced with permission from [21]. Copyright (2017) by Elsevier

Since the volume should be conserved exactly its presentation can verify the precision of numerical evolution. The numerical values of this invariant shown in Fig. 3 are constant up to 10 digits.

6.1 Momentum (Non)Conservation and Adiabatic Invariant $I_{ad}^{(2)}$

To study approximate invariants $I_{\rm ad}^{(2\beta)}$ and $I_{\rm ad}^{(2\alpha)}$ we write each of them as the sum of two terms

$$I_{\rm ad}^{(2\alpha)} = \int_{-\infty}^{\infty} \eta^2\, dx + \int_{-\infty}^{\infty} \frac{1}{4}\alpha\, \eta^3\, dx =: Ie(t) + Ia(t), \tag{96}$$

$$I_{\rm ad}^{(2\beta)} = \int_{-\infty}^{\infty} \eta^2\, dx + \int_{-\infty}^{\infty} \frac{1}{12}\beta\, \eta_x^2\, dx =: Ie(t) + Ib(t). \tag{97}$$

The first terms in (96) and (97) are the same as the exact KdV invariant.

The changes of adiabatic invariants $I_{\rm ad}^{(2\alpha)}$ (96) and $I_{\rm ad}^{(2\beta)}$ (97) presented in Fig. 4 correspond to waves displayed in Fig. 2. In this scale both adiabatic invariants look perfectly constant. To verify how good these invariants are we show how they change with respect to the initial values.

Figure 5 shows changes in the quantities Ie, Ia and Ib for all three 1-, 2-, and 3-soliton waves presented in Fig. 2. These relative changes are defined as

$$Ie = \frac{Ie(t) - Ie(0)}{Ie(0) + Ia(0)}, \quad Ia = \frac{Ia(t) - Ia(0)}{Ie(0) + Ia(0)}, \quad Ib = \frac{Ib(t) - Ib(0)}{Ie(0) + Ia(0)}.$$

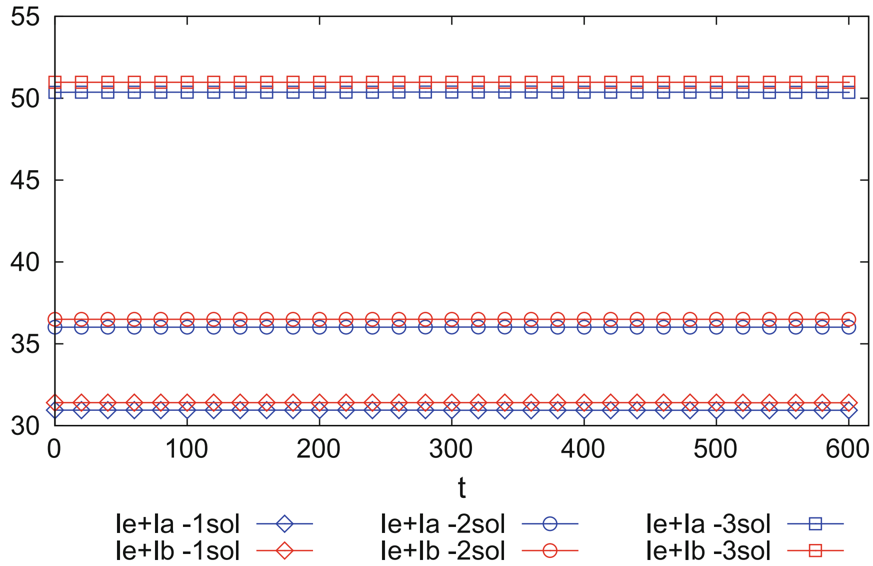

Fig. 4 Absolute values of the adiabatic invariants (96) and (97) for the time evolution shown in Fig. 2. Reproduced with permission from [21]. Copyright (2017) by Elsevier

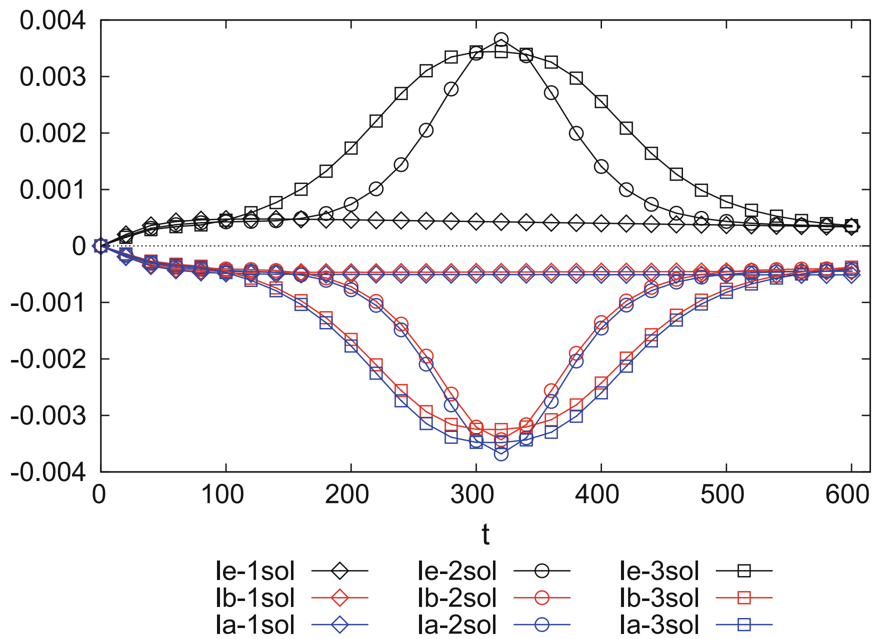

Fig. 5 Relative changes of *Ia* and *Ib* as a functions of time for the three waves presented in Fig. 2. Reproduced with permission from [21]. Copyright (2017) by Elsevier

The figure shows that the corrections *Ia*, *Ib* to the KdV invariant *Ie* have almost the same absolute values as *Ie* but with opposite sign. Therefore one can expect that their summations with *Ie* should only produce small variations of approximate invariants $I_{\rm ad}^{(2\alpha)}$ and $I_{\rm ad}^{(2\beta)}$.

Figure 6 confirms this expectation. For long term evolution, the relative variations of presented approximate invariants are less than the order of 0.00025.

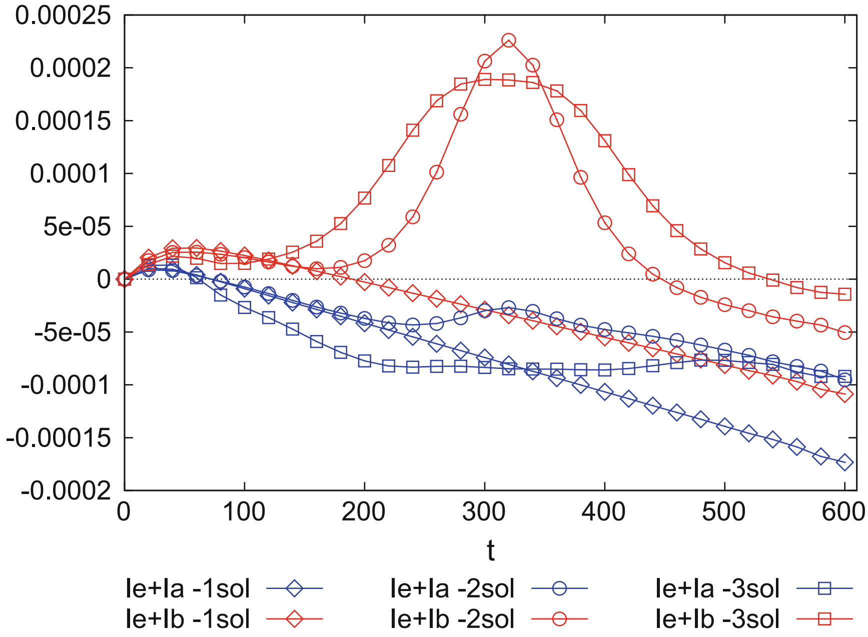

Fig. 6 Relative changes of the approximate invariants: $I_{\mathrm{ad}}^{(2\alpha)}$, denoted as $Ie + Ia$ and $I_{\mathrm{ad}}^{(2\beta)}$ denoted as $Ie + Ib$ for the three waves displayed in the Fig. 2. Reproduced with permission from [21]. Copyright (2017) by Elsevier

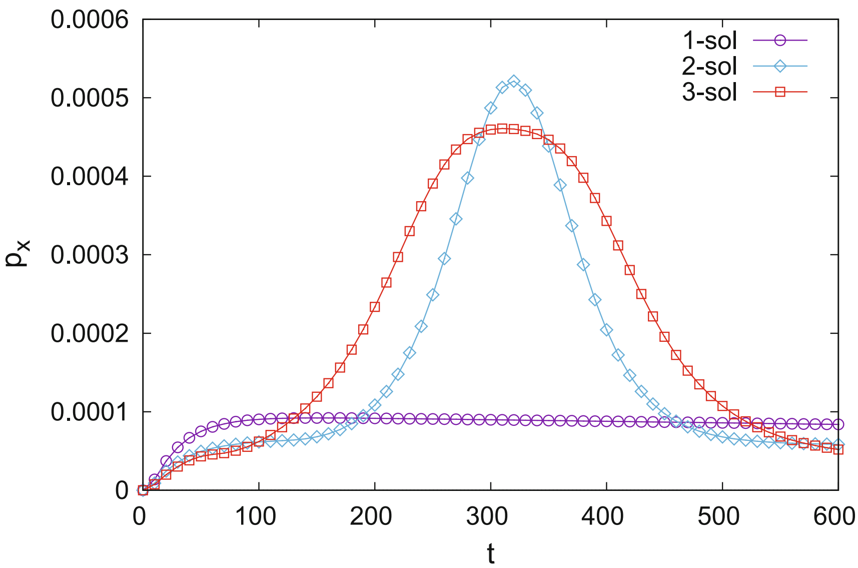

Fig. 7 Relative changes of p_x (92) as a function of time for the three waves presented in Fig. 2

As we have already mentioned the fluid momentum is related to the adiabatic invariant $I_{\mathrm{ad}}^{(2)}$. Let us compare the momentum given by definition (92) with its approximation expressed by adiabatic invariant (93). The former is presented in Fig. 7. In the latter, displayed in Fig. 8, for $I_{\mathrm{ad}}^{(2)}$ we used (56) with $\varepsilon = \frac{1}{2}$. It is clear that the approximate momentum expressed by exact first invariant and adiabatic invariant $I_{\mathrm{ad}}^{(2)}$ suffers much smaller fluctuations than the exact momentum (92).

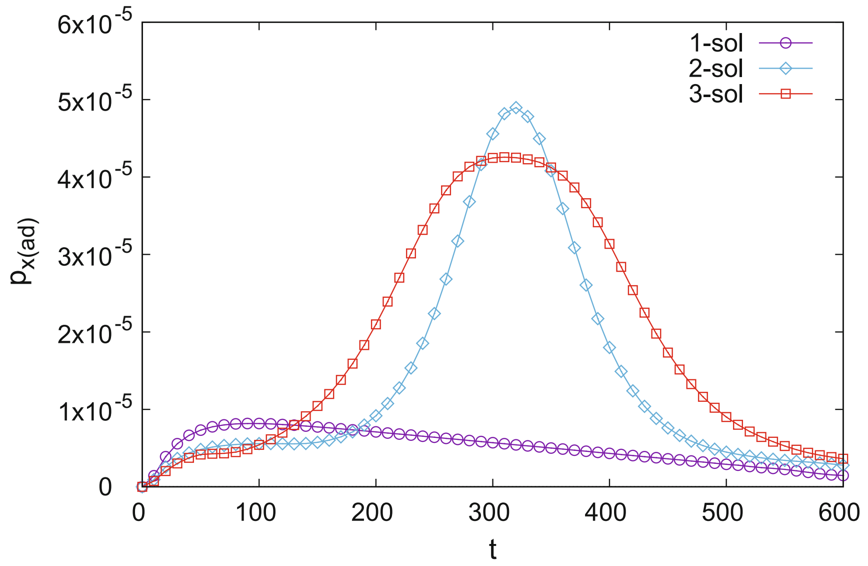

Fig. 8 Relative changes of $p_{x(\mathrm{ad})}$ (93) as a function of time for the three waves presented in Fig. 2

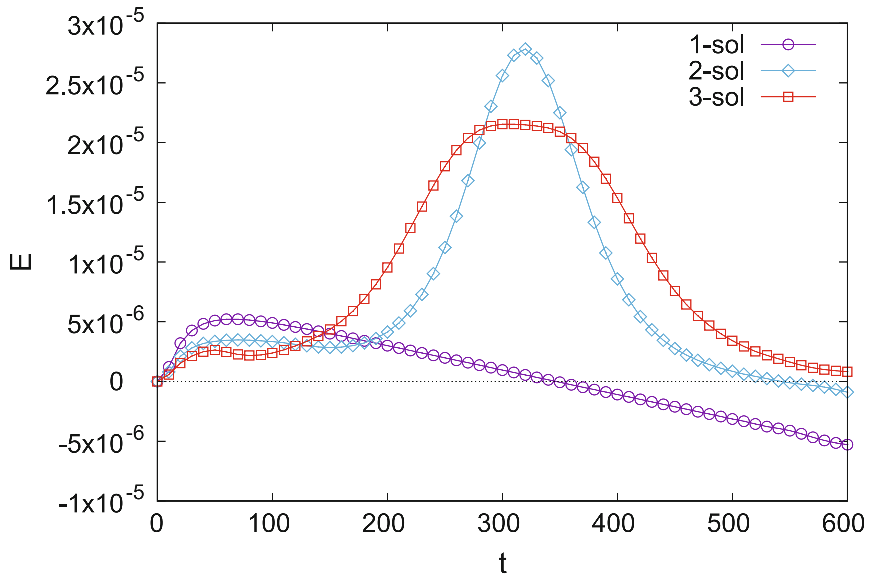

Fig. 9 Relative changes of energy as a function of time for the three waves presented in Fig. 2

6.2 Energy (Non)Conservation and Adiabatic Invariant $I^{(3)}_{ad}$

How close to constant values are adiabatic invariants $I^{(2)}_{\mathrm{ad}}$ and $I^{(3)}_{\mathrm{ad}}$? The relative changes of the energy (94) for three waves shown in Fig. 2 are displayed in Fig. 9.

The energy (94) can be approximated by a linear combination of three terms (95), exact invariant I_1 and adiabatic invariants $I^{(2)}_{\mathrm{ad}}$ and $I^{(3)}_{\mathrm{ad}}$. Relative changes of that approximate energy (95) are displayed in Fig. 10. Comparing Figs. 9 and 10 we see, that, as in the case of momentum, the approximate energy expressed by adiabatic invariants varies less than the exact one.

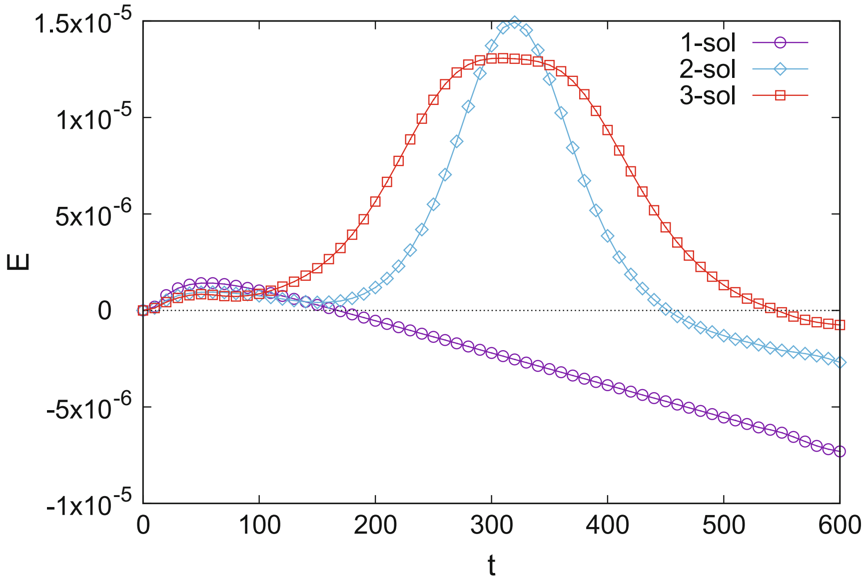

Fig. 10 Relative changes of energy approximated by adiabatic invariants (95) for the three waves presented in Fig. 2

Other than volume conservation, which maintains virtually numerical precision (see, Fig. 3), the adiabatic invariants presented in Figs. 6 and 10, and the energy shown in Fig. 9 over longer periods slowly decrease with time. In our assessment the reason can be found in the fact that 1-, 2-, 3-soliton solutions of the KdV equation, taken as initial conditions, **are not exact** solutions of the KdV2 equation. The 1-soliton analytic solution of KdV2 equation derived in [23] preserves exactly its shape and then possesses the infinite number of invariants. The same is true for recently found [21] exact analytic periodic solutions of KdV2. However, we doubt the existence of exact n-soliton solutions for KdV2 as it does not belong to a hierarchy of integrable equations. Additionally the 2- or 3-soliton solutions of an integrable equation like those obtained through NIT are likewise not exact solutions of (47). Therefore the deviations from exact solutions will lead to dissipation.

7 Summary and Conclusions

In this chapter several properties of solutions to the KdV2 equation (extended KdV equation) are discussed. First, the shallow water problem is formulated within the framework of the motion of ideal fluid under gravitational force with proper boundary conditions. After introducing appropriate scaled variables this model can be considered at different stages of approximation. Next, a general method for derivation of the wave equation is described which can be applied up to arbitrary order in the perturbative approach. Limitation to the first order results in the KdV equation (19), while a second order approach gives KdV2 (26). Then, solutions to KdV are referred to and compared to analytic solutions to KdV2 found by us recently (solitonic [23] and periodic [21] ones). Next, in the main part of the paper, invariants of KdV and adiabatic invariants of KdV2 are described in detail.

Presented is a means of direct calculation of adiabatic invariants for KdV2 which was developed in [25]. This method can be applied directly to equations expressed in the fixed reference frame. Small parameters of $\alpha \neq \beta$ should be of similar order but not necessarily equal.

The method does not depend on a transformation to a particular moving frame, nor on a near-identity transformation. This makes calculations of second invariant simpler. It can be used also to calculate invariants of higher order.

The NIT-based method, developed in Sect. 5, seems to be more suitable for the adiabatic invariant related to energy since it gives directly the most general form of this invariant.

Numerical tests have verified that the second and third adiabatic invariants related to momentum and energy, respectively, have indeed almost constant values. The small deviations from these almost constant values are largest during soliton collisions.

Because the KdV2 equation has non-integrable form, the energy is not an exact constant (see, e.g., Fig. 9).

There is, however, an intriguing kind of paradox with KdV2 invariants. On the one hand, exact invariants related to momentum and energy do **not** exist, only adiabatic ones are found. On the other hand, despite the non-integrability of KdV2, there exist exact analytic solutions of KdV2. The form of the single soliton solution of KdV2 was found in [23, Sect. IV]. Recently, in [21], we found analytic periodic solutions of KdV2 known as *cnoidal waves*. These KdV2 solutions have the same form as corresponding KdV solutions, but with different coefficients. Both of these solutions preserve their shapes during motion, so for such initial conditions the infinite number of invariants like those given by (46) exist. When initial conditions have the form different from analytic solutions only adiabatic invariants are left.

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