INTRODUCTION TO COMPLEX ANALYSIS — SECOND PART
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ALBERTO ABBONDANDOLO – alberto.abbondandolo@rub.de
GABRIELE BENEDETTI – gbenedetti@mathi.uni-heidelberg.de

Worksheet 2

In the following exercises we denote with $D'_r(z_0) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$ an open disc with radius r and center z_0 removed. Recall also the notation $A_{r,R}(z_0)$ for the open annulus with center z_0, inner radius r and outer radius R.

Exercise 1. *Find the singularities of the following functions and determine their type. If they are poles, find also their order.*

(a) $f(z) = \dfrac{(z-1)^3}{z^3-1}$, $\qquad$ (b) $f(z) = \dfrac{2(\cos z)^2 - 1}{(4z-\pi)^2}$,

(c) $f(z) = \sin\left(\dfrac{\pi}{(4z-i)^3}\right)$, $\qquad$ (d) $f(z) = \dfrac{\sin(\pi z) - 2z}{(1-2z)^2}$,

(e) $f(z) = \dfrac{\sin(z) - 1}{(z - \pi/2)^2}$, $\qquad$ (f) $f(z) = \cot(z^2)$,

Exercise 2. *Determine all the singularities of the function*

$$f(z) = \frac{1}{1 - e^{1/z}}.$$

Determine the type of the isolated singularities. Is 0 *an isolated singularity? determine the order of the poles.*

Exercise 3. *Let* $f : A_{r,\infty}(0) \to \mathbb{C}$ *be a holomorphic function which is not a polynomial. Show that* $z \mapsto f(1/z)$ *has an essential singularity in* 0.

Exercise 4. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an injective entire function. Prove that*

(1) f *is a polynomial;*
(2) f *has degree one.*

Exercise 5. *Let us consider the function* $f(z) = e^{1/z}$. *If* r *is an arbitrary positive number, describe the set* $f(D'_r(0))$.

Exercise 6. *Let* $f : D_r(z_0) \to \mathbb{C}$ *be a non-constant holomorphic function such that* $f(z_0) = 0$.

(1) *Determine the order of the pole of the function* $\frac{1}{f(z)}$ *in* z_0. *Hint: look at the smallest* k *such that* $f^{(k)}(z_0) \neq 0$.
(2) *Let* $g : D_r(z_0) \to \mathbb{C}$ *be a holomorphic function. When does* $\frac{g(z)}{f(z)}$ *have a removable singularity in* z_0*? If the singularity is not removable, determine the order of the pole. Hint: look at the smallest* h *such that* $g^{(h)}(z_0) \neq 0$.

Exercise 7. *Find the singularities of the function*

$$f(z) = \cos\left(\frac{2}{z-i}\right) \cdot \frac{z^3 + z^2 - z - 1}{(\cos z - 1)(z+1)(z^3-1)^2}.$$

Determine their type and compute the order of the poles.

Exercise 1 d-e

Freitag, 27. März 2020 11:40

(d) $f_1(z) = \dfrac{\sin(\pi z) - 2z}{(1-2z)^2}$, (e) $f_2(z) = \dfrac{\sin(z) - 1}{(z - \frac{\pi}{2})^2}$.

f_1 is holomorphic for all z such that $1-2z \neq 0 \iff z \neq 1/2$.

$z = 1/2$ is an isolated singularity. Let's if it is removable:

$$\lim_{z \to 1/2} (z - \tfrac{1}{2})^{\cancel{2}\,1} \cdot \frac{\sin(\pi z) - 2z}{2^2 (z - 1/2)^{\cancel{2}}} = \lim_{z \to 1/2} \frac{\sin(\pi z) - 2z}{4(z - 1/2)} \overset{\text{L'Hôpital}}{=} \lim_{z \to 1/2} \frac{\pi \cdot \cos(\pi z) - 2}{4 \cdot 1}$$

$$= \frac{\pi \cdot \cos(\frac{\pi}{2}) - 2}{4} = -\frac{1}{2}.$$

Since $-1/2 \neq 0$ the singularity is not removable and it will be a pole singularity with order 1.

$f(z) = \dfrac{\sin(z) - 1}{(z - \frac{\pi}{2})^2}$ f has an isolated singularity for $z = \frac{\pi}{2}$.

Let's check if the singularity is removable:

$$\lim_{z \to \frac{\pi}{2}} \cancel{(z - \tfrac{\pi}{2})} \cdot \frac{\sin(z) - 1}{(z - \frac{\pi}{2})^{\cancel{2}}} = \lim_{z \to \frac{\pi}{2}} \frac{\sin(z) - 1}{z - \frac{\pi}{2}} \overset{\text{L'Hop.}}{=} \lim_{z \to \frac{\pi}{2}} \frac{\cos(z)}{1}$$

$$= \cos(\tfrac{\pi}{2}) = 0.$$

Since the limit is zero, the singularity is removable.

$\Rightarrow f$ can be extended to a holomorphic function $\tilde{f}$ on $D_r(\frac{\pi}{2})$.

$$\tilde{f}(\tfrac{\pi}{2}) = \lim_{z \to \frac{\pi}{2}} f_2(z) = \lim_{z \to \frac{\pi}{2}} \frac{\sin(z) - 1}{(z - \frac{\pi}{2})^2} = \lim_{z \to \frac{\pi}{2}} \frac{\cos(z)}{2(z - \frac{\pi}{2})} = \lim_{z \to \frac{\pi}{2}} \frac{-\sin(z)}{2}$$

$$= -1/2.$$

Exercise 1

Donnerstag, 26. März 2020 09:25

Exercise 1. *Find the singularities of the following functions and determine their type. If they are poles, find also their order.*

(a) $f(z) = \frac{(z-1)^3}{z^3-1}$, (b) $f(z) = \frac{2(\cos z)^2 - 1}{(4z-\pi)^2}$,

(c) $f(z) = \sin\left(\frac{\pi}{(4z-i)^3}\right)$, (d) $f(z) = \frac{\sin(\pi z) - 2z}{(1-2z)^2}$,

(e) $f(z) = \frac{\sin(z) - 1}{(z-\pi/2)^2}$, (f) $f(z) = \cot(z^2)$.

Three types of singularities:

- removable $\lim_{z \to z_0} (z - z_0) f(z) = 0$
- pole $\exists K \geq 1$ $\lim_{z \to z_0} (z-z_0)^{K+1} f(z) = 0$ $\circledast$ Equivalent to: $(z-z_0)^K f(z)$ has removable singularity

  order of the pole: smallest K such that the limit $\circledast$ holds

  The order of the pole is also the only K such that $\lim_{z \to z_0} (z-z_0)^K \cdot f(z) \in \mathbb{C} \setminus \{0\}$

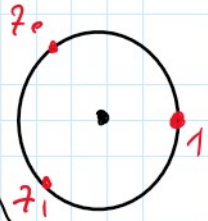
  $f(z) = \frac{a_{-K}}{(z-z_0)^K} + \frac{a_{-K+1}}{(z-z_0)^{K-1}} + \dots \quad a_{-K} \neq 0$

  $(z-z_0)^K f(z) = \underbrace{a_{-K}}_{\neq 0} + a_{-K+1} \cdot (z-z_0) + \dots$
- essential: principal part infinitely many terms

a) $f(z) = \frac{(z-1)^3}{z^3-1}$

f has singularities when $\boxed{z^3 - 1 = 0}$.

$z^3 - 1 = (z-1)(z^2+z+1)$ $z = 1$, $z = z_0$, $z = z_1$



$1, z_0, z_1$ are three distinct roots of $z^3 - 1 = 0$

$z_0 = \frac{-1 + i\sqrt{3}}{2}$

$z_1 = \frac{-1 - i\sqrt{3}}{2}$

$z^3 - 1 = (z-1)(z-z_0)(z-z_1)$

$\Rightarrow f(z) = \frac{(z-1)^{\cancel{3}\,2}}{\cancel{(z-1)}(z-z_0)(z-z_1)}$

$= \frac{(z-1)^2}{(z-z_0)(z-z_1)}$

1 is a removable singularity

$\longrightarrow \lim_{z \to 1} (z-1) \cdot \frac{(z-1)^2}{(z-z_0)(z-z_1)} = 0$ $f(1) = \frac{(1-1)^2}{(1-z_0)(1-z_1)} = \underline{0}$.

z_0 is a singularity.

$\lim_{z \to z_0} \cancel{(z-z_0)} \cdot \frac{(z-1)^2}{\cancel{(z-z_0)}(z-z_1)} = \frac{(z_0-1)^2}{(z_0-z_1)} \neq 0$.

z_0 is a pole of order $k=1$.

[Also possible: $\lim_{z \to z_0} (z-z_0)^{1+1} \frac{(z-1)^2}{\cancel{(z-z_0)}(z-z_1)} = 0$]

z_1 is a pole of order $k=1$ (Same argument)

$$f(z) = \frac{2\cdot(\cos z)^2 - 1}{(4z-\pi)^2}$$

$4z - \pi = 0$ $\boxed{z = \pi/4}$ Singularity

$$\boxed{\lim_{z \to \frac{\pi}{4}} \cancel{\left(z - \frac{\pi}{4}\right)} \cdot \frac{2\cdot(\cos z)^2 - 1}{4^2\left(z - \frac{\pi}{4}\right)^{\cancel{2}}}}$$

$$\lim_{z \to \frac{\pi}{4}} \frac{2\cdot(\cos z)^2 - 1}{4^2 \underbrace{\left(z - \frac{\pi}{4}\right)}_{\to 0}} \overset{\text{Hopital}}{=}$$

$$\lim_{z \to \frac{\pi}{4}} 2\cdot(\cos z)^2 - 1 = 2\cdot\left(\frac{\sqrt{2}}{2}\right)^2 - 1 = 0.$$

Problem: limit is undetermined because both the numerator and the denominator go to zero.

Solution: take Taylor expansion of the numerator at $z = \frac{\pi}{4}$:

Hopital

$$= \lim_{z \to \frac{\pi}{4}} \frac{2\cdot 2\cos z \cdot(-\sin z)}{4^2 \cdot 1} = \frac{4\cdot \cos\frac{\pi}{4}\cdot\left(-\sin\frac{\pi}{4}\right)}{4^2} = \frac{4\cdot\left(-\frac{1}{2}\right)}{4^2}$$

$$= -\frac{1}{8} \neq 0$$

$\Rightarrow \frac{\pi}{4}$ is a pole of order 1.

$$g(z) = 2\cdot(\cos z)^2 - 1 \qquad g(z) = \underbrace{g\left(\frac{\pi}{4}\right)}_{\overset{\shortparallel}{0}} + g'\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right) + O\left(z - \frac{\pi}{4}\right)^2$$

$$g(z) = -2\left(z - \frac{\pi}{4}\right) + O\left(\left(z - \frac{\pi}{4}\right)^2\right)$$

$$\lim_{z \to \frac{\pi}{4}} \cancel{\left(z - \frac{\pi}{4}\right)} \cdot \frac{-2\cdot\left(z - \frac{\pi}{4}\right) + O\left(z - \frac{\pi}{4}\right)^2}{4^2\left(z - \frac{\pi}{4}\right)^{\cancel{2}}}$$

$$= \lim_{z \to \frac{\pi}{4}} \left[-\frac{2}{4^2} + O\left(z - \frac{\pi}{4}\right)\right] = -\frac{2}{4^2} = -\frac{1}{8}.$$

$O\left(\left(z - \frac{\pi}{4}\right)^k\right)$ means that $\lim O\left(\left(z - \frac{\pi}{4}\right)^k\right) = 0$

$O\left(\left(z-\frac{\pi}{4}\right)^k\right)$ means that $\boxed{\lim_{z\to\frac{\pi}{4}} \frac{O\left(\left(z-\frac{\pi}{4}\right)^k\right)}{\left(z-\frac{\pi}{4}\right)^{k-1}} = 0}$

$k=0$: $O\left(z-\frac{\pi}{4}\right) \xrightarrow{z\to\frac{\pi}{4}} 0$

$$\lim_{z\to\frac{\pi}{4}} \left(z-\frac{\pi}{4}\right)\cdot f(z) = -\frac{1}{8}$$

$$\lim_{z\to\frac{\pi}{4}} \left(z-\frac{\pi}{4}\right)^2 f(z) = \underbrace{\lim_{z\to\frac{\pi}{4}}\left(z-\frac{\pi}{4}\right)}_{=0} \cdot \underbrace{\lim_{z\to\frac{\pi}{4}}\left(z-\frac{\pi}{4}\right)\cdot f(z)}_{-1/8} = 0$$

$$f(z) = \sin\left(\frac{\pi}{(4z-i)^3}\right)$$

$4z-i=0 \iff z=\frac{i}{4}$

Function f is holomorphic on $\mathbb{C}\setminus\left\{\frac{i}{4}\right\}$.
Let's find what type of singularity we have at $\frac{i}{4}$.

Idea: find Laurent expansion.

$$\sin w = \sum_{n=0}^{+\infty} (-1)^n \frac{w^{2n+1}}{(2n+1)!}$$

$$w = \frac{\pi}{(4z-i)^3} = \frac{\pi}{4^3\left(z-\frac{i}{4}\right)^3}$$

$$f(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4^3\left(z-\frac{i}{4}\right)^3}\right)^{2n+1}$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!\,4^{3(2n+1)}} \frac{1}{\left(z-\frac{i}{4}\right)^{3\cdot(2n+1)}}.$$

$\Rightarrow a_{-3(2n+1)} \neq 0$ for all $n \geq 0$.

$-3(2n+1)$ represents infinitely many negative integers

$\Longrightarrow \frac{i}{4}$ is an essential singularity.

$$f(z) = \cot(z^2) = \frac{\cos(z^2)}{\sin(z^2)}.$$

$\sin z^2 = 0$

$z^2 = n\pi \quad n\in\mathbb{Z}.$

$z = \pm\sqrt{n\pi} \quad n>0$

We check $z=0$:

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We check $z=0$:

$$\lim_{z\to 0} z \cdot \frac{\cos(z^2)}{\sin(z^2)} \overset{\text{L'Hopital}}{=} \dots$$

Let's use the identity

$$\boxed{\lim_{w\to 0} \frac{\sin w}{w} = 1}$$

$$\lim_{z\to 0} \frac{z^2 \cos(z^2)}{\sin(z^2)}$$

$$= \lim_{z\to 0} \frac{z^2}{\sin z^2} \cdot \lim_{z\to 0} \cos(z^2)$$

$$= 1 \cdot \cos(0) = \underline{\underline{1 \neq 0.}}$$

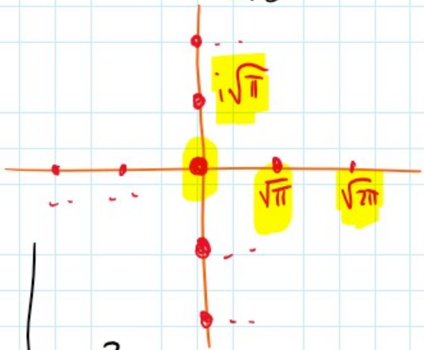
$z = n\pi \quad n \in \mathbb{Z}$.

$z = \pm\sqrt{n\pi} \quad n>0$

$z = 0$

$z = \pm i \underbrace{\sqrt{-n\pi}}_{>0}, \; n<0$

$$z = \pm\sqrt{n\pi}$$



$z^2 = n\pi$

$z = \pm\sqrt{n\pi}$

$= \pm\sqrt{(-1)\cdot(-n\pi)}$

$= \pm\sqrt{-1}\cdot\sqrt{-n\pi}$

$= \pm i\sqrt{-n\pi}$

$z^2 = \underline{-3\pi}$

$z = \pm\sqrt{-3\pi} = \pm\sqrt{-1}\cdot\sqrt{3\pi}$

$= \pm i\sqrt{3\pi}$

$$\lim_{z\to 0} \frac{z^3 \cdot \cos(z^2)}{\sin(z^2)} = \lim z \cdot \frac{z^2 \cos(z^2)}{\sin(z^2)}$$

$$= \lim z \cdot \lim \frac{z^2 \cos(z^2)}{\sin z^2}$$

$$= 0 \cdot 1 = 0$$

$3 = 2+1 \implies$ Pole of order 2

What kind of singularity is $\sqrt{\pi}$?

$$\lim_{z\to\sqrt{\pi}} (z-\sqrt{\pi})^1 \cdot \frac{\cos(z^2)}{\sin(z^2)} = \underbrace{\lim_{z\to\sqrt{\pi}} \cos(z^2)}_{\cos(\pi) = -1} \cdot \boxed{\lim_{z\to\sqrt{\pi}} \frac{z-\sqrt{\pi}}{\sin(z^2)}}$$

L'Hopital

$$= (-1)\cdot \lim_{z\to\sqrt{\pi}} \frac{1}{\cos(z^2)\cdot 2z} = (-1)\cdot \frac{1}{\cos(\pi)\cdot(2\sqrt{\pi})} = \frac{1}{2\sqrt{\pi}} \neq 0.$$

$\implies$ Pole of order 1.

$$\lim_{z\to\sqrt{\pi}} (z-\sqrt{\pi})^2 \frac{\cos(z^2)}{\sin(z^2)} = \left[\lim_{z\to\sqrt{\pi}} (z-\sqrt{\pi})\right] \cdot \lim_{z\to\sqrt{\pi}} (z-\sqrt{\pi}) \cdot \frac{\cos(z^2)}{\sin(z^2)}$$

$$= \underline{\underline{0}} \cdot \underline{\underline{\frac{1}{2\sqrt{\pi}}}} = 0.$$

Same argument shows that $z = \pm\sqrt{n\pi}$ for $n \neq 0$ is a pole of order 1.

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Exercise 2

Donnerstag, 26. März 2020 09:25

Exercise 2. *Determine all the singularities of the function*

$$f(z) = \frac{1}{1 - e^{1/z}}.$$

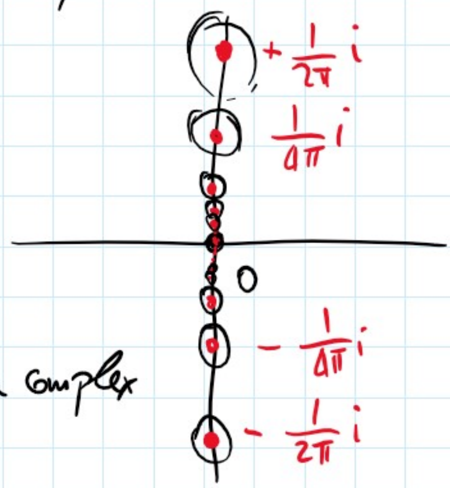
Determine the type of the isolated singularities. Is 0 an isolated singularity? determine the order of the poles.

$$f(z) = \frac{1}{1 - e^{1/z}}$$

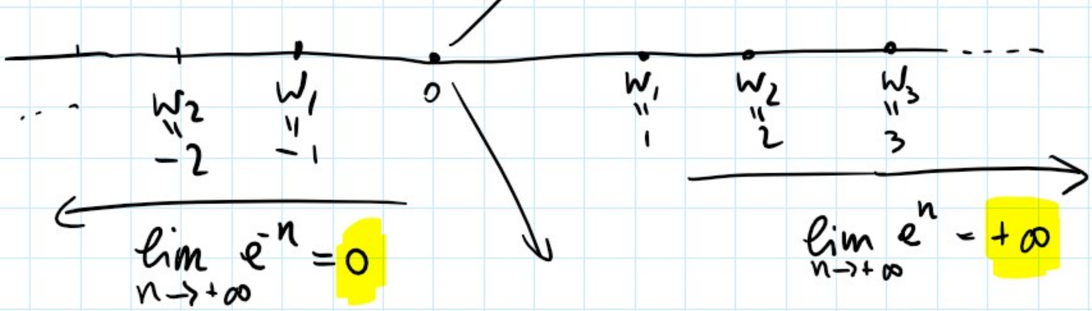
- f is not defined for $z = 0$ (because you have $1/z$ in the formula)
- f is not defined if $1 - e^{1/z} = 0$.

Solutions to: $e^{1/z} = 1$ $\quad \boxed{1/z = 2\pi \cdot k i}$, $k \in \mathbb{Z}$.

$1/z \neq 0 \Rightarrow \boxed{k \neq 0 \text{ and } z = -\frac{1}{2\pi k} i}$



$\lim_{z \to 0} e^{1/z} = \boxed{\lim_{w \to \infty} e^{w}}$: this limit does not exist because w is a complex number



0 is not an isolated singularity because $-\frac{1}{2\pi k} i \xrightarrow{k \to \infty} 0$.

The point $-\frac{1}{2\pi k} i$ is an isolated singularity for all $k \neq 0$.

Let's see what kind of singularity it is.

$$\lim_{z \to -\frac{1}{2\pi k} i} \left[z - \left(-\frac{1}{2\pi k} i \right) \right] \cdot \frac{1}{1 - e^{1/z}}$$

$$= \lim_{z \to -\frac{1}{2\pi k} i} \frac{\left(z + \frac{1}{2\pi k} i \right)}{1 - e^{1/z}}$$

L'Hôpital

$$= \lim_{z \to -\frac{1}{2\pi k} i} \frac{1}{- e^{1/z} \cdot \left(- \frac{1}{z^2} \right)}$$

L'Hôpital

$$= \lim_{z \to -\frac{1}{2\pi k} i} \frac{1}{-e^{1/z} \cdot \left(-\frac{1}{z^2}\right)}$$

$$= \lim_{z \to -\frac{1}{2\pi k} i} \frac{z^2}{e^{1/z}}$$

$$= \frac{\left(-\frac{1}{2\pi k} i\right)^2}{e^{+2\pi k i}} = -\frac{1}{(2\pi k)^2} \neq 0$$

$\Rightarrow$ Singularity is not removable

Pole of order 1: $\lim_{z \to -\frac{1}{2\pi k} i} \left[\left(z - \left(-\frac{1}{2\pi k} i\right)\right)^2 \cdot \frac{1}{1 - e^{1/z}} \right]$

$$= \lim_{z \to -\frac{1}{2\pi k} i} \left(z + \frac{1}{2\pi k} i\right) \cdot \lim_{z \to -\frac{1}{2\pi k} i} \left(z + \frac{1}{2\pi k} i\right) \frac{1}{1 - e^{1/z}}$$

$$= 0 \cdot \left(-\frac{1}{(2\pi k)^2}\right) = 0.$$

Exercise 3

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Exercise 3. *Let f be an entire function which is not a polynomial. Show that $z \mapsto f(1/z)$ has an essential singularity in 0.*

$f: \mathbb{C} \to \mathbb{C}$ holomorphic, f is not a polynomial.

The power series of f centered at $z_0 = 0$ has infinite radius of convergence

$f(z) = \sum_{n=0}^{+\infty} a_n \cdot z^n$. $a_n \neq 0$ for infinitely many n.

$f(1/z)$ is holomorphic in $A_{0,\infty}(0)$ (since f is entire)

$f\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} a_n \left(\frac{1}{z}\right)^n \underset{m=-n}{=} \sum_{m=-\infty}^{0} a_{-m} z^m$ and we know that a_{-m} is different from zero for infinitely many $m < 0$.

$\Rightarrow f\left(\frac{1}{z}\right)$ has an essential singularity at zero.

Exercise 4

Donnerstag, 26. März 2020 09:25

Exercise 4. *Let $f : \mathbb{C} \to \mathbb{C}$ be an injective entire function. Prove that*

(1) *f is a polynomial;*
(2) *f has degree one.*

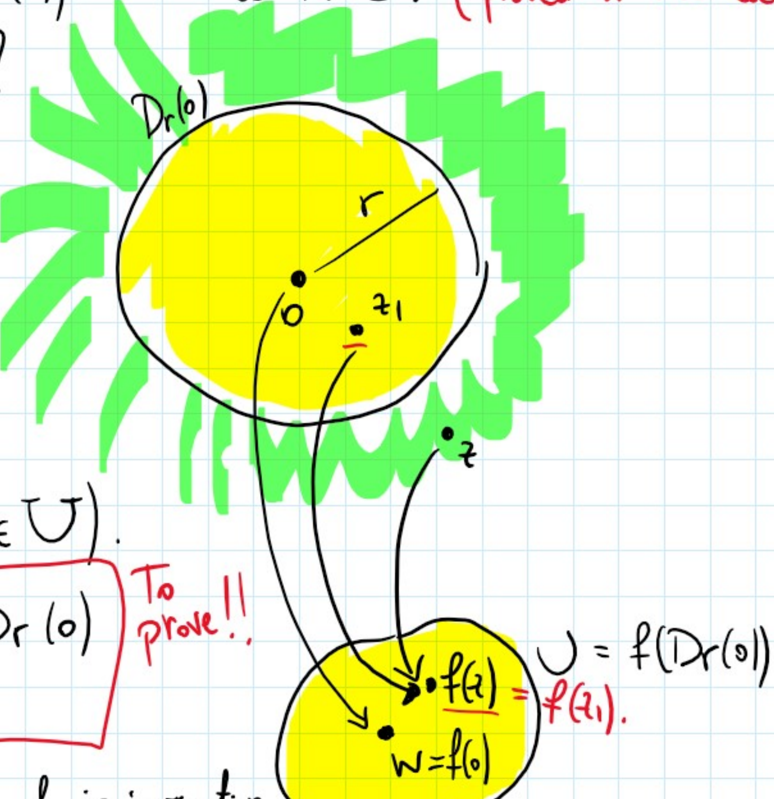
Prove (1) by contradiction. Assume that f is not a polynomial.

For every $r>0$ $f(\mathbb{C} \setminus D_r(0))$ is dense in $\mathbb{C}$. (proved in the lecture).

What does it mean to be dense?

$\forall w \in \mathbb{C}$ and for every neighborhood U of w, there exists an element of $f(\mathbb{C} \setminus D_r(0))$ in U.

$(\exists z \in \mathbb{C}, |z|>r, f(z) \in U)$.



$\boxed{\text{We now find a point } z_1 \in D_r(0) \text{ such that } f(z_1) = f(z).}$ To prove!!

This is a contradiction because f is injective.

Let's use the open mapping theorem: if $g : \underline{\underline{V}} \to \mathbb{C}$, V open, g holomorphic and not constant, then $g(V)$ open.

We apply this theorem with $g = f$ (f is not constant because is injective) and $V = D_r(0)$. Then $f(D_r(0))$ is open in $\mathbb{C}$, therefore is a neighborhood of $f(0) =: \underline{\underline{w}}$. Taking $U = f(D_r(0))$ we see that

$f(z) \ni U = f(D_r(0)) \Rightarrow \underline{f(z) = f(z_1) \text{ for some } z_1 \in D_r(0)}$.

We have shown that f is a polynomial.

Let's show that f has degree 1.

By the fundamental theorem of algebra: $f(z) = c \cdot (z-z_0) \cdot (z-z_1) \cdot \ldots \cdot (z-z_n)$

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If f has two distinct roots z_n, z_k, then $f(z_n)=0$ and $f(z_k)=0$

$\Longrightarrow$ f is not injective: a contradiction.

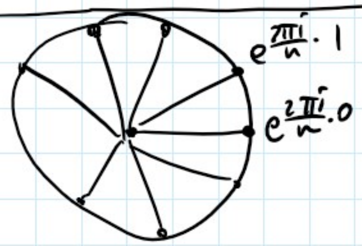
Then all the roots must be the same: $z_0 = z_1 = \ldots = z_n$

$f(z) = c\cdot(z-z_0)^n$. Let's consider the equation $f(z)=c$.

This equation has n solutions: $\cancel{c}\cdot(z-z_0)^n = \cancel{c}$, $c\neq 0$

$(z-z_0)^n = 1 \iff \boxed{z = z_0 + e^{\frac{2\pi i}{n}\cdot k} \quad \text{for } k=0,1,\ldots,n-1.}$

Since f is injective $f(z)=c$ can have only one solution. Hence $n=1$.



Exercise 5

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Exercise 5. *Let us consider the function* $f(z) = e^{1/z}$. *If* r *is an arbitrary positive number, describe the set* $f(D'_r(0))$.

During the lecture you saw that f has essential singularity at $z=0$.

Thm (Casorati - Weierstrass)

$\forall r>0$, $f(D'_r(0))$ is dense in $\mathbb{C}$, if f has essential singularity at $z=0$.

Find the set $f(D'_r(0))$ for $f(z)=e^{1/z}$.

Question: if $w \in \mathbb{C}$, does there exist $z \in D'_r(0)$ such that $e^{1/z} = w$?

We have to understand first what is the image of the exponential function $\exp: \mathbb{C} \to \mathbb{C}$, $u \mapsto e^u$

$e^u = e^{\operatorname{Re} u} \cdot e^{i \operatorname{Im} u}$. If $w \in \mathbb{C}$, $w = r \cdot e^{i\theta}$

Let's find u such that $e^u = w$. | $\operatorname{Log} w = \log|w| + i \operatorname{Arg} w$

$u := \operatorname{Log} w + 2\pi k i \quad \forall k \in \mathbb{Z}$.

$\operatorname{Log}: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ principal branch of Logarithm.

We find that $e^u = w$ has the solution $u = \operatorname{Log} w + 2\pi k i$ for all $w \neq 0$.

Therefore, $e^{1/z} = w$ has the solution $1/z = \operatorname{Log} w + 2\pi k i$

Therefore, $e^{1/z} = w$ has the solution $\boxed{1/z = \operatorname{Log} w + \pi k i}$ if $w \neq 0$.

$$\Updownarrow$$

$$z = \frac{1}{\operatorname{Log} w + 2\pi k i}$$

Question: is $\dfrac{1}{\operatorname{Log} w + 2\pi k i}$ an element of $D_r'(0)$?

This is equivalent to say that $\left|\overbrace{\dfrac{1}{\operatorname{Log} w + 2\pi k i}}^{z}\right| < r$.

Idea: we can take any $k \in \mathbb{Z}$:

"If k is big, then $\left|\dfrac{1}{\operatorname{Log} w + 2\pi k i}\right|$ is small"

$$\lim_{k \to +\infty} \frac{1}{|\operatorname{Log} w + 2\pi k i|} = 0$$ because

$$\lim_{k \to +\infty} |\operatorname{Log} w + 2\pi k i| \geq \lim_{k \to +\infty} |2\pi k i| - |\operatorname{Log} w|$$

$$= \lim_{k \to +\infty} 2\pi k - |\operatorname{Log} w|$$

$$= +\infty.$$

This means that for k sufficiently large

$$\left|\frac{1}{\operatorname{Log} w + 2\pi k i}\right| < r.$$

Therefore $z = \dfrac{1}{\operatorname{Log} w + 2\pi k i} \in D_r'(0)$

$\Rightarrow w \in f(D_r'(0))$

Therefore $\quad \log w + 2\pi k i \quad \Rightarrow w \in f(D_r'(0))$

$\bullet \; e^{1/z} = w. \qquad$ if $w \neq 0$.

Summing up: $\boxed{f(D_r'(0)) = \mathbb{C} \setminus \{0\}.}$

Exercise 6

Donnerstag, 26. März 2020 09:25

Exercise 6. *Let $f : D_r(z_0) \to \mathbb{C}$ be a non-constant holomorphic function such that $f(z_0) = 0$.*

(1) *Determine the order of the pole of the function $\frac{1}{f(z)}$ in z_0. Hint: look at the smallest k such that $f^{(k)}(z_0) \neq 0$.*

(2) *Let $g : D_r(z_0) \to \mathbb{C}$ be a holomorphic function. When does $\frac{g(z)}{f(z)}$ have a removable singularity in z_0? If the singularity is not removable, determine the order of the pole. Hint: look at the smallest h such that $g^{(h)}(z_0) \neq 0$.*

$h(z) = \frac{1}{f(z)}$, this function has a singularity at z_0 since $f(z_0)=0$.
The singularity is isolated because f has isolated zeros since it is not constant.

The power series expansion of f at z_0 is given by $f(z) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$.
Let k_0 be the smallest k such that $f^{(k)}(z_0) \neq 0$ ($k_0 \geq 1$).
It exists because not all coefficients in the power series expansion are zero (otherwise f constant)

$$f(z) = \frac{f^{(k_0)}(z_0)}{k_0!}(z-z_0)^{k_0} + o\left((z-z_0)^{k_0}\right) = (z-z_0)^{k_0} \cdot \left(\frac{f^{(k_0)}(z_0)}{k_0!} + o\left((z-z_0)^0\right)\right)$$

$$h(z) = \frac{1}{f(z)} = \frac{1}{(z-z_0)^{k_0} \cdot \left(\frac{f^{(k_0)}(z_0)}{k_0!} + o\left((z-z_0)^0\right)\right)}$$

$$\lim_{z \to z_0} \cancel{(z-z_0)}^{k_0} \cdot \frac{1}{\cancel{(z-z_0)}^{k_0} \cdot \left(\frac{f^{(k_0)}(z_0)}{k_0!} + o\left((z-z_0)^0\right)\right)} = \frac{1}{\frac{f^{(k_0)}(z_0)}{k_0!} + 0} = \frac{k_0!}{f^{(k_0)}(z_0)} \neq 0$$

since $\lim_{z \to z_0} o\left((z-z_0)^0\right) = 0$.

Since $\frac{k_0!}{f^{(k_0)}(z_0)} \neq 0$ the function $1/f(z)$ has a pole of order k_0.

Example $f(z) = \sin(z^2)$ on $D_r(0)$. $f(0) = 0$.

$f'(z) = 2z \cdot \cos(z^2)$, $f'(0) = 0$

$f''(z) = 2\cos(z^2) + 2z \cdot (\cos(z^2))'$

$f''(0) = 2 \cdot \cos(0) + \underbrace{2 \cdot 0 \cdot (\ldots)'}_{=0} = 2 \neq 0 \Rightarrow k_0 = 2$.

$\Rightarrow \frac{1}{\sin(z^2)}$ has a pole of order 2 at $z_0 = 0$.

$\Rightarrow \frac{1}{\sin(z^2)}$ has a pole of order 2 at $z_0 = 0$.

Alternative way to compute order of the pole (not using the criterion above)

$$h(z) = \frac{1}{\sin(z^2)} = \frac{1}{z^2 - \frac{(z^2)^3}{3!} + \frac{(z^2)^5}{5!} + \dots} = \frac{1}{z^2(1 + o(z^0))}$$

$(z^2)^3 = z^6$, $(z^2)^5 = z^{10}$

$$\Rightarrow \lim_{z\to 0} z^2 \cdot \frac{1}{z^2(1+o(z^0))} = 1 \neq 0 \Rightarrow \text{Pole of order two.}$$

Let $h(z) = \frac{g(z)}{f(z)}$ $\quad f(z_0) = 0$ f not constant.

If $g \equiv 0$, then $h(z) \equiv 0 \Rightarrow z_0$ is a removable singularity.

If $g \not\equiv 0$, $g(z) = \sum_{h=0}^{+\infty} \frac{g^{(h)}(z_0)}{h!}(z-z_0)^h$. Let h_0 be the smallest integer h such that $g^{(h)}(z_0) \neq 0$. (h_0 exists since $g \not\equiv 0$).

$$g(z) = (z - z_0)^{h_0} \cdot \left(\frac{g^{(h_0)}(z_0)}{h_0!} + o((z-z_0)^0) \right).$$

$$h(z) = \frac{g(z)}{f(z)} = \frac{(z-z_0)^{h_0}\left(\frac{g^{(h_0)}(z_0)}{h_0!} + o((z-z_0)^0)\right)}{(z-z_0)^{k_0}\left(\frac{f^{(k_0)}(z_0)}{k_0!} + o((z-z_0)^0)\right)}$$

$$= \frac{1}{(z-z_0)^{k_0 - h_0}} \cdot \frac{g^{(h_0)}(z_0)/h_0! + o((z-z_0)^0)}{\frac{f^{(k_0)}(z_0)}{k_0!} + o((z-z_0)^0)}$$

If $k_0 \leq h_0$, then the singularity is removable:

$$\lim_{z\to z_0} (z-z_0)h(z) = \lim_{z\to z_0} (z-z_0)^{\overset{h_0 - k_0 + 1}{>0}} \cdot \frac{g^{(h_0)}(z_0)/h_0! + \dots}{\frac{f^{(k_0)}(z_0)}{k_0!} + \dots}$$

$$= 0 \cdot \frac{g^{(h_0)}(z_0)/h_0!}{\frac{f^{(k_0)}(z_0)}{k_0!}} = 0. \Rightarrow \text{singularity is removable.}$$

If $k_0 > h_0$ then z_0 is a pole of order $k_0 - h_0$.

$k_0!$

If $k_0 > h_0$, then z_0 is a pole of order $k_0 - h_0$:

$$\lim_{z \to z_0} \cancel{(z-z_0)^{k_0-h_0}} \cdot \frac{1}{\cancel{(z-z_0)^{k_0-h_0}}} \cdot \frac{g^{(h_0)}(z_0)/h_0! + \dots}{f^{(k_0)}(z_0)/k_0!}$$

$$= \frac{g^{(h_0)}(z_0) \cdot k_0!}{f^{(k_0)}(z_0)\, h_0!} \neq 0 \Rightarrow z_0 \text{ pole of order } k_0 - h_0.$$

Example of application:

$$h(z) = \frac{e^z - 1}{\sin(z^3)}. \quad g(z) = e^z - 1 \quad f(z) = \sin(z^3).$$

$h_0 = ?$ $\quad k_0 = ?$

For h_0:

$g(0) = e^0 - 1 = 0 \Rightarrow h_0 > 0.$

$g'(z) = e^z \Rightarrow g'(0) = 1 \Rightarrow h_0 = 1.$

$\Rightarrow h(z)$ has a pole of order $3 - 1 = 2$ at $z = 0$.

For k_0:

$f(z) = \sin(z^3) = z^3 - \frac{(z^3)^3}{3!} + \dots$

$\Rightarrow f(0) = f'(0) = f''(0) = 0$

$f'''(0) \neq 0$

$\overset{\shortparallel}{3!} = 6 \Rightarrow k_0 = 3$

Exercise 7. *Find the singularities of the function*

$$f(z)=\cos\left(\frac{2}{z-i}\right)\left[\frac{z^3+z^2-z-1}{(\cos z-1)(z+1)(z^3-1)^2}\right]$$

Determine their type and compute the order of the poles.

f is holomorphic for all z such that:

- $z-i\neq 0 \iff z\neq i$
- $\cos z-1\neq 0 \iff z\neq 2\pi\cdot k\,,\ k\in\mathbb{Z}$.
- $z+1\neq 0 \iff z\neq -1$
- $z^3-1\neq 0 \iff (z-1)(z-z_0)(z-z_1)\neq 0\,,\quad z_0=\frac{-1+i\sqrt{3}}{2}\,,\ z_1=\frac{-1-i\sqrt{3}}{2}$.
  $z\neq 1,\ z\neq z_0,\ z\neq z_1$

f has the singularities $\boxed{i,\ 2\pi k \text{ for } k\in\mathbb{Z},\ -1,\ 1,\ z_0,\ z_1}$.

- The singularity i: we know that $\cos\left(\frac{2}{z-i}\right)$ has an essential singularity at $z=i$:

$$\cos\left(\frac{2}{z-i}\right)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\cdot\left(\frac{2}{z-i}\right)^{2n} \quad (\text{Laurent expansion})$$

This expansion has infinitely many terms with $(z-i)^m$ where m is a negative integer.

$\Rightarrow \cos\left(\frac{2}{z-i}\right)$ has an essential singularity at $z=i$.

$$\cos\left(\frac{2}{z-i}\right)=f(z)\cdot g(z)\,,\quad \text{where}\quad g(z)=\frac{(\cos z-1)(z+1)(z^3-1)^2}{z^3+z^2-z-1}.$$

Suppose that f does not have an essential singularity at $z=i$.

$\Rightarrow f(z)=a_m(z-i)^m+o\left((z-i)^m\right)$ $\quad m\in\mathbb{Z}$. (Laurent expansion of f)

We know that g is holomorphic in a neighborhood of $z=i$,

because $i^3+i^2-i-1=-i-1-i-1=-2(i+1)\neq 0$.

$g(z)=b_n(z-i)^n+o\left((z-i)^n\right)$ for some $n\in\mathbb{N}$.

$$\Rightarrow \underbrace{f\cdot g\,(z)}_{=\cos\left(\frac{2}{z-i}\right)}=a_m\cdot b_n(z-i)^{m+n}+o\left((z-i)^{m+n}\right)$$

This is a contradiction because $\cos\left(\frac{2}{z-i}\right)$ has infinitely many terms in the Laurent expansion with negative exponent.

$\Rightarrow f$ has essential singularity at $z=i$

in the Laurent expansion with negative exponent.

$\Rightarrow f$ has essential singularity at $z=i$.

The singularities $\underline{\underline{2\pi k}}$, $k \in \mathbb{Z}$.

$$\cos z - 1 = \underbrace{\cos(2\pi k) - 1}_{=0} + (-\underbrace{\sin(2\pi k)}_{=0}) \cdot (z - 2\pi k) + \frac{1}{2}\left[-\underbrace{\cos(2\pi k)}_{=1}\right] \cdot (z-2\pi k)^2 + o((z-2\pi k)^2)$$

$$\Rightarrow \cos z - 1 = -\frac{1}{2}(z-2\pi k)^2 + o((z-2\pi k)^2)$$

$$= (z-2\pi k)^2 \left(-\frac{1}{2} + o((z-2\pi k)^0)\right).$$

$$f(z) = \cos\left(\frac{2}{z-i}\right) \frac{z^3+z^2-z-1}{(z-2\pi k)^2 \left(-\frac{1}{2} + o((z-2\pi k)^0)\right) \cdot (z+1)(z^3-1)^2}$$

$$= \frac{1}{(z-2\pi k)^2} \cdot g(z) \qquad g(z) = \cos\left(\frac{2}{z-i}\right) \cdot \frac{z^3+z^2-z-1}{\left(-\frac{1}{2} + o(\ldots)\right) \cdot (z+1)(z^3-1)^2}$$

g is holomorphic around $2\pi k$ since $g(2\pi k) = \cos\left(\frac{2}{2\pi k - i}\right) \cdot \frac{(2\pi k)^3 + (2\pi k)^2 - (2\pi k) - 1}{-\frac{1}{2}(2\pi k+1)((2\pi k)^3-1)^2} \in \mathbb{C}$

$\neq 0$

Want to show $g(2\pi k) \neq 0$:
- $\cos\left(\frac{2}{2\pi k - i}\right) \neq 0$ since $\frac{2}{2\pi k - i} \neq \frac{\pi}{2} + h \cdot \pi$, $h \in \mathbb{Z}$

  (it is true because $\frac{2}{2\pi k - i} \notin \mathbb{R}$)

- $(2\pi k)^3 + (2\pi k)^2 - (2\pi k) - 1 \neq 0$

  Let's factorize $z^3+z^2-z-1 = z^2(z+1) - 1(z+1)$

  $= (z+1)(z^2-1)$

  $= (z+1)^2(z-1)$.

  Roots are $z=-1$, $z=+1$.

$2\pi k$ is a pole of order 2 for f: $\lim_{z \to 2\pi k} (z-2\pi k)^2 f(z) = \lim_{z \to 2\pi k} g(z) = g(2\pi k) \neq 0$.

The singularities $-1, 1, z_0, z_1$.

$$f(z) = \cos\left(\frac{2}{z-i}\right) \cdot \frac{(z+1)^{\cancel{2}} \cancel{(z-1)}}{(\cos(z)-1)\cancel{(z+1)}(z-1)^{\cancel{2}}(z-z_0)^2(z-z_1)^2}$$

$$= \frac{\cos\left(\frac{2}{z-i}\right)}{\cos(z)-1} \cdot \frac{z+1}{(z-1)(z-z_0)^2(z-z_1)^2}.$$

$z=-1$ is a removable singularity: $\lim_{z \to -1} \frac{\cos\left(\frac{2}{z-i}\right)}{\cos(z)-1} \frac{z+1}{(z-1)(z-z_0)^2(z-z_1)^2} = 0$.

$z=1$ is a pole of order 1:

$$\cdots \quad z \to -1 \quad \cos(z)-1 \quad (z-1)(z-z_0)^2(z-z_1)^2$$

$z=1$ is a pole of order 1:

$$\lim_{z\to 1} \frac{\cos\left(\frac{2}{z-i}\right)}{\cos(z)-1} \frac{z+1}{\cancel{(z-1)}(z-z_0)^2(z-z_1)^2} \cdot \cancel{(z-1)}$$

$$= \frac{\cos\left(\frac{2}{1-i}\right)}{\cos(1)-1} \frac{2}{(1-z_0)^2(1-z_1)^2} \neq 0$$

$z=z_0$ is a pole of order 2:

$$f(z) = \boxed{\frac{1}{(z-z_0)^2}} \cdot \underbrace{\boxed{\frac{\cos\left(\frac{2}{z-i}\right)}{\cos(z)-1} \cdot \frac{z+1}{(z-1)(z-z_1)^2}}}_{=:g(z)}$$

$g(z_0) \neq 0$ (easy to check)

$\lim_{z\to z_0} (z-z_0)^2 \cdot f(z) = g(z_0) \neq 0 \Rightarrow z=z_0$ pole of order 2.

$z=z_1$ is a pole of order 2 (same argument as with $z=z_0$).