Examples 1

1. see Prop (2.2). If $f, g : X \to Y$ are continuous and Y Hausdorff then $\{x \in X | f(x) = g(x)\}$ is closed in X.

2. Let G be a group and $\mathcal{L}$ a non-empty family of normal subgroups s.t. if $K_1, K_2 \in \mathcal{L}$ and K_3 is a normal subgroup containing $K_1 \cap K_2$ then $K_3 \in \mathcal{L}$. Let $\mathcal{T}$ be the family of all unions of sets of cosets Kg with $K \in \mathcal{L}, g \in G$. Show that $\mathcal{T}$ is a topology on G and that G is a topological group with respect to this topology. Show also that $\mathcal{L}$ is the set of open normal subgroups of G with respect to this topology.

3. Lemma (1.1)(b). G a topological group. To prove:
Every open subgroup of G is closed.
Every closed subgroup of finite index is open.
If G compact, every open subgroup of G has finite index.

4. Recall if $S \subseteq G$ then the centralizer of S is

$$C_G(S) = \{g \in G | gs = sg \ \ \forall s \in S\}$$

and the normalizer of a subgroup H is

$$N_G(H) = \{g \in G | g^{-1}hg \in H \ \& \ ghg^{-1} \in H \ \ \forall h \in H\}.$$

Suppose G is a Hausdorff topological group.
(a) Prove that centralizers of subsets and normalizers of subgroups are closed.
(b) Prove that each closed abelian subgroup A is contained in a maximal closed abelian subgroup (i.e. the family of closed abelian subgroups containing A, partially ordered with respect to inclusion, has a maximal element).
(c) Prove that if there is a series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that G_i/G_{i+1} is abelian for each i, then there is such a series consisting of closed subgroups. (Consider closures).

5. G a profinite group and $X \leq G$, then

$$\overline{X} = \cap\{K | X \leq K \leq_o G\}.$$

6. Let X be a dense subgroup of a topological group G (i.e. $\overline{X} = G$). Prove that $N = \overline{N \cap X}$ for each open subgroup N of G.

7. If G is finite what is the profinite completion of G?

8. Give an example of a group which has trivial pro-p completions for each p but a non-trivial profinite completion (or describe the properties such a group should have).

Examples 2

1. Let G be a finite group and $x, y \in G$.
(i) Prove

$$(yx)^n \equiv y^n x^n [x,y]^{\binom{n}{2}} [x,y,y]^{\binom{n}{3}} \dots [x,_{n-1} y] \bmod M$$

where M is the normal closure in G of the group generated by the set of all commutators in $\{x, y\}$ of weight at least 2 in x (i.e. contain at least two xs).

(ii) Prove

$$[x, y^n] \equiv [x,y]^n [x,y,y]^{\binom{n}{2}} [x,y,y,y]^{\binom{n}{3}} \dots [x,_n y] \bmod N$$

where N is the normal closure in G of the group generated by the set of all commutators in $\{x, [x,y]\}$ of weight at least 2 in $[x,y]$.

2. Let G be a profinite group. A sequence (g_i) converges in G iff it is Cauchy. (See Lemma 6.1).

3. Let G be a profinite group, and let $w(X_1, \dots, X_n)$ be a group word.
(i) Show that the set $w(G) = \{w(x_1, \dots, x_n) | x_1, \dots, x_n \in G\}$ is closed in G.
(ii) Deduce that if $g \in G$ and if for every $N \lhd_o G$ there exist $x_1(N), \dots, x_n(N) \in G$ with $g \equiv w(x_1(N), \dots, x_n(N)) \bmod N$, then there exist $x_1, \dots, x_n \in G$ such that $g = w(x_1, \dots, x_n)$.
(We use this result in Lemma 8.3).

4. Show that the natural map $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$ is surjective, for all m and n. Denoting its kernel by $K_n(m)$, show that

$$\varprojlim (\mathrm{SL}_n(\mathbb{Z})/K_n(p^i))_{i \in \mathbb{N}} \cong \mathrm{SL}_n(\mathbb{Z}_p)$$

$$\varprojlim (\mathrm{SL}_n(\mathbb{Z})/K_n(m))_{m \in \mathbb{N}} \cong \mathrm{SL}_n(\hat{\mathbb{Z}})$$

where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.
(Hint: for the first part find a simple generating set for $\mathrm{SL}_n(\mathbb{Z}/m\mathbb{Z})$.)

5. Fix a prime p and a positive integer n. For each j put

$$\Gamma_j = \{g \in \mathrm{SL}_n(\mathbb{Z}_p) | g \equiv 1_n \bmod p^j\}.$$

(i) Show that Γ_1 is a pro-p group (with the subspace topology induced by the p-adic topology on $\mathrm{M}_n(\mathbb{Z}_p)$.
(ii) Show that Γ_1 is topologically finitely generated. Deduce that every subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z}_p)$ contains Γ_j for some j.

6. Let G be a pro-p group. Show that G satisfies the ascending chain condition for closed subgroups (i.e. every such chain becomes stationary after finitely many steps) iff every closed subgroup of G is topologically finitely generated.

7. Write -1 as an element of $\mathbb{Z}_p$ i.e. as an infinite sum of the form $\sum_{i\geq 0} a_i p^i$ with $0 \leq a_i \leq p-1$. Also is $1/2 \in \mathbb{Z}_3$ and is $1/3 \in \mathbb{Z}_3$?

8. Calculate the lower central series of the Nottingham group.

Examples 3

1. Prove that in a pro-p group an element of finite order has p-power order. (We use this in (9.4)).

2. Prove that a powerful 2-generator finite p-group is metacyclic (i.e. has a cyclic normal subgroup with cyclic quotient). (Hint: show that the derived group is cyclic).

3. Prove that if p is odd then every metacyclic p-group is powerful.

4. A pro-p group is meta-procyclic if it has a procyclic normal subgroup with procyclic quotient. Show that a pro-p group is meta-procyclic iff it is an inverse limit of metacyclic p-groups.

5. Show that a powerful pro-p group that can be topologically generated by 2 elements is meta-procyclic. Show also that such a group either has an open normal procyclic subgroup or else is torsion-free.

6. Let G be a profinite group and let $N \lhd G$ be closed. Show that

$$\max\{\mathrm{rk}(N), \mathrm{rk}(G/N)\} \leq \mathrm{rk}(G) \leq \mathrm{rk}(N) + \mathrm{rk}(G/N).$$

Deduce that if $H \leq G$ is open and $\mathrm{rk}(H)$ is finite then $\mathrm{rk}(G)$ is finite. (We use this in (8.11)).

7. Prove that for a non-Archimedean norm

$$\|a + b\| = \max\{\|a\|, \|b\|\} \text{ unless } \|a\| = \|b\|.$$

8.(i) Show that every linear map $\mathbb{Q}_p^n \to \mathbb{Q}_p^m$ is continuous (both spaces have the product topology). Now let V be a finite-dimensional vector space over $\mathbb{Q}_p$; deduce that V has a unique topology - the p-adic topology - given by identifying V with $\mathbb{Q}_p^n$ by choosing an arbitrary basis and taking the product topology on $\mathbb{Q}_p^n$. Show that every vector subspace of V is closed in the p-adic topology.
(ii) Let A be a normed $\mathbb{Q}_p$-algebra and V a finite-dimensional vector subspace over A. Show that the inclusion map $V \to A$ is continuous with respect to the p-adic topology on V and the norm topology on A. Does the latter topology necessarily induce the former topology on V?
(We use this in (12.2)).