THE REPRESENTATION OF BAIRE FUNCTIONS

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SUMMARY

Gelfand [1][1] has shown that a real Banach algebra in which for every element we have $||x^2|| = ||x||^2$, is isomorphic and isometric to the ring of continuous functions on some compact Hausdorff space. Since he was concerned with an abstract Banach algebra, his representation for this space is necessarily quite complicated; indeed, it is in terms of a space of maximal ideals of the Banach algebra. One would expect, then, that for a particular Banach algebra a simpler characterization of this space would be obtained. It is the purpose of this paper to find such a simpler representation for the collection of Baire functions of class α, for each $\alpha \geq 1$, over a topological space S. These collections satisfy the conditions of Gelfand's theorem. Our representation, which is done in terms of lattice, instead of ring, operations, will give the space as a Boolean space associated with a Boolean algebra of subsets of the original space S.

The paper is divided into two parts. In part I, we define the Baire functions of class α and obtain some results connecting them and the Boolean algebra. Part II is concerned with the representation theorem, some of its consequences, and examples to show that the theory is non-vacuous.

1. References to the literature are indicated by numbers in square brackets.

PART I

BAIRE FUNCTIONS

1.1. *Baire functions.* Let s be a collection of bounded real-valued[2] functions on a topological space S. Such a collection is called a *complete system* [2] if it satisfies the following conditions:

(1.1.1 a) Every constant function is in s.

(1.1.1 b) Pointwise sup and inf of two functions in s is also in s.

(1.1.1 c) Sums, differences, products, and (bounded) quotients (with non-vanishing denominators) of two functions in s are again in s.

(1.1.1 d) The limit of a uniformly convergent sequence of functions in s is in s.

Thus, under pointwise operations, every complete system is both a lattice and a ring. Further, under the usual norm definition $||f|| = \sup_{x \in S} |f(x)|$, a complete system forms a real Banach algebra in which $||f^2|| = ||f||^2$. Gelfand's theorem then says that a complete system is isomorphic and isometric to the continuous functions on a compact Hausdorff space; and a fortiori, is lattice and ring isomorphic to the set of continuous functions on this space.

Now, for any collection of functions s, the *Baire functions over* s are defined [3] as the smallest family of functions, $B(s)$, such that:

(i), $s \subseteq B(s)$; (ii), a limit of functions belonging to $B(s)$ again belongs to $B(s)$. We shall always suppose that s is a complete system. Then $B(s)$ is arranged

2. Unless otherwise stated, all functions shall henceforth be assumed bounded and real-valued.

into classes of functions in the following manner [2].

(1.1.2 a) The functions belonging to s are of class zero.

(1.1.2 b) Functions of class $\alpha + 1$ are limits of convergent sequences of functions of class α.

(1.1.2 c) If α is a limit ordinal, the functions of class α are those of the smallest complete system containing all of the functions of class less than α[3].

Denoting the functions of class α by $B_\alpha(s)$, it is clear from (1.1.2 a) - (1.1.2 c) that $B_\alpha(s) \subseteq B_\beta(s)$ for all α, β such that $\alpha \leqslant \beta$; and that for each α, $B_\alpha(s)$ is a complete system. Hence $B_\alpha(s)$ is lattice (ring) isomorphic to the lattice (ring) of continuous functions on some compact Hausdorff space. In order to prepare the way for characterizing this space by means of the representation theorem, the remainder of part I is devoted to expressing some properities of functions of class α in terms of subsets S, for suitable collections s and spaces S.

1.2. *Borel Classes.* Let $\mathfrak{S}$ be any collection of subsets of a topological space S. *The Borel ring over* $\mathfrak{S}$ is the smallest family of sets, $\Phi(\mathfrak{S})$, such that

(1.2.1 a) $\mathfrak{S} \subseteq \Phi(\mathfrak{S})$

(1.2.1 b) $A_n \epsilon \Phi(\mathfrak{S}) \longrightarrow \bigcup_n A_n \epsilon \Phi(\mathfrak{S})$ $\qquad (n = 1, 2, \ldots)$

(1.2.1 c) $A_n \epsilon \Phi(\mathfrak{S}) \longrightarrow \bigcap_n A_n \epsilon \Phi(\mathfrak{S})$

while the *Borel field over* $\mathfrak{S}$ is the smallest family of sets, $\Lambda(\mathfrak{S})$, satisfying

(1.2.2 a) $\mathfrak{S} \subseteq \Lambda(\mathfrak{S})$

(1.2.2 b) $A \epsilon \Lambda(\mathfrak{S}) \longrightarrow A^c \epsilon \Lambda(\mathfrak{S})$

(1.2.2 c) $A_n \epsilon \Lambda(\mathfrak{S}) \longrightarrow \bigcup_n A_n \epsilon \Lambda(\mathfrak{S})$ $\qquad (n = 1, 2, \ldots)$

3. Since the limiting processes are denumerable, the ordinal α of any class is always less than Ω, the first ordinal of the third kind (non-denumerable).

Since the properties (1.2.2 b) and (1.2.2 c) imply property (1.2.1 c), it is clear that $\Phi(\mathfrak{S}) \subseteq \Lambda(\mathfrak{S})$. In fact, if $\mathfrak{S}^*$ is the collection of subsets of S which are complementary sets to those of $\mathfrak{S}$, then $\Lambda(\mathfrak{S}) = \Phi(\mathfrak{S}) \times \Phi(\mathfrak{S}^*)$, where a set belonging to $\Phi(\mathfrak{S}) \times \Phi(\mathfrak{S}^*)$ is either an intersection or union of one from $\Phi(\mathfrak{S})$ and one from $\Phi(\mathfrak{S}^*)$.

We classify the sets of $\Phi(\mathfrak{S})$, and hence those of $\Lambda(\mathfrak{S})$, as follows: $\Phi(\mathfrak{S})$ is the sum of a transfinite sequence of (type Ω) families [4]

(1.2.3) $\Phi(\mathfrak{S}) = \Phi_0(\mathfrak{S}) + \Phi_1(\mathfrak{S}) + \cdots + \Phi_\alpha(\mathfrak{S}) + \cdots$

where: (1), $\Phi_0(\mathfrak{S}) = \mathfrak{S}$; (2), the sets of the family $\Phi_\alpha(\mathfrak{S})$ are intersections or unions of denumerable sequences of sets belonging to $\Phi_\xi(\mathfrak{S})$ with $\xi < \alpha$, according to whether α is even or odd, limit ordinals being considered even.

We also have the following dual classification: $\Phi(\mathfrak{S})$ is the sum of a transfinite sequence of families

(1.2.4) $\Phi(\mathfrak{S}) = \Psi_0(\mathfrak{S}) + \Psi_1(\mathfrak{S}) + \cdots + \Psi_\alpha(\mathfrak{S}) + \cdots$

where: (1), $\Psi_0(\mathfrak{S}) = \mathfrak{S}$; (2), the sets of the family $\Psi_\alpha(\mathfrak{S})$ are unions or intersections of denumerable sequences of sets belonging to $\Psi_\xi(\mathfrak{S})$ with $\xi < \alpha$, according to whether α is even or odd.

We note that if $\mathfrak{S}$ is a δ-ring, the classification (1.2.4) collapses into that of (1.2.3); for in that case, we clearly have $\Psi_1(\mathfrak{S}) = \Psi_0(\mathfrak{S}) = \mathfrak{S}$, and so $\Psi_{\alpha+1}(\mathfrak{S}) = \Phi_\alpha(\mathfrak{S})$. On the other hand, if $\mathfrak{S}$ is a σ-ring, (1.2.4) is the appropriate classification.

1.3. We shall henceforth suppose that the collection satisfies the following properties:

(1.3.1 a) S and the null set ϕ are contained in $\mathfrak{S}$.

(1.3.1 b) $\mathfrak{S}$ is a δ-ring.

(1.3.1 c) $\mathfrak{S} \subseteq \Psi_1(\mathfrak{S}^*)$

(1.3.1 d) $\mathfrak{S}$ is closed under finite union.

For want of a better name, a collection of subsets S satisfying (1.3.1 a)-(1.3.1 d) will be called *admissible.*

Since an admissible collection $\mathfrak{S}$ is a δ-ring, $\Phi(\mathfrak{S})$ is classified according to (1.2.3), while $\Phi(\mathfrak{S}^*)$, since $\mathfrak{S}^*$ is a σ-ring, is classified according to (1.2.4). By (1.3.1 c) it is clear that $\Phi(\mathfrak{S}) \subseteq \Phi(\mathfrak{S}^*)$. Also, $\mathfrak{S} \subseteq \Psi_1(\mathfrak{S}^*)$ if and only if $\mathfrak{S}^* \subseteq \Phi_1(\mathfrak{S})$; hence $\Phi(\mathfrak{S}^*) \subseteq \Phi(\mathfrak{S})$. Thus, we have $\Lambda(\mathfrak{S}) = \Phi(\mathfrak{S}) = \Phi(\mathfrak{S}^*)$. For convenience, however, we will want to use both classifications (1.2.3) and (1.2.4). Then $\Lambda(\mathfrak{S})$ divided into classes of sets with properties as noted below.

P 1.3.1. The families $\Phi_\alpha(\mathfrak{S})$ with even index and $\Psi_\alpha(\mathfrak{S}^*)$ with odd index are multiplicative in a denumerable sense; that is, they are closed under denumerable intersections. Sets belonging to these families are called *of class α multiplicative.*

P 1.3.2. Dually, the families $\Phi_\alpha(\mathfrak{S})$ with odd index and $\Psi_\alpha(\mathfrak{S}^*)$ with even index are closed under denumerable unions. Sets belonging to these families are of *class α additive.*

P 1.3.3. A set which is both class α additive and class α multiplicative is called *ambiguous of class α.* Since $[\Phi_\alpha(\mathfrak{S})]^* = \Psi_\alpha(\mathfrak{S}^*)$ the ordinary set complement of a set of class α additive is of class α multiplicative and vice versa. Consequently, the complement of a set ambiguous of class α is again ambiguous of class α. Denote the family of sets ambiguous of class α by $\Phi_\alpha^\alpha(\mathfrak{S})$.

P 1.3.4. By an elementary transfinite induction on (1.3.1 c) we see that $\Phi_\alpha(\mathfrak{S}) \subseteq \Psi_{\alpha+1}(\mathfrak{S}^*)$, and since always $\Phi_\alpha(\mathfrak{S}) \subseteq \Phi_{\alpha+1}(\mathfrak{S})$, we have $\Phi_\alpha(\mathfrak{S}) \subseteq \Phi_{\alpha+1}(\mathfrak{S}) \cap \Psi_{\alpha+1}(\mathfrak{S}^*)$ Also, $\mathfrak{S} \subseteq \Psi_1(\mathfrak{S}^*) \longrightarrow \mathfrak{S}^* \subseteq \Phi_1(\mathfrak{S}) \longrightarrow \Psi_\alpha(\mathfrak{S}^*) \subseteq \Phi_{\alpha+1}(\mathfrak{S}) \rightarrow \Psi_\alpha(\mathfrak{S}^*) \subseteq \Phi_{\alpha+1}(\mathfrak{S}) \cap \Psi_{\alpha+1}(\mathfrak{S}^*)$ since always $\Psi_\alpha(\mathfrak{S}^*) \subseteq \Psi_{\alpha+1}(\mathfrak{S})$. Thus, every set of class α, additive or multiplicative, is ambiguous of class $\alpha + 1$; and, a fortiori, ambiguous of class β for $\beta \geq \alpha + 1$.

P 1.3.5. Since, by (1.3.1 b) and (1.3.1 d), $\mathfrak{S}$ is a distributive lattice under the usual set operations, so are $\Phi_\alpha(\mathfrak{S})$ and $\Psi_\alpha(\mathfrak{S}^*)$ for each α. Also, under set complementation, $\Phi_\alpha^\alpha(\mathfrak{S})$ is a Boolean algebra for each α. One should note, however, that $\Phi_0^0(\mathfrak{S})$ can be, and usually is, empty of sets other than S and ϕ.

We shall make extensive use of the following separation property, due to Sierpinski [5], [6], expressed in

Lemma 1.3.1. If $\mathfrak{S}$ is admissible, and if A, B, are any two sets of class α multiplicative, $\alpha > 0$, such that $A \cap B = \phi$, then there exists a set H, ambiguous of class α, such that $A \subseteq H$ while $B \cap H = \phi$.

Proof: We have $A = \bigcap_n A_n$, $B = \bigcap_n B_n$, where A_n, B_n are of class $\alpha_n < \alpha$. Clearly, we can assume $A_n \supseteq A_{n+1}$, $B_n \supseteq B_{n+1}$. We define

$$(*) \qquad H = \bigcup_n [A_n \cap B_n^c].$$

We first notice that

$$(**) \qquad H^c = \bigcup_n [B_{n-1} \cap A_n^c]. \qquad (B_0 = S).$$

For, let $x \in H^c$. By (*), $H^c = \bigcap [A_n^c \cup B_n] \longrightarrow x \notin A_n \cap B_n^c$ for every n. Now if $x \in A_n$ for every n, then this implies $x \notin B_n^c$ every $n \longrightarrow x \in B_n$ every $n \longrightarrow x \in (\bigcap_n A_n) \cap (\bigcap_n B_n)$, a contradiction of the hypothesis $A \cap B = \phi$. Hence, there exist numbers n such that $x \notin A_n$. Let m be the smallest of these numbers. If $m = 1$, $x \notin A_1 \longrightarrow x \in B_0 \cap A_1^c$. If $m > 1$, $x \notin A_m$, while $x \in A_{m-1}$. Taking $n = m - 1$, we have, since $x \notin A_n \cap B_n^c$, $x \notin B_{m-1}^c$, or $x \in B_{m-1}$. Hence, $x \in B_{m-1} \cap A_m^c$. Thus, in either case, $H^c \subseteq \bigcup_n [B_{n-1} \cap A_n^c]$. On the other hand, let $x \in \bigcup_n [B_{n-1} \cap A_n^c]$. If $x \in B_0 \cap A_1^c$, $x \in A_1^c \longrightarrow x \in A_n^c$ all n, since $A_1 \supseteq A_2 \cdots \longrightarrow A_1^c \subseteq A_2^c \subseteq \cdots$. But this implies $x \in H^c = \bigcap_n [A_n^c \cup B_n]$. If $x \in B_{n-1} \cap A_n^c$ for $n > 1$, then $x \in B_{n-1}$, $x \notin A_n \longrightarrow x \in B_\kappa$ for $\kappa \leq n - 1$ and

$x \notin A_\kappa$ for $\kappa \geq n$; hence, $x \notin A_\kappa \cap B_\kappa^c$ for all κ implies $x \in H^c$. In either case, $\bigcup_n [B_{n-1} \cap A_n^c] \subseteq H^c$ and (**) is proved.

But now (*) and (**) assert that H is ambiguous of class α. For A_n, B_n, being of class $\alpha_n < \alpha$ are by P 1.3.4 ambiguous of class α; thus $A_n \cap B_n^c$ is ambiguous of class α and, a fortiori, is of class α additive. Hence, H is of class α additive. Similarly, (**) shows H^c is of class α additive. Thus, H is ambiguous of class α.

To complete the proof, we observe that if $x \in A$, then $x \notin B$ and so there exists an integer m such that $x \notin B_m$; that is, $x \in A_m \cap B_m^c \longrightarrow x \in H$. So, $A \subseteq H$. Finally, if $x \in B_n$ all n there exists an integer m such that $x \notin A_m$; thus $x \in B_{m-1} \cap A_m^c \longrightarrow x \in H^c \longrightarrow B \subseteq H^c \longrightarrow B \cap H = \phi$.

1.4. *Measurable functions.* If $\mathfrak{S}$ is any admissible collection of subsets of a topological space S, a real-valued functions f on S is said to be *measurable of class α with respect to* $\mathfrak{S}$ if, given any open set of real numbers, U, $f^{-1}(U)$ is of Borel class α additive over $\mathfrak{S}$. The collection of measurable functions over $\mathfrak{S}$ will be denoted by $D(\mathfrak{S})$; the functions of class α by $D_\alpha(\mathfrak{S})$:

Since the reals are separable, every open set is a denumerable union of intervals. Hence, one can phrase the definition of measurable functions in any of the following equivalent forms:

D 1.4.1 For every real number λ, the spectral sets $\{x \mid f(x) < \lambda\}$ and $\{x \mid f(x) > \lambda\}$ are of class α additive over $\mathfrak{S}$.

D 1.4.2 For every real number λ, the spectral sets $\{x \mid f(x) \leq \lambda\}$ and $\{x \mid f(x) \geq \lambda\}$ are of class α multiplicative over λ.

D 1.4.3 For every real number λ, the spectral sets $\{x \mid f(x) < \lambda\}$ are of class α additive over $\mathfrak{S}$, while the spectral sets $\{x \mid f(x) \leq \lambda\}$ are of class α multiplicative over $\mathfrak{S}$.

We will need the following properties of the measurable functions.

Lemma 1.4.1 If $\alpha > 0$, f is a measurable function of class α over $\mathfrak{S}$ if and only if given any two real numbers λ_1, λ_2 such that $\lambda_1 > \lambda_2$, there exists an ambiguous set of class α, $A_f(\lambda_1 ; \lambda_2)$, such that $\{x \mid f(x) \leq \lambda_2\} \subseteq A_f(\lambda_1 ; \lambda_2) \subseteq \{x \mid f(x) < \lambda_1\}$.

Proof: Let $f \in D_\alpha(\mathfrak{S})$. Since $\lambda_1 > \lambda_2$, we have $\{x \mid f(x) \leq \lambda_2\} \cap \{x \mid f(x) \geq \lambda_1\} = \phi$ and both sets are of class α multiplicative. By lemma 1.3.1 there exists $A_f(\lambda_1 ; \lambda_2) \in \Phi_\alpha^\alpha(\mathfrak{S})$ such that $\{x \mid f(x) \leq \lambda_2\} \subseteq A_f(\lambda_1 ; \lambda_2)$ while $\{x \mid f(x) \geq \lambda_1\} \cap A_f(\lambda_1 ; \lambda_2) = \phi$. Thus, $A_f(\lambda_1 ; \lambda_2)$ is the required set. Conversely, suppose the property is satisfied. Now if λ is any real number, then $\{x \mid f(x) < \lambda\} = \bigcup_{r<\lambda} \{x \mid f(x) \leq r\}$ where r is a rational number. Since $\{x \mid f(x) \leq r\} \subseteq A_f(r; \lambda) \subseteq \{x \mid f(x) < \lambda\}$, we have $\{x \mid f(x) < \lambda\} = \bigcup_{r<\lambda} A_f(r; \lambda)$. But $A_f(r; \lambda)$, being ambiguous of class α is automatically additive of class α; and so $\{x \mid f(x) < \lambda\}$ is of class α additive, since the rationals are denumerable. Similarly, $\{x \mid f(x) \leq \lambda\}$ being equal to $\bigcap_{r<\lambda} A_f(\lambda; r)$ is class α multiplicative. Hence, by D 1.4.3 $f \in D_\alpha(\mathfrak{S})$ and the lemma is proved.

Lemma 1.4.2. If $\alpha > 0$, and $f \in D_\alpha(\mathfrak{S})$, then $|f|$ and $f - \mu$ for any real number μ, are also contained in $D_\alpha(\mathfrak{S})$.

Proof: Let $\lambda_1 > \lambda_2$, then $\lambda_1 + \mu > \lambda_2 + \mu$. Also, we have $\{x \mid f(x) - \mu \leq \lambda_2\} = \{x \mid f(x) \leq \lambda_2 + \mu\}$, $\{x \mid f(x) - \mu < \lambda_1\} = \{x \mid f(x) < \lambda_1 + \mu\}$. Hence, the set $A_f(\lambda_1 + \mu; \lambda_2 + \mu)$ has the required properties for $A_{f-\mu}(\lambda_1 ; \lambda_2)$. By lemma 1.4.1 then, $f - \mu \in D_\alpha(\mathfrak{S})$.

Now we note that $\{x \mid |f| \geq \lambda_1\} = \{x \mid f \geq \lambda_1\} \cup \{x \mid f \leq -\lambda_1\}$ while $\{x \mid |f| \leq \lambda_2\} = \{x \mid f \leq \lambda_2\} \cap \{x \mid f \geq -\lambda_2\}$. Since $\lambda_1 > \lambda_2$, $-\lambda_1 < -\lambda_2$. Hence,

$$\{x \mid f \leq -\lambda_1\} \subseteq A_f(-\lambda_2 ; -\lambda_1), \qquad \{x \mid f \geq -\lambda_2\} \cap A_f(-\lambda_2 ; -\lambda_2) = \phi$$

Now, let $B = A_f(\lambda_1 ; \lambda_2) \cap [A_f(-\lambda_2 ; -\lambda_1)]^c$. Then $B \in \Phi_\alpha^\alpha(\mathfrak{S})$ and

$$\{x \mid |f| \geq \lambda_1\} \cap B = [\{x \mid f \geq \lambda_1\} \cap A_f(\lambda_1 ; \lambda_2) \cap [A_f(-\lambda_2 ; -\lambda_1)]^c] \cup$$

$$[\{x \mid f \leq -\lambda_1\} \cap A_f(\lambda_1 ; \lambda_2) \cap \{A_f(-\lambda_2 ; -\lambda_1)\}^c] = \phi \cup \phi = \phi$$

While,

$$\{x \mid |f| \leq \lambda_2\} \subseteq \{x \mid f \leq \lambda_2\} \subseteq A_f(\lambda_1 ; \lambda_2)$$

$$\{x \mid |f| \leq \lambda_2\} \subseteq \{x \mid f \geq \lambda_2\} \subseteq [A_f(-\lambda_2 ; -\lambda_1)]^c$$

So, $\{x \mid |f| \leq \lambda_2\} \subseteq A_f(\lambda_1 ; \lambda_2) \cap [A_f(-\lambda_2 ; -\lambda_1)]^c = B$. Hence, B will serve as $A_{|f|}(\lambda_1 ; \lambda_2)$; and so by lemma 1.4.1, $|f| \in D_\alpha(\mathfrak{S})$.

Lemma 1.4.3. If $\alpha > 0$, and $f \in D_\alpha(\mathfrak{S})$, then f is the limit of a uniformly convergent sequence of functions $f_n \in D_\alpha(\mathfrak{S})$, such that f_n is of the form $f_n(x) = \sum_{r=1}^{n} \lambda_r \varphi(x; F_r)$, where λ_r is a real number, and $\varphi(x; F_r)$ is the characteristic function of the ambiguous set of class α, F_r.[4]

Proof: Let $|f| \leq M$. Given $\epsilon > 0$, there exists a finite number of points $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that if μ is contained in the closed interval $<-M, M>$, then $|\mu - \lambda_r| < \epsilon$ for some λ_r. We can, for instance, find $\lambda_1, \lambda_2, \ldots, \lambda_m$ by taking open intervals of length ϵ about each rational in $<-M, M>$. Since the closed interval is compact, a finite number of these open sets cover $<-M, M>$, and so the rationals at the center of these subintervals will serve. Now, let

$$A_r = \{x \mid |f(x) - \lambda_r| \leq \epsilon\} \qquad B_r = \{x \mid |f(x) - \lambda_r| \geq 2\epsilon\}$$

Since $f \in D_\alpha(\mathfrak{S})$, lemma 1.4.2 shows that $|f - \lambda_r| \in D_\alpha(\mathfrak{S})$, and hence by lemma 1.3.1, for each $r = 1, 2, \ldots, m$, there exists a set $G_r \in \Phi_\alpha^\alpha(\mathfrak{S})$ such that $A_r \subseteq G_r$, $B_r \cap G_r = \phi$. Now if $x \in S$, $f(x) \in <-M, M> \longrightarrow |f(x) - \lambda_r| < \epsilon$ for

4. Lemma 1.4.3 will be found, in a slightly different form, on page 186 of Reference [4]. Kuratowski's elegant proof is followed almost exactly here.

some $\lambda_r \longrightarrow x \in A_r$ for some r. Hence, $S = A_1 \cup A_2 \cup \cdots \cup A_m$; and thus $S = G_1 \cup G_2 \cup \cdots \cup G_m$. We now define the sets F_r by

$$F_1 = G_1, \qquad F_r = G_r \cap G_1^c \cap G_2^c \cap \cdots \cap G_{r-1}^c$$

Then $F_r \in \Phi_\alpha^\alpha(\mathfrak{S}^*)$; and note that if $r \neq s$, then $F_r \cap F_s = \phi$.
For, if (say) $r < s$, $r + \kappa = s$, then

$$F_r \cap F_s = G_r \cap G_1^c \cap \cdots \cap G_{r-1}^c \cap G_{r+\kappa} \cap \cdots \cap G_r^c \cap \cdots \cap G_{r-1+\kappa}^c = \phi.$$

Also, $F_r = \bigcup_r G_r \cap G_1^c \cap \cdots \cap G_{r-1}^c = (\bigcup_r G_r) \cap (\bigcap G_r \cup G_r^c) = S \cap S = S.$
Further, $F_r \cap B_r \subseteq G_r \cap B_r = \phi$. Now, let

$$f_\epsilon(x) = \sum_{r=1}^{m} \lambda_r \varphi(x; F_r)$$

where $\varphi(x; F_r)$ is the characteristic function of F_r. Then the above remarks show that $f_\epsilon(x) = \lambda_r$ if and only if $x \in F_r$. Thus $f_\epsilon^{-1}(\lambda_r) = F_r \in \Phi_\alpha^\alpha(\mathfrak{S})$. Since the values of the function f_ϵ are a finite set, this implies f_ϵ^{-1}(any set) $\in \Phi_\alpha^\alpha(\mathfrak{S})$. Hence, f_ϵ is certainly contained in $D_\alpha(\mathfrak{S})$. Further, if $x \in S$, then $x \in F_r$ some $r \longrightarrow f_\epsilon(x) = \lambda_r$. Also, $F_r \cap B_r = \phi$ implies $x \in B_r^c \longrightarrow |f(x) - \lambda_r| = |f(x) - f_\epsilon(x)| < 2\epsilon$. Since the bound 2ϵ is independent of x, we see that $f(x) = \lim_{n \to \infty} f_n(x)$ and that the approach to the limit is uniform, where we have taken $\epsilon = 1/2n$ and have renamed $f_\epsilon(x)$ to be $f_n(x)$.

Lemma 1.4.4. If $\mathfrak{S}$ is any admissible collection of subsets of S, then $D_0(\mathfrak{S})$ is a complete system.

Proof: (1.1.1 a) follows immediately from (1.3.1 a). For (1.1.1 b), we observe that $\{x \mid (f_1 \vee f_2)(x) < \lambda\} = \{x \mid f_1(x) < \lambda\} \cap \{x \mid f_2(x) < \lambda\}$, and $\{x \mid f_1 \vee f_2 > \lambda\} = \{x \mid f_1 > \lambda\} \cup \{x \mid f_2 > \lambda\}$. Dually for $f_1 \wedge f_2$. Hence, (1.1.1 b) follows from (1.3.1 b) and (1.3.1 d). Now, let $f = \lim_{n \to \infty} f_n$, where $f_n \in D_0(\mathfrak{S})$ and the convergence is uniform. Then, there exists an increasing sequence of integers m_n such that $|f(x) - f_{m_n+\kappa}(x)| < 1/n$ for all x and $\kappa \geq 0$. Hence, for any real number λ, we have

$$(*) \qquad \{x|f(x) \leq \lambda\} = \bigcap_n \bigcap_\kappa \{x|f_{m_n+\kappa}(x) \leq \lambda + 1/n\}$$

$$(**) \qquad \{x|f(x) \geq \lambda\} = \bigcap_n \bigcap_\kappa \{x|f_{m_n+\kappa}(x) \geq \lambda - 1/n\}$$

We prove only (*), since (**) follows from a dual argument. Let $x \epsilon \{x|f(x) \leq \lambda\}$. Then, since $|f(x) - f_{m_n+\kappa}(x)| < 1/n$ all n, κ, we have $f_{m_n+\kappa}(x) < \lambda + 1/n$ and hence $x \epsilon \{x|f_{m_n+\kappa}(x) \leq \lambda + 1/n\}$ for all n, κ. Conversely, if we suppose the latter to be true, $f_{m_n+\kappa}(x) \leq \lambda + 1/n$ for all n, κ. If $f(x) > \lambda$ there exists an n such that $f(x) > \lambda + 1/n$, which, since $f = \lim f_n$, implies that for some κ, $f_{m_n+\kappa}(x) > \lambda + 1/n$, a contradiction. So, $f(x) \leq \lambda$. Thus (*) and (**) are established. But then (1.1.1 d) follows at once from (*), (**), (1.3.1 b) and D 1.4.2.

Finally, to prove (1.1.1 c) we proceed as follows. (i), for sums, note that $f_1 + f_2 < \lambda$ is the same as $f_1 > \lambda - f_2$, and hence there is a rational r such that $f_1 > r > \lambda - f_2$; or $f_1 > r$, $f_2 > \lambda - r$. Hence

$$\{x|f_1 + f_2 > \lambda\} = \bigcup_r [\{x|f_1 > r\} \cap \{x|f_2 > \lambda - r\}]$$

$$\{x|f_1 + f_2 < \lambda\} = \bigcup_r [\{x|f_1 < r\} \cap \{x|f_2 < \lambda - r\}]$$

Thus, $D_0(\mathfrak{S})$ is closed under sums by D 1.4.1, and the fact that $\mathfrak{S}^*$ is closed under finite intersection and denumerable union. (ii), for differences, we observe that $\{x| -f \lessgtr \lambda\} = \{x|f \gtrless -\lambda\}$ and so the negative of a function in $D_0(\mathfrak{S})$ is again in $D_0(\mathfrak{S})$. (ii) then follows from (i). (iii), that $D_0(\mathfrak{S})$ is closed under products follows at once from (i) and (ii), since

$$f_1 f_2 = \tfrac{1}{4}[(f_1 + f_2)^2 - (f_1 - f_2)^2],$$

if we can show $f^2 \in D_0(\mathfrak{S})$. But this is easy, for $\{x|f^2 < \lambda\}$ is ϕ or $\{x|f > \sqrt{\lambda}\}$ $\cap$ $\{x|f > -\sqrt{\lambda}\}$, according to whether $\lambda \leq 0$ or $\lambda > 0$; while $\{x|f^2 > \lambda\}$ is S or $\{x|f > \sqrt{\lambda}\} \cup \{x|f < -\sqrt{\lambda}\}$, according to whether $\lambda < 0$ or $\lambda \geq 0$. (iii) is thus established. (iv), for quotients, if $f, g \in D_0(\mathfrak{S})$ and g never vanishes we have

$$\{x|f/g < \lambda\} = [\{x|g > 0\} \cap \{x|f - \lambda g < 0\}] \cup [\{x|g < 0\} \cap \{x|f - \lambda g > 0\}].$$

$$\{x|f/g > \lambda\} = [\{x|g > 0\} \cap \{x|f - \lambda g > 0\}] \cup [\{x|g < 0\} \cap \{x|f - \lambda g < 0\}].$$

By (i), (iii), and D 1.4.1, all of these sets are in $\mathfrak{S}^*$. Hence, (iv) is established, and the proof of the lemma is complete.

Corollary: If $\mathfrak{S}$ is admissible, then for each α, $D_\alpha(\mathfrak{S})$ is a complete system.

Proof: Let $M_\alpha(\mathfrak{S})$, for the moment be the collection of sets which are of class α multiplicative over $\mathfrak{S}$. Then P 1.3.1 and P 1.3.5 imply that M_α is an admissible collection. Also, it is clear from D 1.4.2 that $D_\alpha(\mathfrak{S}) = D_0(M_\alpha)$. Hence, the corollary follows at once from lemma 1.4.4.

We also obtain a converse of lemma 1.4.4. If s is any collection of functions on S let $\mathfrak{C}$ be the collection of 'closed' spectral sets of s. That is, $A \in \mathfrak{C}$ means that there exists a real number λ and a function f in s such that $A = \{x|f \leq \lambda\}$. Then we have

Lemma 1.4.5. If s is a complete system, then $\mathfrak{C}$ is an admissible collection of subsets S.

Proof: Since s is a complete system, $(f - \lambda) \vee 0$ is in s whenever f is. Hence we can say that $\mathfrak{C}$ consists of all sets A such that $A = \{x|f = 0\}$, where f is non-negative and $f \epsilon s$. Thus, $\mathfrak{S}^*$ is the collection of all sets B such that $B = \{x | f > 0\}$, for some non-negative f in s.

Now, we observe that

$$\{x \mid f = 0\} = \bigcap_{r > 0} \{x | (r - f) \vee 0 > 0\} \qquad (r \text{ rational})$$

If $f \epsilon s$, then $(r - f) \vee 0 \epsilon s$. Hence, $\mathfrak{S}$ satisfies property (1.3.1 c). That it satisfies (1.3.1 a) follows at once from (1.1.1 a). (1.3.1 d) follows from the fact that, for non-negative f, g

$$\{x | f(x) = 0\} \cup \{x | g(x) = 0\} = \{x | f \wedge g = 0\}$$

and (1.1.1 b). It only remains to prove that $\mathfrak{S}$ satisfies (1.3.1 b).

Accordingly, let $A = \bigcap A_n$, where $A_n = \{x | f_n = 0\}$, and $f_n \epsilon s$. Let ϵ_n be a sequence of positive numbers such that $\Sigma \epsilon_n$ converges. Define

$$f = \sum_{r=1}^{\infty} \epsilon_r (f_r \wedge 1)$$

Then $A = \{x | f = 0\}$. For, if $x \epsilon A$, then $x \epsilon A_n$ all n and so $f_n(x) = 0$ all n; whence, $f(x) = 0$. On the other hand, if $x \notin A$, then $x \notin A_n$ for some n implies $f_n(x) > 0$; whence, $f(x) \geq \epsilon_n (f_n(x) \wedge 1) > 0$. Now, to complete the proof, we need only show that $f \epsilon s$. But, observe that, given any $\epsilon > 0$,

$$|f(x) - \sum_{r=1}^{N} \epsilon_r (f_r(x) \wedge 1)| = | \sum_{N}^{\infty} \epsilon_r (f_r(x) \wedge 1)| \leq \sum_{N}^{\infty} \epsilon_r < \epsilon$$

for sufficiently large N, since $\Sigma \epsilon_r$ is convergent. This bound is independent of x, and hence f is the uniform limit of the functions $h_n = \sum_{r=1}^{n} \epsilon_n (f_r \wedge 1)$. Clearly, each h_n is in s and so by (1.1.1 d), f is contained in s. The proof of the lemma is thus complete.

1.5. *Connection between Baire and measurable functions.*

Let $\mathfrak{S}$ be any admissible collection of subsets of a topological space S, and denote by s the collection of functions $D_0(\mathfrak{S})$. The lemma 1.4.4 implies that s is a complete system. It makes sense, then to talk about the Baire functions of class α over s, $B_\alpha(s)$. In fact, we can say at once

Lemma 1.5.1. For each α, $B_\alpha(s) \subseteq D_\alpha(\mathfrak{S})$.

Proof: For $\alpha = 0$ this follows at once from the definition of s. In fact, we note that $s = D_0(\mathfrak{S})$ and (always) $s = B_0(s)$ imply the stronger relation $B_0(s) = D_0(\mathfrak{S})$.

Suppose, now that the theorem is true for all $\xi < \alpha$. If α has a predecessor, then, $f \in B_\alpha(s)$ means that $f = \lim_{n\to\infty} f_n$ where $f_n \in B_{\alpha-1}(s)$. Let λ be any real number. Then

$$(*) \qquad \{x|f(x) \leq \lambda\} = \bigcap_n \bigcup_K \{x|f_{n+K}(x) < \lambda + 1/n\}$$

$$(**) \qquad \{x|f(x) \geq \lambda\} = \bigcap_n \bigcup_K \{x|f_{n+K}(x) > \lambda - 1/n\}$$

We prove only (*), since the other follows from a dual argument: If $f(x) \leq \lambda$, all the points $f_m(x)$ with index sufficiently large satisfy the inequality $|f(x) - f_m(x)| < 1/n$, and hence $f_m(x) < \lambda + 1/n$; thus, for all n there exists a κ such that $f_{n+\kappa}(x) < \lambda + 1/n$, which proves (*) one way. Conversely, if $f(x) > \lambda$, there exists an N such that $f(x) > \lambda + 1/N$. Then $f = \lim f_n$ implies that for some $n > N$ all the points $f_{n+\kappa}(x)$ are such that $f_{n+\kappa}(x) > \lambda + 1/N > \lambda + 1/n$; thus, the supposition that for all n there exists a κ such that $f_{n+\kappa}(x) < \lambda + 1/n$ implies that $f(x) \leq \lambda$. Whence, (*) — and similarly (**) — is established. But now by the induction hypothesis, $\{x|f_{n+\kappa}(x) < \lambda + 1/n\}$ and $\{x|f_{n+\kappa}(x) > \lambda - 1/n\}$ are of class $\alpha - 1$ additive. Hence, (*) and (**) show that $\{x|f(x) \leq \lambda\}$ and $\{x|f(x) \geq \lambda\}$ are of class α multiplicative. Thus, by D 1.4.2, $f \in D_\alpha(\mathfrak{S})$.

To complete the proof, we observe that if α is a limit ordinal, then the induction hypothesis implies that every Baire function of class less than α is contained in $D_\alpha(\mathfrak{S})$. But, by the corollary to lemma 1.4.4, $D_\alpha(\mathfrak{S})$ is a complete system; it must therefore contain as a subset the smallest complete system containing every function of Baire class less than α. But by the definition (1.1.2 c), this smallest system is just $B_\alpha(s)$. Hence, $B_\alpha(s) \subseteq D_\alpha(\mathfrak{S})$, and the lemma follows.

In proving lemma 1.5.1, we note that $B_0(s) = D_0(\mathfrak{S})$. If we seek a condition for this to be true in general, we are led to adopt the following definition.

D 1.5.1. S is called $\mathfrak{S}$-*normal*, if whenever $A, B \in \mathfrak{S}$ are such that $A \cap B = \phi$, there exists a function f in s such that $0 \leq f \leq 1$, and $f(x) = 0$ if $x \in A$, while $f(x) = 1$ if $x \in B$.

Lemma 1.5.2. If $\alpha > 0$, S is $\mathfrak{S}$-normal, and $A \in \Phi_\alpha^\alpha(\mathfrak{S})$, then the characteristic function of A, $\varphi(x; A)$, is contained in $B_\alpha(s)$.

Proof: For $\alpha = 1$, we have $A = \bigcup B_n$, where $B_n \in \mathfrak{S}$ and $A^c = \bigcup C_n$, $C_n \in \mathfrak{S}$. Clearly, we can assume $B_n \subseteq B_{n+1}$, $C_n \subseteq C_{n+1}$. Now, $A \cap A^c = \phi \longrightarrow \bigcup_{n,m} B_n \cap C_m = \phi \longrightarrow B_n \cap C_m = \phi$ all n, m. By $\mathfrak{S}$-normality there exist functions $f_n \in s$ such that $0 \leq f_n \leq 1$, while $f_n(x) = 1$ if $x \in B_n$, and $f_n(x) = 0$ if $x \in C_n$. Then, if $x \in S$, either $x \in A$ or $x \in A^c$, but not both. If $x \in A$, then $x \in B_{N+\kappa}$ for some N and all κ, whence $f_{N+\kappa}(x) = 1$ all κ. If $x \in A^c$, then $x \in C_{M+\kappa}$ for some M and all κ, whence $f_{M+\kappa}(x) = 0$ all κ. Thus, for any x, we have $|\varphi(x; A) - f_{N+\kappa}(x)| = 0 < \epsilon$ for some N and all κ. Hence, $\varphi(x; A) = \lim_{n \to \infty} f_n(x)$, and the lemma is established for $\alpha = 1$.

We assume that now the lemma holds for all $\xi < \alpha$. If α has a predecessor, then $A = \bigcup_n B_n$, $A^c = \bigcup_n C_n$ where B_n, C_n are of class $\alpha - 1$ multiplicative. Again, $A \cap A^c = \phi \longrightarrow B_n \cap C_n = \phi$. By lemma 1.3.1, there exist sets F_n, ambiguous of class $\alpha - 1$, such that $B_n \subseteq F_n$, while $C_n \cap F_n = \phi$. By the induction hypothesis, the characteristic function of F_n, $\varphi(x; F_n)$, is of Baire class $\alpha - 1$; and by arguments similar to those above, it is clear that $\varphi(x; A) = \lim_{n \to \infty} \varphi(x; F_n)$. Hence by (1.1.2 b), $\varphi(x; A) \in B_\alpha(s)$.

Finally, if α is a limit ordinal, we proceed as follows. We first construct a function h, of Baire class α, which vanishes on A and which is positive on A^c. To do this, we observe that $A^c = \bigcup B_n$, where the B_n are of class $\alpha_n < \alpha$. Since α is a limit ordinal, $\alpha_n + 1 < \alpha$; hence the B_n are ambiguous of class less than α. By the induction hypothesis, the characteristic functions $\varphi(x; B_n)$ of the B_n are contained in $B_{\alpha_n+1}(s) \subseteq B_\alpha(s)$. Now, let ϵ_n be a sequence of positive numbers such that $\Sigma\, \epsilon_n$ converges, and consider the function

$$h = \sum_{r=1}^{\infty} \epsilon_n \varphi(x; B_n) = \lim_{n \to \infty} h_n$$

Note first that, if $\epsilon > 0$, $|h(x) - h_N(x)| = |\sum_N^\infty \epsilon_n \varphi(x; B_n)| \leq \sum_N^\infty \epsilon_n < \epsilon$ if N is sufficiently large, since $\Sigma\, \epsilon_n$ converges. The bound being independent of x, the convergence is uniform, and hence h belongs to $B_\alpha(s)$ since it is a complete system. Further, if $x \in A$, then $x \notin B_n$ for all n implies $\varphi(x; B_n) = 0$ for all n, and hence that $h(x) = 0$. While if $x \in A^c$, then $x \in B_n$ for some n, implies that $\varphi(x; B_n) = 1$, *and so* $h(x) \geq \epsilon_n > 0$. Thus h is the required function. In a similar way, we construct a function $g \in B_\alpha(s)$ which vanishes A^c and is positive on A. Then it is clear that

$$(*) \qquad \varphi(x; A) = \frac{h(x)}{h(x) + g(x)}$$

But $A \cap A^c = \phi$ implies that $h + g$ never vanishes. Thus (*) shows that $\varphi(x; A) \in B_\alpha(s)$, since $B_\alpha(s)$ is a complete system. The lemma is thus established.

We are finally able to prove the main result of Part I, namely:

Theorem 1.5.1. If $\mathfrak{S}$ is admissible, $s = D_\alpha(\mathfrak{S})$, and S is $\mathfrak{S}$-normal, then for each α, $B_\alpha(s) = D_\alpha(\mathfrak{S})$.

Proof: We have already noted that the theorem is true for $\alpha = 0$. For $\alpha > 0$, in view of lemma 1.5.1, it only remains to prove $D_\alpha(\mathfrak{S}) \subseteq B_\alpha(s)$. Accordingly, we note that if $f \in D_\alpha(\mathfrak{S})$ then by lemma 1.4.3 we have f as the uniform limit of functions f_n of the form $\sum_{r=1}^{m} \lambda_r \varphi(x; F_r)$ where F_r is ambiguous of class α over $\mathfrak{S}$. But by lemma 1.5.2, $\varphi(x; F_r) \in B_\alpha(s)$. Since $B_\alpha(s)$ is a complete system this implies $f \in B_\alpha(s)$. Hence, the theorem is established.

1.6. In order to get a clearer idea of the meaning of the condition of $\mathfrak{S}$-normalit y, suppose that $\mathfrak{S}$ is admissible, $s = D_0(\mathfrak{S})$, and $\mathfrak{F}$, as in lemma 1.4.5, is the collection of 'closed' spectral sets of s. Then, by that lemma $\mathfrak{F}$ is also admissible. Hence, we can talk about $D_\alpha(\mathfrak{F})$. But even more, S is $\mathfrak{F}$-normal. For if $A, B \in \mathfrak{F}$, then

$$A = \{x \mid f(x) = 0\} \qquad B = \{x \mid g(x) = 0\}$$

where f, g are in s and non-negative. Then, if $A \cap B = \phi$, the function

$$\varphi = \frac{f}{f+g}$$

is in s, since s is a complete system and $A \cap B = \phi$ implies that $f + g$ never vanishes. But clearly $\varphi(x) = 0$ if $x \in A$ while $\varphi(x) = 1$ if $x \in B$ and $0 \leq \varphi \leq 1$. This shows that S is $\mathfrak{F}$-normal. Hence, by theorem 1.5.1, we have $B_\alpha(s) = D_\alpha(\mathfrak{F})$. If now, S is also $\mathfrak{S}$-normal, then $B_\alpha(s) = D_\alpha(\mathfrak{S})$.

Hence, in this case $D_\alpha(\mathfrak{S}) = D_\alpha(\mathfrak{F})$. Thus, the condition of $\mathfrak{S}$-normality assures us that the spectral sets, $\mathfrak{F}$, are 'dense' in $\mathfrak{S}$, in the sense that the measurable functions over them fill out the possible measurable functions over $\mathfrak{S}$. For the Baire functions, $\mathfrak{S}$-normality enables us to work with the given admissible set, instead of the (possibly) harder to characterize collection of spectral sets.

PART II

THE REPRESENTATION THEOREM

2.1. Let $\mathfrak{S}$ be an admissible collection of subsets of the topological space S, and let s denote $D_0(\mathfrak{S})$. If S is $\mathfrak{S}$-normal, then the main result of the paper can be stated as

Theorem 2.1.1 (*The Representation Theorem*). If $a > 0$, then the Baire functions of class a over s are isomorphic, as a lattice (ring), to the continuous functions on the Boolean space associated with the Boolean algebra $\Phi_a^a(\mathfrak{S})$ of the ambiguous sets of class a over $\mathfrak{S}$.

2.2. Before proceeding to the proof of the representation theorem, we shall need some preliminary results. In what follows we shall denote the Boolean space associated with Φ_a^a by Σ_a. We shall suppose a arbitrary, but fixed, and greater than zero. Also, to avoid confusion, lattice operations will always be indicated by the pointed symbols $\vee$, $\wedge$, and $\leq$; while set operations will be indicated by rounded symbols $\cup$, $\cap$, and $\subseteq$.

First, of all, we recall [7] that any distributive lattice L has a topological space $\mathfrak{L}$ associated with it which is always compact, and, at least T_1. The points of $\mathfrak{L}$ are minimal dual ideals of the lattice L, and the closure operation in $\mathfrak{L}$ is defined by $\overline{\mathfrak{A}} = \{\mathfrak{p} | \mathfrak{p} \leq \bigvee\mathfrak{A}\}$, where $\mathfrak{A}$ is a set of points of $\mathfrak{L}$. (small german letters will always denote minimal dual lattice ideals). If L is a Boolean algebra, then $\mathfrak{L}$ is a totally-disconnected, compact, Hausdorff space; that is, a Boolean space [8].

The following theorem, which will be useful in extending the representation theorem, should serve to illustrate the ideas and methods involved in going from L to $\mathfrak{L}$.

Theorem 2.2.1. Let L be a sublattice (with zero z) of the distributive lattice M (with the same zero z), the elements of which separate those of M in the following sense: if $m, n \in M$ are such that $m \wedge n = z$, then there exists an element $h \in L$, such that $m \leq h$, while $n \wedge h = z$. Then the space $\mathfrak{L}$ is homeomorphic to the space $\mathfrak{M}$.

Proof: If $\mathfrak{p} \in \mathfrak{L}$, we define

$$\text{(i)} \quad \mathfrak{p}^* = \{m \in M \mid m \wedge k \neq z \text{ for all } k \in \mathfrak{p}\}$$

(1). *$\mathfrak{p}^*$ is a minimal dual ideal of M.* For, first of all, $\mathfrak{p}^*$ is not void, since, L being a sublattice of M, every member of $\mathfrak{p}$ is contained in it. Secondly, if $m, n \in \mathfrak{p}^*$, then $m \wedge n \neq z$. For if $m \wedge n = z$, then by the separation hypothesis, there exists $h \in L$ such that $m \leq h$, $n \wedge h = z$. But then $h \in \mathfrak{p}$, since for any $k \in \mathfrak{p}$, $h \wedge k \geq m \wedge k \neq z$ and $\mathfrak{p}$ is minimal in L. Hence, $n \wedge h = z$ contradicts the fact that $n \in \mathfrak{p}$. Thirdly, if $m, n \in \mathfrak{p}^*$, then $m \wedge n \in \mathfrak{p}^*$. For if not, there exists a $k \in \mathfrak{p}$ such that $m \wedge n \wedge k = z$. So, $m \wedge (n \wedge k) = z$. But $n \wedge k \in \mathfrak{p}^*$ since $n \wedge k \wedge k' = n \wedge k'' \neq z$ for every $k' \in \mathfrak{p}$. Hence, $m \wedge (n \wedge k) = z$ implies $m \notin \mathfrak{p}^*$ by what has just been proved; this contradiction shows that $m \wedge n \in \mathfrak{p}^*$. Fourthly, if $m \in \mathfrak{p}^*$ and $n \geq m$, then $n \wedge k \geq m \wedge k \neq z$ for all $k \in \mathfrak{p}$ and so $n \in \mathfrak{p}^*$. Thus, $\mathfrak{p}^*$ is a proper dual ideal of M. It is obviously minimal, since $m \wedge n \neq z$ all $n \in \mathfrak{p}^*$ implies $m \wedge k \neq z$ all $k \in \mathfrak{p}$, and so $m \in \mathfrak{p}^*$.

(2). $\mathfrak{p}_1 \neq \mathfrak{p}_2 \longrightarrow \mathfrak{p}_1^* \neq \mathfrak{p}_2^*$. For $\mathfrak{p}_1 \neq \mathfrak{p}_2$ implies there exists $k_1 \in \mathfrak{p}_1$, $k_2 \in \mathfrak{p}_2$ such that $k_1 \wedge k_2 = z$. Hence, $k_1 \notin \mathfrak{p}_2^*$ and so $\mathfrak{p}_1^* \neq \mathfrak{p}_2^*$. Now, if $\mathfrak{q} \in \mathfrak{M}$, we define

$$\text{(ii)} \quad \mathfrak{q}_* = \{h \in L \mid h \in \mathfrak{q}\}.$$

(3). *$\mathfrak{q}_*$ is a minimal dual ideal of L, and $(\mathfrak{q}_*)^* = \mathfrak{q}$.* For first, of all, $\mathfrak{q}_*$ is non-empty, since if $h \in L$ is not contained in $\mathfrak{q}_*$ there exists $m \in \mathfrak{q}$ such that $h \wedge m = z$. But then, by the separation property, there exists $k \in L$ such that $h \wedge k = z$ while $k \geq m$. Then, $k \geq m$ implies $k \in \mathfrak{q}$ and so $k \in \mathfrak{q}_*$. Secondly,

$h, k \in \mathfrak{q}_*$ if and only if $h, k \in \mathfrak{q}$. Whence, $\mathfrak{q}_*$ is clearly a proper dual ideal of L. Thirdly, it is minimal. For, let $h \in L$ be such that $h \wedge k \neq z$ all $k \in \mathfrak{q}$. If $h \notin \mathfrak{q}_*$, then as in the proof that $\mathfrak{q}_*$ is non-empty, the separation property yields a $k' \in \mathfrak{q}_*$ such that $h \wedge k' = z$. This is a contradiction, and so establishes the minimality of $\mathfrak{q}_*$. Finally, it is clear that $(\mathfrak{q}_*)^* \leq \mathfrak{q}$, and since both ideals are minimal in M, this implies $(\mathfrak{q}_*)^* = \mathfrak{q}$.

But now the statements (1), (2), and (3) show that the mapping $\mathfrak{p} \longrightarrow \mathfrak{p}^*$ is a 1-1 mapping of $\mathfrak{L}$ onto $\mathfrak{M}$. To complete the proof of the theorem, we need only show that the topologies are preserved under this mapping. Accordingly, let $\mathfrak{p} \leq \bigvee_{\sigma} \mathfrak{p}_{\sigma}$ and suppose $m \in \bigvee_{\sigma} \mathfrak{p}_{\sigma}^*$. If $m \notin \mathfrak{p}^*$, then by (i) there exists $k \in \mathfrak{p}$ such that $m \wedge k = z$. By the separation property, there exists $h \in L$ such that $h \geq m$ while $h \wedge k = z$. But $h \geq m$ implies $h \in \bigvee_{\sigma} \mathfrak{p}_{\sigma}^*$, implies $h \in \mathfrak{p}^*$, and so $h \in \mathfrak{p}$. Then $h \wedge k = z$ is a contradiction of the fact that both h and k are in $\mathfrak{p}$. Hence, $m \in \mathfrak{p}^*$, and so $\mathfrak{p}^* \leq \bigvee_{\sigma} \mathfrak{p}_{\sigma}^*$.

Finally, if $\mathfrak{q} \leq \bigvee_{\sigma} \mathfrak{q}_{\sigma}$, let $h \in \bigvee_{\sigma} (\mathfrak{q}_{\sigma})_*$. Then $h \in \bigvee_{\sigma} \mathfrak{q}_{\sigma}$ and so $h \in \mathfrak{q}$ which implies $h \in \mathfrak{q}_*$. Hence, $\mathfrak{q}_* \leq \bigvee_{\sigma} (\mathfrak{q}_{\sigma})_*$, and the theorem is established.

2.3. We now focus our attention on Σ_{α}. If $x \in S$, let

$$\mathfrak{p}_x = \{ A \in \Phi_{\alpha}^{\alpha}(\mathfrak{S}) \mid x \in A \}.$$

Then $\mathfrak{p}_x \in \Sigma_{\alpha}$. For $\mathfrak{p}_x$ is clearly a proper dual ideal of Φ_{α}^{α}. It is also minimal, for let $P \cap A \neq \phi$ all $A \in \mathfrak{p}_x$ and suppose $P \notin \mathfrak{p}_x$. Then $x \notin P \longrightarrow x \in P^c \longrightarrow P^c \in \mathfrak{p}_x \longrightarrow P \cap P^c \neq \phi$ a contradiction. So, $\mathfrak{p}_x$ is a minimal dual ideal of Φ_{α}^{α}. Let $\mathfrak{B}$ denote the totality of $\mathfrak{p}_x$.

Lemma 2.3.1. $\mathfrak{B}$ is dense in Σ_{α}.

Proof: $\overline{\mathfrak{B}} = \{ \mathfrak{p} | \mathfrak{p} \leq \bigvee \mathfrak{B} \}$. Now, if $B \in \bigvee \mathfrak{B}$, then $B \in \mathfrak{p}_x$ all $x \in S$ implies $x \in B$ all $x \in S \longrightarrow B = S$. So, $\bigvee \mathfrak{B} = (S)$. Since every dual ideal of Φ_{α}^{α} contains S, we have $\overline{\mathfrak{B}} = \Sigma_\alpha$, which proves the lemma.

Lemma 2.3.2. Let X be any T_1 -space, and let Y be dense in X. Then, if $f(x)$, $g(x)$ are continuous real-valued functions on X such that $f(y) = g(y)$ for all $y \epsilon Y$, then $f(x) = g(x)$ for all $x \epsilon X$.

Proof: Suppose there exists a point x_0 where $f(x_0) \neq g(x_0)$, say $f(x_0) > g(x_0)$. Let $N = \{x | f(x) > g(x)\}$. Then, since $f - g$ is a continuous function, N is an open set containing x_0. But since Y is dense in X, there exists a point $y \epsilon Y$ which is contained in N, a contradiction of the definition of N. Thus, $f(x_0) \not> g(x_0)$. Similarly, $g(x_0) \not> f(x_0)$. Hence $f(x_0) = g(x_0)$, which contradiction establishes the lemma.

Corollary: A continuous function on Σ_α is determined by its values at $\mathfrak{B}$.

2.4. Denote by $K(R, \mathfrak{X})$, the set of all (bounded real-valued) functions on the topological space $\mathfrak{X}$; and by $C(R, \mathfrak{X})$ the set of continuous (bounded real-valued) functions on $\mathfrak{X}$.

We now define a pair of correspondence between $K(R, S)$ and $K(R, \Sigma_\alpha)$. The correspondence σ is a mapping of $K(R, S)$ into $K(R, \Sigma_\alpha)$ defined by

$$\text{(2.4.1)} \qquad \text{If } f \epsilon K(R, S), \text{ then } (\sigma f)(\mathfrak{p}) = \inf_{A \epsilon \mathfrak{p}} \sup_{x \epsilon A} f(x)$$

While τ maps $K(R, \Sigma_\alpha)$ into $K(R, S)$ as follows:

$$\text{(2.4.2)} \qquad \text{If } F \epsilon K(R, \Sigma_\alpha) \text{ then } (\tau F)(x) = F(\mathfrak{p}_x).$$

Lemma 2.4.1. If $f \epsilon B_\alpha(s)$, then $\sigma f \epsilon C(R, \Sigma_\alpha)$.

Proof: Let $\mathfrak{A}_\lambda = \{\mathfrak{p} | (\sigma f)(\mathfrak{p}) < \lambda\}$. Then, if $\mathfrak{p} \epsilon \mathfrak{A}_\lambda$, $\inf_{A \epsilon \mathfrak{p}} \sup_{x \epsilon A} f(x) < \lambda$.

So there exists $A \in \mathfrak{p}$ such that $\sup_{x \in A} f(x) < \lambda$. Now if $A \in \mathfrak{q}$, then $(\sigma f)(\mathfrak{q})$ $= \inf_{Q \in \mathfrak{q}} \sup_{x \in Q} f(x) \leq \sup_{x \in A} f(x) < \lambda$ implies $\mathfrak{q} \in \mathfrak{U}_\lambda$. Thus, $\{\mathfrak{q} | A \in \mathfrak{q}\}$ is an open (and closed) set containing $\mathfrak{p}$ and contained in $\mathfrak{U}_\lambda$. Since $\mathfrak{p}$ was arbitrary, every point of $\mathfrak{U}_\lambda$ is surrounded by an open set contained in $\mathfrak{U}_\lambda$. Whence, $\mathfrak{U}_\lambda$ is open. Now, let

$$\mathfrak{U}^*_\lambda = \{\mathfrak{p} | (\sigma f)(\mathfrak{p}) > \lambda\}$$

To complete the proof of the lemma, we need to show that $\mathfrak{U}^*_\lambda$ is open. Let $\mathfrak{p} \in \mathfrak{U}^*_\lambda$, then $(\sigma f)(\mathfrak{p}) > \lambda$. Choose λ_1, λ_2 such that $(\sigma f)(\mathfrak{p}) > \lambda_1 > \lambda_2 > \lambda$. Define

$$V = \{x | f(x) \geq \lambda_1\}$$

$$W = \{x | f(x) \leq \lambda_2\}$$

Since $f \in B_\alpha(s)$, and so $f \in D_\alpha(\mathfrak{S})$, V and W are of class α multiplicative. Clearly, $V \cap W = \phi$. By lemma 1.3.1, there exists a set $B \in \Phi^\alpha_\alpha(\mathfrak{S})$ such that $W \subseteq B$, $V \cap B = \phi$. Since $(\sigma f)(\mathfrak{p}) > \lambda_1$, $\inf_{P \in \mathfrak{p}} \sup_{x \in P} f(x) > \lambda_1 \longrightarrow \sup_{x \in P} f(x) > \lambda_1$ all $P \in \mathfrak{p} \longrightarrow P \cap V \neq \phi$ for all $P \in \mathfrak{p}$.

Suppose, now, that for all $P \in \mathfrak{p}$, there exists a $\mathfrak{q} \leq (P)$ such that $(\sigma f)(\mathfrak{q}) \leq \lambda$. Then $(\sigma f)(\mathfrak{q}) < \lambda_2 \longrightarrow \inf_{Q \in \mathfrak{p}} \sup_{x \in Q} f(x) < \lambda_2$ implies there exists a $Q \in \mathfrak{q}$ such that $\sup_{x \in Q} f(x) < \lambda_2$; and hence that $Q \subseteq W \subseteq B$, which implies $B \in \mathfrak{q}$. Also, $\mathfrak{q} \leq (P) \longrightarrow P \in \mathfrak{q}$. So, $P \cap B \neq \phi$, for otherwise $\mathfrak{q}$ would not be a proper dual ideal of Φ^α_α. Now, our supposition was that for every $P \in \mathfrak{p}$ a $\mathfrak{q}$ existed such that $(\sigma f)(\mathfrak{q}) \leq \lambda$. Since B is a fixed set independent of P, this implies $B \cap P \neq \phi$ all $P \in \mathfrak{p} \longrightarrow B \in \mathfrak{p}$, since $\mathfrak{p}$ is minimal. Thus $B \cap V = \phi$ contradicts the fact that $P \cap V \neq \phi$ for all $P \in \mathfrak{p}$. Hence, for some $P \in \mathfrak{p}$, $(\sigma f)(\mathfrak{q}) > \lambda$ for all $\mathfrak{q} \in \{\mathfrak{q} | \mathfrak{q} \leq (P)\}$. Thus $\mathfrak{p}$ is contained in an open set contained in $\mathfrak{U}^*_\lambda$. Since $\mathfrak{p}$ was arbitrary, this implies

$\mathfrak{A}^*_\lambda$ is open and the lemma is proved.

Lemma 2.4.2. If $f \epsilon B_a(s)$, then $(\sigma f)(\mathfrak{p}_x) = f(x)$.

Proof: We have $(\sigma f)(\mathfrak{p}_x) = \inf_{P \epsilon \mathfrak{p}_x} \sup_{y \epsilon P} f(y)$. For any $P \epsilon \mathfrak{p}_x$, $x \epsilon P$ implies $\sup_{y \epsilon P} f(y) \geq f(x)$ and hence $(\sigma f)(\mathfrak{p}_x) \geq f(x)$. Suppose $(\sigma f)(\mathfrak{p}_x) > f(x)$. Choose λ_1, λ_2 such that $(\sigma f)(\mathfrak{p}_x) > \lambda_1 > \lambda_2 > f(x)$. Let $V = \{y|f(y) \leq \lambda_2\}$, $W = \{y|f(y) \geq \lambda_1\}$. Since $f \epsilon B_a(s) = D_a(\mathfrak{S})$, there exists a set $A \epsilon \Phi^a_a(\mathfrak{S})$, such that $V \subseteq A$, $W \cap A = \phi$. Now, $x \epsilon V \subseteq A \longrightarrow A \epsilon \mathfrak{p}_x$. Also $A \subseteq W^c \longrightarrow \sup_{y \epsilon A} f(y) \leq \sup_{y \epsilon W^c} f(y) \leq \lambda_1$. Hence,

$(\sigma f)(\mathfrak{p}_x) = \inf_{P \epsilon \mathfrak{p}_x} \sup_{y \epsilon P} f(y) \leq \sup_{y \epsilon A} f(y) \leq \lambda_1 < (\sigma f)(\mathfrak{p}_x)$. This contradiction proves the lemma.

Corollary: If $f_1, f_2 \epsilon B_a(s)$, then $\sigma f_1 = \sigma f_2 \longrightarrow f_1 = f_2$.

Proof: $\sigma f_1 = \sigma f_2 \longrightarrow (\sigma f_1)(\mathfrak{p}_x) = \sigma f_2(\mathfrak{p}_x)$ all $x \epsilon S$. Hence, by lemma 2.4.2, $f_1(x) = f_2(x)$ for all $x \epsilon S$, whence $f_1 = f_2$.

Lemma 2.4.3. If $F \epsilon C(R, \Sigma_a)$, then $\tau F \epsilon B_a(s)$. Further, $\sigma \cdot \tau F = F$.

Proof: Let $V_\lambda = \{x|(\tau F)(x) < \lambda\}$, $W_\lambda = \{x|(\tau f)(x) > \lambda\}$. By theorem 1.5.1 we need only show that V_λ, W_λ are of class a multiplicative. We shall show this for V_λ, a dual proof holding for W_λ. Now, if $x \epsilon V_\lambda$, $F(\mathfrak{p}_x) = (\tau F)(x) < \lambda$. Then $F(\mathfrak{p}_x) < r < \lambda$, for some rational r less than λ. Let

$$\mathfrak{A}_\lambda = \{\mathfrak{p}|F(\mathfrak{p}) < \lambda\}, \qquad \mathfrak{A}^*_r = \{\mathfrak{p}|F(\mathfrak{p}) \leq r\}.$$

Since $F(\mathfrak{p})$ is continuous, $\mathfrak{A}_\lambda$ is open and $\mathfrak{A}^*_r$ is closed. Also, $\mathfrak{A}^*_r \subseteq \mathfrak{A}_\lambda$ for $r < \lambda$. For each $r < \lambda$ we construct an open-and-closed set, $\overline{\mathfrak{A}}_r$, such that $\mathfrak{A}^*_r \subseteq \overline{\mathfrak{A}}_r \subseteq \mathfrak{A}_\lambda$. This can be done, for instance, as follows: since the open-and-closed sets form a basis for the open sets in Σ_a, around each point of $\mathfrak{A}^*_r$ we can put an open-and-closed set contained in $\mathfrak{A}_\lambda$; since Σ_α is compact, a finite number of these cover $\mathfrak{A}^*_r$; the union of this finite number of sets is then our $\overline{\mathfrak{A}}_r$.

Now, the open-and-closed sets of Σ_α are in 1-1 correspondence with the elements of $\Phi^\alpha_\alpha(\mathfrak{S})$; that, is, there exists a $P_r \epsilon \Phi^\alpha_\alpha$ such that $\overline{\mathfrak{A}}_r = \{\mathfrak{p} | \mathfrak{p} \lesssim (P_r)\}$. Then we have

$$(*) \qquad V_\lambda = \bigcup_{r<\lambda} P_r$$

For $x \epsilon V_\lambda \longrightarrow F(\mathfrak{p}_x) < r < \lambda \longrightarrow \mathfrak{p}_x \epsilon \mathfrak{A}^*_r \subseteq \overline{\mathfrak{A}}_r \longrightarrow x \epsilon P_r$ for some r. Whence, $\bigcup_{r<\lambda} P_r \supseteq V_\lambda$. But if $y \epsilon P_r$, then $\mathfrak{p}_y \epsilon \overline{\mathfrak{A}}_r \subseteq \mathfrak{A}_\lambda \longrightarrow F(\mathfrak{p}_y) < \lambda \longrightarrow (\tau F)(y) < \lambda \rightarrow y \epsilon V_\lambda$. So, $P_r \subseteq V_\lambda$ all $r < \lambda \longrightarrow \bigcup_{r<\lambda} P_r \subseteq V_\lambda$. Hence (*) is proved. But the P_r are of class α ambiguous and so (*) asserts that V_λ is of class α additive. Hence, $\tau F \epsilon D_\alpha(\mathfrak{S}) = B_\alpha(s)$.

Finally, we observe that, by lemma 10, $(\sigma\tau F)(\mathfrak{p}_x) = (\tau F)(x)$, and so by the definition 2.4.2, we have $(\sigma\tau F)(\mathfrak{p}_x) = F(\mathfrak{p}_x)$. Thus $\sigma\tau F = F$ at all points of $\mathfrak{B}$. Since by lemma 2.4.1 and what has been proved above both functions are continuous, the corollary to lemma 2.3.2 implies $\sigma\tau F = F$ for all points of Σ_α.

2.5. *Proof of the Representation theorem:* By lemma 2.4.1 and the corollary to lemma 2.4.2, the mapping $f \longrightarrow \sigma f$ is a 1-1 mapping of $B_\alpha(s)$ into $C(R, \Sigma_\alpha)$. By lemma 2.4.3, every function in $C(R, \Sigma_\alpha)$ is the image of some function in $B_\alpha(s)$, so the mapping in 1-1 onto. It is clearly order preserving. Hence, the theorem follows.

2.6. Now $\Phi^\alpha_\alpha(\mathfrak{S})$ is a sublattice of the lattice of sets of class α multiplicative. Hence, by lemma 1.3.1 and theorem 2.2.1, we immediately get the following form of the representation theorem.

Theorem 2.6.1. If $\alpha > 0$, then the Baire functions of class α over s are isomorphic, as a lattice, to the continuous functions on the Wallman space associated with the distributive lattice of the sets of class α multiplicative over $\mathfrak{S}$.

Also, in case we start with a complete system s instead of an admissible set $\mathfrak{S}$, or in case the space S is *not* $\mathfrak{S}$-normal, then in view of the remarks of section 1.6 (and in the notation of that section) the representation theorem takes the following form:

Theorem 2.6.2. Let s be any complete system of functions on the topological space S. Then, if $\alpha > 0$, the Baire functions of class α over s are isomorphic, as a lattice, to the continuous functions on the Boolean space associated with the Boolean algebra $\Phi_{\alpha}^{\alpha}(\mathfrak{S})$ of sets ambiguous of class α over $\mathfrak{S}$.

2.7. *Examples.* Let $\mathfrak{S}$ be the collection of closed subsets of a topological space S, which are, at the same time, denumerable intersections of open sets. It is clear that $\mathfrak{S}$ is an admissible collection. Then the collection of functions $s = D_0(\mathfrak{S})$ is just the set of all continuous functions. The collections $B_{\alpha}(s)$ are then the usual Baire functions of class α. The condition of $\mathfrak{S}$-normality is then normality in the ordinary sense. Applied to the functions of class 1, the representation theorem gives the result that the functions of Baire class 1, on a normal space S, are isomorphic to the lattice of continuous functions on the Boolean space associated with the Boolean algebra of subsets of S which are both denumerable unions of closed sets (in $\mathfrak{S}$) and denumerable intersections of open sets (in $\mathfrak{S}$). Corresponding results hold for functions of higher classes.

In a metric space, which is always normal, every closed set is an intersection of open sets. Hence, the remarks above apply, for a metric space, to the collection $\mathfrak{S}$ of all closed subsets of S. We also note, that for a metric space $\mathfrak{G} = \mathfrak{F}$. For if A is any closed set, then

$$A = \{x | p(x, A) = 0\}$$

where $p(x, A)$ is the continuous function measuring the distance of the point x from the closed set A. Applying these results to the real line, we conclude with:

Theorem 2.7.1. Let R be the closed real interval $\langle 0, 1 \rangle$. Then the Baire functions of class 1 on R are isomorphic, as a lattice, to the continuous functions on the Wallman space associated with the subsets of R which are denumerable intersections of open sets.

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