

# Handbook of Teichmüller Theory Volume I

Athanase Papadopoulos  
Editor



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Teichmüller Theory  
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