5.4. Limited formulas, limited indiscernibles, x-definability, normal form.

We fix $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...)$ as given by Lemma 5.3.18.

DEFINITION 5.4.1. Let L(E) be the first order predicate calculus with equality, using $<,0,1,+,-,\bullet,\uparrow,\log,E$, where E is 1-ary. The c's are not included in L(E). We will always write $t \in E$ instead of $E(t)$.

We follow the convention that $\varphi(v_1,...,v_k)$ represents a formula of L(E) whose free variables are among $v_1,...,v_k$. This does not require that v_k be free or even appear in φ. Recall that all variables are of the form v_n, where $n \geq 1$.

In this section, we will only be concerned with what we call the E formulas of L(E).

DEFINITION 5.4.2. The E formulas of L(E) are inductively defined as follows.

i) Every atomic formula of L(E) is an E formula;
ii) If φ,ψ are E formulas then $(\neg\varphi)$, $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $(\varphi\rightarrow\psi)$, $(\varphi\leftrightarrow\psi)$ are E formulas;
iii) If φ is an E formula and $n \geq 1$, then $(\exists v_n \in E)(\varphi)$, $(\forall v_n \in E)(\varphi)$ are E formulas.

DEFINITION 5.4.3. We take

$$(\exists v_n \in E)(\varphi),\ (\forall v_n \in E)(\varphi)$$

to be abbreviations of

$$(\exists v_n)(v_n \in E \wedge \varphi),\ (\forall v_n)(v_n \in E \rightarrow \varphi).$$

Although general formulas of L(E) will arise in this section, attention will be focused on their relativizations, which are E formulas of L(E).

DEFINITION 5.4.4. Let $\varphi(v_1,...,v_k)$ be a formula of L(E) and v be a variable not among $v_1,...,v_k$. We let $\varphi(v_1,...,v_k)^v$ be the result of bounding all quantifiers in $\varphi(v_1,...,v_k)$ to

$$E \cap [0,v].$$

I.e., we replace each quantifier

$(\forall u)$ by $(\forall u \in E \cap [0,v])$
$(\exists u)$ by $(\exists u \in E \cap [0,v])$.

These bounded quantifiers should be expanded in the usual way to create an actual formula in L(E).

We now define a very important 6-ary relation.

DEFINITION 5.4.5. We define $A(r,n,m,\varphi,a,b)$ if and only if

i) $r,n,m,a,b \in N\backslash\{0\}$, $n < m$;
ii) $\varphi = \varphi(v_1,\ldots,v_r)$ is a formula of L(E); i.e., all free variables of φ are among $v_1,\ldots,v_r$;
iii) Let $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then $\varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$ $aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E$.

LEMMA 5.4.1. Let $r,n \geq 1$ and $\varphi(v_1,\ldots,v_r)$ be a quantifier free formula of L. There exist a,b such that $A(r,n,n+1,\varphi,a,b)$.

Proof: Let r,n,φ be as given. Note that $\varphi^{c\_n} = \varphi$.

By Lemma 5.3.18 vii), let $a,b \in N\backslash\{0\}$ be such that the following holds. Let $n \geq 1$ and $x_1,\ldots,x_r \in E \cap [0,c_n]$.

$$(\exists y \in E)(y \leq |c_n,x_1,\ldots,x_r|\uparrow\uparrow \wedge y \leq |x_1,\ldots,x_r| \wedge \varphi(c_n,x_1,\ldots,x_r,y))$$
$$\leftrightarrow$$
$$(\exists y \in E)(y \leq |c_n,x_1,\ldots,x_r|\uparrow\uparrow \wedge \rho(c_n,x_1,\ldots,x_r,y))$$
$$\leftrightarrow$$
$$aCODE(c_{n+1};c_n,x_1,\ldots,x_r)+b \in E.$$

$$\varphi(x_1,\ldots,x_r) \leftrightarrow$$
$$aCODE(c_{n+1};c_n,x_1,\ldots,x_r)+b \in E.$$

Hence $A(r,n,n+1,\varphi,a,b)$. QED

Note that in the proof of Lemma 5.4.1, the second displayed formula is subject to Lemma 5.3.18 vii). The formula ρ used can be read off easily from the first displayed formula. We will be using this style of exposition throughout this section.

LEMMA 5.4.2. Let $r \geq 1$ and φ be $t \in E$, where $t(v_1,\ldots,v_r)$ is a term of L. There exist a,b such that the following holds. Let $n \geq 1$. Then $A(r,n,n+1,\varphi,a,b)$.

Proof: Let r,φ,t be as given.

Let $p \geq 2$ be such that for all $x_1,\ldots,x_r \in A$, $t(x_1,\ldots,x_r) \leq {\uparrow}p(|x_1,\ldots,x_r|)$. By Lemma 5.3.18 vii), let $a,b \in N\backslash\{0\}$ be such that the following holds. Let $n \geq 1$ and $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then

$$(\exists y \in E)(y \leq {\uparrow}p(|c_n,x_1,\ldots,x_r|) \wedge y = t(x_1,\ldots,x_r))$$
$$\leftrightarrow$$
$$(\exists y \in E)(y \leq {\uparrow}p(|c_n,x_1,\ldots,x_r|) \wedge \rho(c_n,x_1,\ldots,x_r,y))$$
$$\leftrightarrow$$
$$a\mathrm{CODE}(c_{n+1};c_n,x_1,\ldots,x_r)+b \in E.$$

$$t(x_1,\ldots,x_r) \in E \leftrightarrow$$
$$a\mathrm{CODE}(c_{n+1};c_n,x_1,\ldots,x_r)+b \in E.$$

Hence A(r,n,n+1,φ,a,b). QED

LEMMA 5.4.3. Let A(r,n,m,φ,a,b). There exist d,e such that A(r,n,m,$\neg\varphi$,d,e).

Proof: Let A(r,n,m,φ,a,b). By Lemma 5.3.18 vi), fix $i,j \in N\backslash\{0\}$ such that the following holds. Let $x_1 \in \alpha(E;1,<\infty)$. Then

$$(\exists x_2 \in E)(x_2 \leq x_1 \wedge x_2 = x_1) \leftrightarrow$$
$$ix_1+j \notin E.$$

Clearly for all $x_1 \in \alpha(E;1,<\infty)$,

$$1)\ x_1 \in E \leftrightarrow ix_1+j \notin E.$$

Now let $x_1,\ldots,x_r \in E \cap [0,c_n]$. By A(r,n,m,$\varphi$,a,b),

$$2)\ \varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$a\mathrm{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E.$$

By Lemma 5.3.18 viii),

$$\mathrm{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r) \in E.$$

By 1),

$$3)\ a(\mathrm{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r))+b \in E \leftrightarrow$$
$$ia(\mathrm{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r))+ib+j \notin E.$$

By 2),3),

$$\neg\varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$ia(\text{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r))+ib+j \in E.$$

Therefore A(r,n,m,¬φ,ia,ib+j). QED

LEMMA 5.4.4. Let a,b,d,e ∈ N\{0}. There exist f,g ∈ N\{0} such that the following holds. Let w ∈ α(E;1,<∞). Then (aw+b ∈ E ∧ dw+e ∈ E) ↔ fw+g ∈ E.

Proof: Let a,b,d,e ∈ N\{0}. Let p = max(a,b,d,e).

By Lemma 5.3.18 vi), let f,g ∈ N\{0} such that the following holds. Let w ∈ α(E;1,<∞). Then

$$(\exists y,z \in E)(y,z \leq pw+p \wedge y = aw+b \wedge z = cw+d) \leftrightarrow$$
$$fw+g \in E.$$

$$(\exists y,z \in E)(y = aw+b \wedge z = cw+d) \leftrightarrow$$
$$fw+g \in E.$$

$$(aw+b \in E \wedge cw+d \in E) \leftrightarrow$$
$$fw+g \in E.$$

QED

LEMMA 5.4.5. Let A(r,n,m,φ,a,b) and A(r,n,m,ψ,d,e). There exist f,g such that A(r,n,m,φ∧ψ,f,g).

Proof: Assume A(r,n,m,φ,a,b), A(r,n,m,ψ,d,e). Let $x_1,\ldots,x_r \in E \cap [0,c_n]$, Then

$$1)\ \varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$a\text{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E.$$

$$\psi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$d\text{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_e)+e \in E.$$

Let f,g be given by Lemma 5.4.4 using a,b,d,e. By Lemma 5.3.18 viii),

$$\text{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r) \in E.$$

Hence by Lemma 5.4.4,

$$2)\ a\text{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E \wedge$$

$$dCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+e \in E$$
$$\leftrightarrow$$
$$fCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+g \in E.$$

By 1),2),

$$((\varphi\wedge\psi)(x_1,\ldots,x_r))^{c\_n}$$
$$\leftrightarrow$$
$$\varphi(x_1,\ldots,x_r)^{c\_n} \wedge \psi(x_1,\ldots,x_r)^{c\_n}$$
$$\leftrightarrow$$
$$fCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+g \in E.$$

QED

LEMMA 5.4.6. Let $1 \leq i \leq r$ and $A(r,n,m,\varphi,a,b)$. There exists d,e such that $A(r,n,m+1,(\exists v_i)(\varphi),d,e)$.

Proof: Let $1 \leq i \leq r$ and $A(r,n,m,\varphi,a,b)$. Let $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then

$$1)\ \varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E.$$

By Lemma 5.3.18 vii), let $d,e \in N\backslash\{0\}$ be such that the following holds, using m for n. Let $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then

$$(\exists x_i,w \in E)(x_i,w \leq c_m\uparrow\uparrow \wedge x_i \leq c_n \wedge$$
$$w = aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b))$$
$$\leftrightarrow$$
$$(\exists z,w \in E)(z,w \leq c_m\uparrow\uparrow \wedge z \leq c_n \wedge$$
$$w = aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_{i-1},z,x_{i+1},\ldots x_r)+b))$$
$$\leftrightarrow$$
$$(\exists z,w \in E)(z,w \leq c_m\uparrow\uparrow \wedge \rho(c_n,\ldots,c_m,x_1,\ldots,x_r,z,w))$$
$$\leftrightarrow$$
$$dCODE(c_{m+1};c_n,\ldots,c_m,x_1,\ldots,x_r)+e \in E.$$

$$(\exists x_i,w \in E)(x_i \leq c_n \wedge$$
$$w = aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b))$$
$$\leftrightarrow$$
$$dCODE(c_{m+1};c_n,\ldots,c_m,x_1,\ldots,x_r)+e \in E.$$

$$2)\ (\exists x_i \in E)(x_i \leq c_n \wedge$$
$$aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E)$$
$$\leftrightarrow$$
$$dCODE(c_{m+1};c_n,\ldots,c_m,x_1,\ldots,x_r)+e \in E.$$

By 1),2),

$$(\exists x_i \in E)(x_i \leq c_n \wedge \varphi(x_1,\ldots,x_r)^{c\_n}) \leftrightarrow$$
$$dCODE(c_{m+1};c_n,\ldots,c_m,x_1,\ldots,x_r)+e \in E.$$

$$(\exists x_i)(\varphi(x_1,\ldots,x_r))^{c\_n}$$
$$\leftrightarrow$$
$$dCODE(c_{m+1};c_n,\ldots,c_m,x_1,\ldots,x_r)+e \in E.$$

Hence $A(r,n,m+1,(\exists v_i)(\varphi),d,e)$. QED

LEMMA 5.4.7. Let $m \leq m'$ and $A(r,n,m,\varphi,a,b)$. There exist d,e such that $A(r,n,m',\varphi,d,e)$.

Proof: Let $m < m'$ and $A(r,n,m,\varphi,a,b)$. Let $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then

$$1)\ \varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E.$$

By Lemma 5.3.18 vii), let $d,e \in N\backslash\{0\}$ be such that the following holds. Let $x_1,\ldots,x_r \in E \cap [0,c_n]$. Then

$$(\exists y \in E)(y \leq c_{m'}\uparrow\uparrow \wedge y = aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b)$$
$$\leftrightarrow$$
$$(\exists y \in E)(y \leq c_{m'}\uparrow\uparrow \wedge \rho(c_n,\ldots,c_{m'-1},x_1,\ldots,x_r,y)$$
$$\leftrightarrow$$
$$dCODE(c_{m'};c_n,\ldots,c_{m'-1},x_1,\ldots,x_r)+e \in E.$$

$$(\exists y \in E)(y = aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b) \leftrightarrow$$
$$dCODE(c_{m'};c_n,\ldots,c_{m'-1},x_1,\ldots,x_r)+e \in E.$$

$$2)\ aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E \leftrightarrow$$
$$dCODE(c_{m'};c_n,\ldots,c_{m'-1},x_1,\ldots,x_r)+e \in E.$$

By 1),2),

$$\varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow$$
$$dCODE(c_{m'};c_n,\ldots,c_{m'-1},x_1,\ldots,x_r)+e \in E.$$

Therefore $A(r,n,m',\varphi,d,e)$. QED

LEMMA 5.4.8. Let $r \leq r'$ and $A(r',n,m,\varphi,a,b)$, where all free variables of φ are among $v_1,\ldots,v_r$. There exist d,e such that $A(r,n,m+1,\varphi,d,e)$.

Proof: Let r,r'n,m,φ,a,b be as given. By A(r',n,m,φ,a,b), for all $x_1,...,x_{r'} \in E \cap [0,c_n]$,

$$1)\ \varphi(x_1,...,x_{r'})^{c\_n} \leftrightarrow aCODE(c_m;c_n,...,c_{m-1},x_1,...,x_{r'})+b \in E.$$

Note that $\varphi(x_1,...,x_{r'})^{c\_n} = \varphi(x_1,...,x_r)^{c\_n}$. Hence for all $x_1,...,x_r \in E \cap [0,c_n]$,

$$2)\ \varphi(x_1,...,x_r)^{c\_n} \leftrightarrow aCODE(c_m;c_n,...,c_{m-1},x_1,...,x_r,x_r,...,x_r)+b \in E.$$

By Lemma 5.3.18 vii), let d,e ∈ N\{0} be such that the following holds. Let $x_1,...,x_r \in E \cap [0,c_n]$. Then

$$(\exists z \in E)(z \leq c_m\uparrow\uparrow \wedge z = aCODE(c_m;c_n,...,c_{m-1},x_1,...,x_r,x_r,...,x_r))$$
$$\leftrightarrow$$
$$(\exists z \in E)(z \leq c_n\uparrow\uparrow \wedge \rho(c_n,...,c_m,x_1,...,x_r,z)$$
$$\leftrightarrow$$
$$dCODE(c_{m+1};c_n,...,c_m,x_1,...,x_r)+e \in E.$$

$$3)\ aCODE(c_m;c_n,...,c_{m-1},x_1,...,x_r,x_r,...,x_r)+b \in E \leftrightarrow dCODE(c_{m+1};c_n,...,c_m,x_1,...,x_r)+e \in E.$$

By 2),3), for all $x_1,...,x_r \in E \cap [0,c_n]$,

$$\varphi(x_1,...,x_r)^{c\_n} \leftrightarrow dCODE(c_{m+1};c_n,...,c_m,x_1,...,x_r)+e \in E.$$

Hence A(r,n,m+1,φ,d,e). QED

LEMMA 5.4.9. Let r,n ≥ 1 and $\varphi(x_1,...,x_r)$ be a formula of L(E). There exists m,a,b such that A(r,n,m,φ,a,b).

Proof: By induction on the complexity of φ. Without loss of generality, we can assume that φ uses only the connectives ¬,∧, and only the quantifier ∃. For our purposes, we define c(φ) as the total number of occurrences of connectives and quantifiers in φ.

We prove the following by induction on p ≥ 0. Let r,n ≥ 1 and $\varphi(v_1,...,v_r)$ be a formula of L(E) with c(φ) ≤ p. There exist m,a,b such that A(r,n,m,φ,a,b).

We first handle the basis case p = 0. Let r,n,φ be as given. Then φ has no connectives and no quantifiers, and so

φ is an atomic formula of L(E). Now use Lemmas 5.4.1 and 5.4.2 with m = n+1.

Now assume that the statement holds of $p \geq 0$. Let $r,n \geq 1$ and $\varphi(v_1,\ldots,v_r)$ be a formula of L(E) with $c(\varphi) = p+1$.

case 1. $\varphi(v_1,\ldots,v_r) = \neg\psi(v_1,\ldots,v_r)$. By the induction hypothesis, let $A(r,n,m,\psi,a,b)$. By Lemma 5.4.3, there exist d,e such that $A(r,n,m,\varphi,d,e)$.

case 2. $\varphi(v_1,\ldots,v_r) = \psi(v_1,\ldots,v_r) \wedge \rho(v_1,\ldots,v_r)$. By the induction hypothesis, let $A(r,n,m,\psi,a,b)$,$A(r,n,m',\rho,d,e)$. By Lemma 5.4.7, let $A(r,n,\max(m,m'),\psi,a',b')$, $A(r,n,\max(m,m'),\rho,d',e')$. By Lemma 5.4.5, there exists f,g such that $A(r,n,\max(m,m'),\varphi,f,g)$.

case 3. $\varphi(v_1,\ldots,v_r) = (\exists v_i)(\psi)$, $1 \leq i \leq r$. Then we can write $\psi = \psi(v_1,\ldots,v_r)$ because has all free variables of φ are among $v_1,\ldots,v_r$. By the induction hypothesis, let $A(r,n,m,\psi,a,b)$. By Lemma 5.4.6, there exist d,e such that $A(r,n,m+1,\varphi,d,e)$.

case 4. $\varphi(v_1,\ldots,v_r) = (\exists v_i)(\psi)$, $i > r$. Then ψ has all free variables among $v_1,\ldots,v_i$, and we an write $\psi = \psi(v_1,\ldots,v_i)$. By the induction hypothesis, let $A(i,n,m,\psi,a,b)$. By Lemma 5.4.6, let $A(i,n,m+1,\varphi,d,e)$. By Lemma 5.4.8, there exists f,g such that $A(r,n,m+2,\varphi,f,g)$.

QED

We now extend the indiscernibility in Lemma 5.3.18 iv) to formulas.

LEMMA 5.4.10. Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula in L(E). Let $1 \leq i_1,\ldots,i_{2r} < n$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and the same min. Let $y_1,\ldots,y_r \in E$, $y_1,\ldots,y_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then $\varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)^{c\_n} \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)^{c\_n}$.

Proof: Let $r,\varphi,i_1,\ldots,i_{2r}$ be as given. Let $n > i_1,\ldots,i_{2r}$. By Lemma 5.4.9, let m,a,b be such that the following holds. For all $x_1,\ldots,x_{2r} \in E \cap [0,c_n]$,

$$\varphi(x_1,\ldots,x_{2r})^{c\_n} \leftrightarrow$$
$$a\mathrm{CODE}(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_{2r})+b \in E.$$

Let $y_1,\ldots,y_r$ be as given. Then

$$1)\ \varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)^{c\_n} \leftrightarrow$$
$$aCODE(c_m{:}c_n,\ldots,c_{m-1},c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \in E.$$

$$\varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)^{c\_n} \leftrightarrow$$
$$aCODE(c_m;c_n,\ldots,c_{m-1},c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \in E.$$

By Lemma 5.3.18 ix),

$$2)\ aCODE(c_m;c_n,\ldots,c_{m-1},c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \in E \leftrightarrow$$
$$aCODE(c_m;c_n,\ldots,c_{m-1},c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \in E.$$

By 1),2),

$$\varphi(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)^{c\_n} \leftrightarrow$$
$$\varphi(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)^{c\_n}.$$

QED

LEMMA 5.4.11. Let $k,n \geq 1$ and $\varphi(v_1,\ldots,v_k)$ be a formula of L(E). There exist $m,a,b \in N\backslash\{0\}$, $n < m$, and $y_n,\ldots,y_m \in E \cap [0,c_{n+1}]$ such that for all $x_1,\ldots,x_k \in E \cap [0,c_n]$,

$$\varphi(x_1,\ldots,x_k)^{c\_n} \leftrightarrow$$
$$aCODE(y_m;y_n,\ldots,y_{m-1},x_1,\ldots,x_k)+b \in E.$$

Proof: Let k,n,φ be as given. By Lemma 5.4.9, there exist $m,a,b \in N\backslash\{0\}$ such that

$$1)\ (\forall x_1,\ldots,x_k \in E \cap [0,c_n])(\varphi(x_1,\ldots,x_r)^{c\_n}$$

$$\leftrightarrow$$

$$aCODE(c_m;c_n,\ldots,c_{m-1},x_1,\ldots,x_r)+b \in E).$$

$$2)\ (\exists y_n,\ldots,y_m \in E)(\forall x_1,\ldots,x_k \in E \cap [0,c_n])$$
$$(\varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow aCODE(y_m;y_n,\ldots,y_{m-1},x_1,\ldots,x_r)+b \in E).$$

$$3)\ (\exists y_n,\ldots,y_m \in E)(\forall x_1,\ldots,x_k \in E \cap [0,c_n])$$
$$(\varphi(x_1,\ldots,x_r)^{c\_n} \leftrightarrow (\exists z \in E)(z = aCODE(y_m;y_n,\ldots,y_{m-1},x_1,\ldots,x_r)+b)).$$

By Lemma 5.3.18 iii), choose t so large that

$$4)\ (\exists y_n,\ldots,y_m \in E \cap [0,c_t])(\forall x_1,\ldots,x_k \in E \cap [0,c_n])(\varphi(x_1,\ldots,x_r)^{c\_n}$$

$\leftrightarrow$

$(\exists z \in E \cap [0,c_{t+1}])(z = aCODE(y_m;y_n,...,y_{m-1},x_1,...,x_r)+b))$.

By Lemma 5.4.10,

5) $(\exists y_n,...,y_m \in E \cap [0,c_{n+1}])(\forall x_1,...,x_k \in E \cap [0,c_n])(\varphi(x_1,...,x_r)^{c\_n} \leftrightarrow$

$(\exists z \in E \cap [0,c_{n+2}])(z = aCODE(y_m;y_n,...,y_{m-1},x_1,...,x_r)+b))$.

6) $(\exists y_n,...,y_m \in E \cap [0,c_{n+1}])(\forall x_1,...,x_k \in E \cap [0,c_n])(\varphi(x_1,...,x_r)^{c\_n} \leftrightarrow$

$(\exists z \in E)(z = aCODE(y_m;y_n,...,y_{m-1},x_1,...,x_r)+b))$.

7) $(\exists y_n,...,y_m \in E \cap [0,c_{n+1}])(\forall x_1,...,x_k \in E \cap [0,c_n])(\varphi(x_1,...,x_r)^{c\_n} \leftrightarrow$

$aCODE(y_m;y_n,...,y_{m-1},x_1,...,x_r)+b \in E)$.

QED

Recall that $\alpha(E)$ is the set of all values of terms in L at arguments from E (Definition 5.3.3).

LEMMA 5.4.12. $\alpha(E) = E-E$. Let $n \geq 1$. $\alpha(E \cap [0,c_n]) \subseteq (E \cap [0,c_{n+1}]) - (E \cap [0,c_{n+1}])$.

Proof: Since $E-E \subseteq \alpha(E)$, it suffices to prove $\alpha(E) \subseteq E-E$. Let $t(v_1,...,v_k)$ be a term in L, and let $x_1,...,x_k \in E$. Let n be such that $x_1,...,x_k < c_n$.

Note that by Lemma 5.3.18 iv),v),

$$t(x_1,...,x_k) < c_{n+1}.$$
$$2c_{n+1}+t(x_1,...,x_k),3c_{n+1}+t(x_1,...,x_k) \in \alpha(E;1,<\infty).$$
$$3(2c_{n+1}+t(x_1,...,x_k))+1,\ 2(3c_{n+1}+t(x_1,...,x_k))+1 \in E.$$
$$6c_{n+1}+3t(x_1,...,x_k)+1,\ 6c_{n+1}+2t(x_1,...,x_k)+1 \in E.$$
$$(6c_{n+1}+3t(x_1,...,x_k)+1)-(6c_{n+1}+2t(x_1,...,x_k)+1) = t(x_1,...,x_k) \in E-E.$$

Thus we have written $t(x_1,...,x_k)$ as the difference between two elements of E. This establishes the first claim.

For the second claim, let $n \geq 1$. Let $t(v_1,...,v_k)$ be a term in L. By the proof of the first claim,

1) $(\forall x_1,...,x_k \in E \cap [0,c_n])(\exists y,z \in E \cap [0,c_{n+2}])$ $(t(x_1,...,x_k) = y-z)$.

By Lemma 5.4.10,

$$2)\ (\forall x_1,\ldots,x_k \in E \cap [0,c_n])(\exists y,z \in E \cap [0,c_{n+1}])\ (t(x_1,\ldots,x_k) = y-z).$$

QED

LEMMA 5.4.13. Let $k,r \geq 1$ and $x_1,\ldots,x_k,y_1,\ldots,y_r \in A$. Then $P(y_1,\ldots,y_r,x_1,\ldots,x_k) = P(P(y_1,\ldots,y_r),x_1,\ldots,x_k)$.

Proof: Recall the definition of P in section 5.3, right after the proof of Lemma 5.3.10. We prove the following by induction on $r \geq 1$:

$$\text{for all } k \geq 1 \text{ and } x_1,\ldots,x_k,y_1,\ldots,y_r \in A,\ P(y_1,\ldots,y_r,x_1,\ldots,x_k) = P(P(y_1,\ldots,y_r),x_1,\ldots,x_k).$$

For the basis case $r = 1$, this asserts that for all $k \geq 1$

$$P(y_1,x_1,\ldots,x_k) = P(P(y_1),x_1,\ldots,x_k)$$

which follows from $P(y_1) = y_1$.

Fix $r \geq 1$. Suppose that for all $k \geq 1$ and $x_1,\ldots,x_k,y_1,\ldots,y_r$,

$$1)\ P(y_1,\ldots,y_r,x_1,\ldots,x_k) = P(P(y_1,\ldots,y_r),x_1,\ldots,x_k).$$

We want to verify that for all $k \geq 1$ and $x_1,\ldots,x_k,y_1,\ldots,y_{r+1}$,

$$P(y_1,\ldots,y_{r+1},x_1,\ldots,x_k) = P(P(y_1,\ldots,y_{r+1}),x_1,\ldots,x_k).$$

Let $k \geq 1$ and $x_1,\ldots,x_k,y_1,\ldots,y_{r+1} \in A$. By the induction hypothesis 1) using $k = k+1$,

$$2)\ P(y_1,\ldots,y_{r+1},x_1,\ldots,x_k) = P(P(y_1,\ldots,y_r),y_{r+1},x_1,\ldots,x_k).$$

By the definition of P,

$$3)\ P(P(y_1,\ldots,y_r),y_{r+1},x_1,\ldots,x_k) = P(P(P(y_1,\ldots,y_r),y_{r+1}),x_1,\ldots,x_k).$$

By the induction hypothesis 1) using k = 1,

$$4)\ P(P(y_1,...,y_r),y_{r+1}) = P(y_1,...,y_{r+1}).$$

By 2),3),4),

$$P(y_1,...,y_{r+1},x_1,...,x_k) = P(P(y_1,...,y_{r+1}),x_1,...,x_k)$$

as required. QED

LEMMA 5.4.14. Let k,n,r ≥ 1, and $\varphi(v_1,...,v_{r+k})$ be a formula of L(E). Let $y_1,...,y_r \in E \cap [0,c_n]$. There exist $d,e,f,g,h,i,j,p \in E \cap [0,c_{n+1}]$ such that for all $x_1,...,x_k \in E \cap [0,c_n]$,

$$\varphi(y_1,...,y_r,x_1,...,x_k)^{c\_n} \leftrightarrow (d-e)\mathrm{CODE}(f-g;h-i,x_1,...,x_k)+(j-p) \in E.$$

Proof: Let k,n,r,φ be as given. By Lemma 5.4.11, let m,a,b ∈ N\{0}, n < m, and $z_n,...,z_m \in E \cap [0,c_{n+1}]$, be such that for all $y_1,...,y_r,x_1,...,x_k \in E \cap [0,c_n]$,

$$\varphi(y_1,...,y_r,x_1,...,x_k)^{c\_n} \leftrightarrow a\mathrm{CODE}(z_m;z_n,...,z_{m-1},y_1,...,y_r,x_1,...,x_k)+b \in E.$$

By the definition of CODE introduced right after the proof of Lemma 5.3.10,

$$1)\ \varphi(y_1,...,y_r,x_1,...,x_k)^{c\_n} \leftrightarrow a(8((\log(z_m))\uparrow+P(z_n,...,z_{m-1},y_1,...,y_r,x_1,...,x_k))+1)+b \in E.$$

Now fix $y_1,...,y_r$ as given. These are in addition to the already fixed $z_n,...,z_m \in E$.

By Lemma 5.4.13 and 1), for all $x_1,...,x_k \in E \cap [0,c_n]$,

$$\varphi(y_1,...,y_r,x_1,...,x_k)^{c\_n} \leftrightarrow a(8((\log(z_m))\uparrow+P(P(z_n,...,z_{m-1},y_1,...,y_r),x_1,...,x_k))+1)+b \in E.$$

$$2)\ \varphi(y_1,...,y_r,x_1,...,x_k)^{c\_n} \leftrightarrow a\mathrm{CODE}(z_m;P(z_n,...,z_{m-1},y_1,...,y_r),x_1,...,x_k)+b \in E.$$

By Lemma 5.4.12, let a = d-e, z_m = f-g, $P(z_n,...,z_{m-1},y_1,...,y_r)$ = h-i, b = j-p, where d,e,f,g,h,i,j,p ∈ E ∩

$[0,c_{n+2}]$. Make these substitutions for $a,z_m,P(z_n,\ldots,z_{m-1},y_1,\ldots,y_r),b$, respectively, in 2).

The Lemma requires that $d,e,f,g,h,i,j,p \in E \cap [0,c_{n+1}]$, and we only have $d,e,f,g,h,i,j,p \in E \cap [0,c_{n+2}]$. However, we can apply Lemma 5.4.10 in the obvious way to reduce to $E \cap [0,c_{n+1}]$. QED

LEMMA 5.4.15. For all $k \geq 1$ there exists a term $t(v_1,\ldots,v_{k+8})$ of L(E) such that the following holds. Let $n,r \geq 1$, $\varphi(v_1,\ldots,v_{r+k})$ be a formula of L(E), and $y_1,\ldots,y_r \in E \cap [0,c_n]$. There exists $w_1,\ldots,w_8 \in E \cap [0,c_{n+1}]$ such that for all $x_1,\ldots,x_k \in E \cap [0,c_n]$,

$$\varphi(y_1,\ldots,y_r,x_1,\ldots,x_k)^{c\_n} \leftrightarrow$$
$$t(x_1,\ldots,x_k,w_1,\ldots,w_8) \in E.$$

Proof: Let $k \geq 1$. Let $t(v_1,\ldots,v_{k+8})$ be the obvious term of L(E) such that for all $x_1,\ldots,x_k,w_1,\ldots,w_8 \in A$,

$$t(x_1,\ldots,x_k,w_1,\ldots,w_8) =$$
$$(w_1-w_2)\mathrm{CODE}(w_3-w_4;w_5-w_6,x_1,\ldots,x_k)+(w_7-w_8).$$

Let $n,r,\varphi,y_1,\ldots,y_r$ be as given. By Lemma 5.4.14, there exist $w_1,\ldots,w_8 \in E \cap [0,c_{n+1}]$ such that for all $x_1,\ldots,x_k \in E \cap [0,c_n]$,

$$\varphi(y_1,\ldots,y_r,x_1,\ldots,x_k)^{c\_n} \leftrightarrow$$
$$(w_1-w_2)\mathrm{CODE}(w_3-w_4;w_5-w_6,x_1,\ldots,x_k)+(w_7-w_8) \in E.$$

QED

DEFINITION 5.4.6. Let $k \geq 1$ and $x \in E$. An x-definable k-ary relation is a relation R of the form

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,x]^k\colon \varphi(x_1,\ldots,x_p)^x\}$$

where $p \geq k$, $\varphi(v_1,\ldots,v_p)$ is a formula of L(E), and $x_{k+1},\ldots,x_p \in E \cap [0,x]$.

It is essential that x-definability requires boundedness. These are the internal relations, and this requirement is in analogy with the set/class distinct in set (class) theory.

LEMMA 5.4.16. Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. Let t_k is the term of L(E) given by Lemma 5.4.15.

There exists $y_1,...,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k: t_k(x_1,...,x_k,y_1,...,y_8) \in E\}$.

Proof: Let $n \geq 1$ and R be a c_n-definable k-ary relation. Write

$$R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k: \varphi(x_1,...,x_p)^{c\_n}\}$$

where $p \geq k \geq 1$, $\varphi(v_1,...,v_p)$ is a formula of L(E), and $x_{k+1},...,x_p \in E \cap [0,c_n]$.

We now apply Lemma 5.4.15 to the formula

$$\varphi'(v_1,...,v_p) = \varphi(v_{p-k+1},...,v_p,v_1,...,v_{p-k}).$$

We use the present $x_{k+1},...,x_p$ for the parameters $y_1,...,y_r$ in Lemma 5.4.15.

Let $y_1,...,y_8 \in E \cap [0,c_{n+1}]$, where for all $x_1,...,x_k \in E \cap [0,c_n]$,

$$\varphi'(x_{k+1},...,x_p,x_1,...,x_k)^{c\_n} \leftrightarrow \varphi(x_1,...,x_p)^{c\_n} \leftrightarrow R(x_1,...,x_k) \leftrightarrow t_k(x_1,...,x_k,z_1,...,z_8) \in E.$$

QED

Below, the new features over Lemma 5.3.18 are items vi) and vii).

LEMMA 5.4.17. There exists a countable structure $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...)$, and terms $t_1,t_2,...$ of L, where for all i, t_i has variables among $v_1,...,v_{i+8}$, such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) Let $r,n \geq 1$ and $t(v_1,...,v_r)$ be a term of L, and $x_1,...,x_r \leq c_n$. Then $t(x_1,...,x_r) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1,\ 3\alpha(E;1,<\infty)+1 \subseteq E$;
vi) Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. There exists $y_1,...,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k: t_k(x_1,...,x_k,y_1,...,y_8) \in E\}$;
vii) Let $r \geq 1$ and $\varphi(v_1,...,v_{2r})$ be a formula of L(E). Let $1 \leq i_1,...,i_{2r} < n$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $y_1,...,y_r \in E$,

$y_1,...,y_r \le \min(c_{i\_1},...,c_{i\_r})$. Then $\varphi(c_{i\_1},...,c_{i\_r},y_1,...,y_r)^{c\_n} \leftrightarrow \varphi(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r)^{c\_n}$.

Proof: The t's are given by Lemma 5.4.15. i)-v) are from Lemma 5.3.18 i)-v). vi) is by Lemma 5.4.16. vii) is by Lemma 5.4.10. QED