Goldstone's theorem and the Higgs Mechanism

Prof. Peter A Boyle

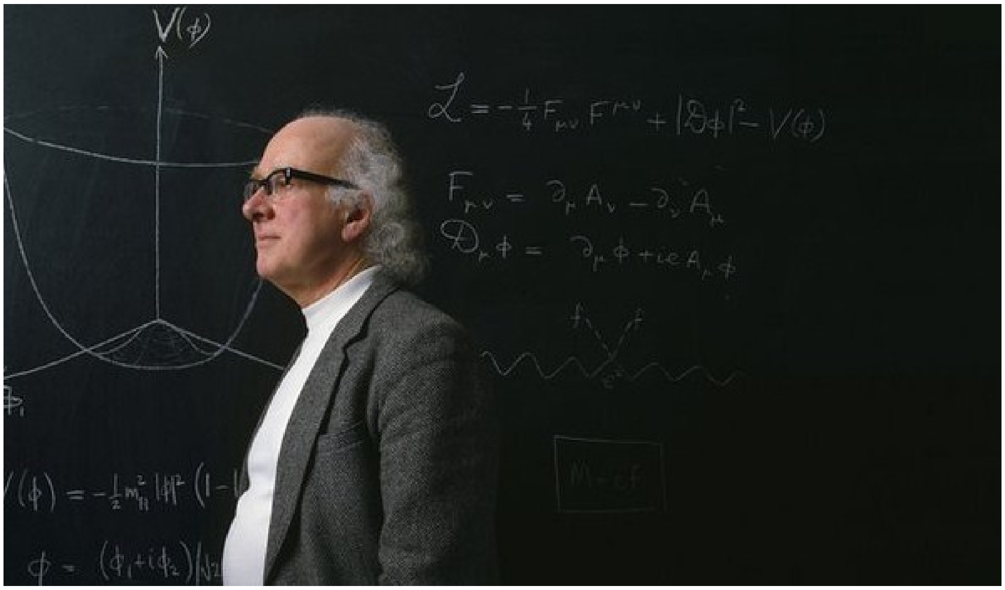
February 22, 2016

1 Outline

1. Mass terms and potentials
2. Spontaneous symmetry breaking and Superconductors
3. Goldstone's theorem
4. Higgs mechanism

2 Mass terms and potentials

The mathematical framework for the Higgs mechanism lies in the Lagrangian approach to field theory. The picture below is the Lagrangian of the simplest Higgs model, introduced by Peter Higgs in 1964.



Our aim will be to piece together how the Higgs mechanism works without using the scary equations. The content of Professor Higgs' model can be simply stated as:

Electromagnetism + Charged Higgs field ϕ + Mexican Hat potential

Here the different terms in Professor Higgs' Lagrangian are

	Equations	Term
Electromagnetism	Maxwell's equations	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$
Higgs field	Wave equation	$\lvert\mathcal{D}\phi\rvert^2$
Mexican hat potential		$V(\phi)$

Luigi del Debbio covered potential and kinetic energy in Week 1. Specifying the potential V and kinetic energy T as a function of position and velocity defines all motion of a system, and this is encoded in the Lagrangian,

$$L = T - V.$$

The *Euler-Lagrange* equations relate a Lagrangian to the corresponding equation of motion.

The approach is really useful for complex systems such as field equations, or indeed gyroscopes. However, we will skip the maths. In this lecture we can always figure out the solutions using a mechanical spring model *which shares the same Lagrangian and so the same wave equations as the Higgs field* instead to figure out how the Higgs field behaves.

That is to say if we give a stretched spring a Mexican hat potential, we can figure out how the fields in Professor Higgs' model will behave.

3 Massless Scalar fields

Richard Ball covered the massless wave equation in week 3.

In the standard model, the Higgs field is a scalar (spinless) field, similar to the field that entered Richard's wave equation. For a massless field the potential $V(\phi)$ in Prof. Higgs' model is zero.

Massless wave equation		No potential
$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dx^2}$	$\Longleftrightarrow$	$V = 0$

Exercise:
Remember the wave quanta have energy $E = h\nu$ and momentum $p = \frac{h}{\lambda}$.

Subsitute the wave

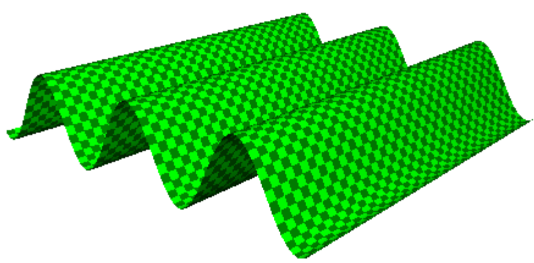
$$\phi(x,t) = \cos(Et - px)$$

into the wave equation above and differentiate to show that these fields have the massless energy-momentum relation

$$E^2 = p^2$$

One dimensional spring model

We can understand the wave equation in 3d in a simple way, by noting that we are always interested in plane wave solutions.



The field remains constant as we move perpendicular to the direction of propagation. So, we can always just consider the wave equation with one space direction, x, which is parallel to the direction of propagation. The field is constant when moving perpendicular to the direction of propagation.

A one dimensional wave equation makes things easy because the one dimensional wave equation is solved automatically by a long spring, often called a slinky. Our instinct for the plane wave solutions of the 3d wave equation can be formed by playing with a slinky.

In the above picture the profile of the end "slice" of the 2d plane wave is the same sine wave as a solution of the 1d wave equation, which we can find with the spring.

In this way, the coordinate along the spring represents *all* of space, while time represents time.

3.1 One component fields are uncharged

We can arrange that the wave displacement has only one direction by placing the slinky on a table. This corresponds to a *real* scalar field which has only one component.

This could represent an uncharged Higgs field. You should view the displacement of the spring $\phi(x,t)$ as corresponding to the value of the Higgs field at each space time point.

- For one component fields is only one class of wave solutions.
  When these are quantized we call them particles.

- If we reverse time the standing wave solutions look just the same. You can either say that the one component field has no anti-particles (or, some say, the particle is the same as the anti-particle; if there is a difference it is philosophical!).

Exercise: Experimentally test $E = p$ for wave equation
Get yourself a slinky (and involve the family! You'll need someone with a stopwatch) in verifying

$$\nu \propto \frac{1}{\lambda}$$

It is easy to verify that a straight slinky doesn't oscillate. That shows the frequency $\nu = 0$ indeed corresponds to $\lambda \to \infty$.
Keep a fixed amount of stretch in the slinky and drive it at different frequencies to hit resonant modes of oscillation. Use the spacing between fixed points to measure the wavelength.

- You should find modes where

  1. the middle oscillates and the ends are fixed
  2. the middle and ends are fixed points
  3. there are fixed points at 1/3 and 2/3 etc...

- Measure the frequency ν and the wavelength λ. Plot ν versus $\frac{1}{\lambda}$ on some graph paper to verify that $\nu \propto \frac{1}{\lambda}$.

- This shows we have the massless energy-momentum relation from our spring (up to the 3×10^8 difference in wave speed, of course!).

3.2 Two component fields are charged

However, we can allow the slinky to lie move in two directions. We'll call these $\phi_1(x,t)$ and $\phi_2(x,t)$.

Unlike the one component case, we can now have oscillations in ϕ_1, oscillations in ϕ_2, or indeed oscillations in both at the same time.

Remember ϕ_1 and ϕ_2 represent two components of the field, and x represents all of space. You can either view these two components as cartesian coordinates, or as real and imaginary parts of "complex" numbers representing the fields value.

There are now two important distinct types of oscillation:

- Clockwise

- Anti-clockwise

This is very important.

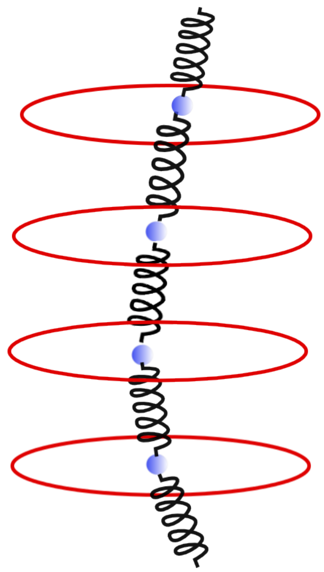
- If we reverse time the clockwise solutions turn into anticlockwise and vice versa.

- We identify the clockwise modes of oscillation with particles, and the anti-clockwise modes with antiparticles. They are related by changing $E \leftrightarrow -E$ (or $t \to -t$).

- The particle and anti-particle are now distinct species.

4 Introducing potentials to our spring model

We introduce a way of thinking that makes it easier to analyse Goldstone's theorem later. We can consider a spring as being composed of an infinite chain of small, massive balls, connected by massless springs. We will call the springs connecting consecutive balls "wave springs".



Each ball is constrained to move on a disc. The coordinate of each ball on its disc represents the field components ϕ_1 and ϕ_2 for that position along the chain.

If we want to introduce a potential $V(\phi)$ to the Lagrangian we think what would happen if we added *extra* springs to the balls, anchored to the disk.

From the de Broglie relation

$$p = \frac{h}{\lambda}$$

we see that zero momentum corresponds to an infinite wavelength mode. Such a mode is constant all over space, but may (or may not) oscillate in time.

The question "does a particle have mass" can be answered quite simply by thinking whether a straight mode oscillates (i.e. independent of the wave springs).

This oscillation in time, even when the momentum is zero, correspond to a quantum having energy when the particle is at rest.

The energy momentum relation,

$$E^2 = p^2 + m^2,$$

makes this the definition of mass.

Massless wave equation (no potential)

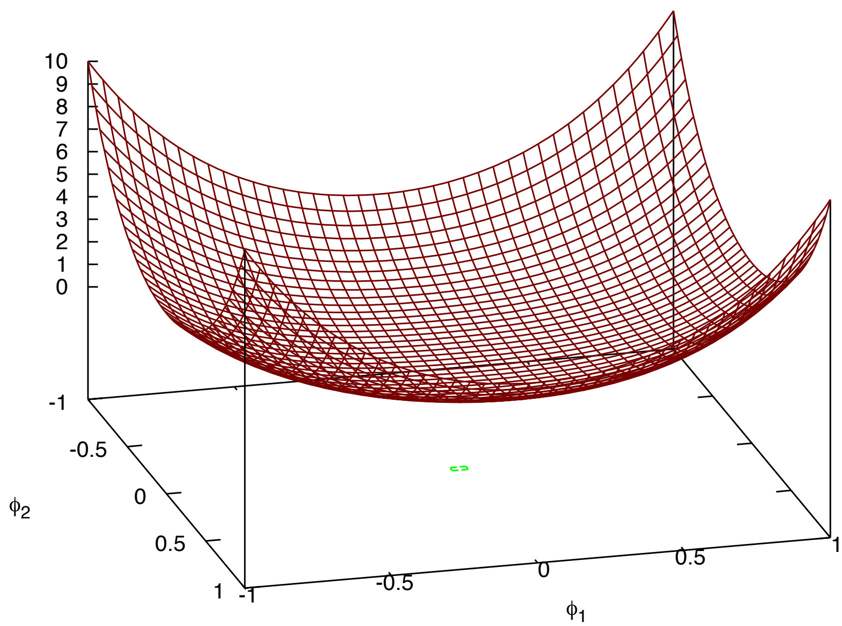
When the balls all lie in a line, the wave springs contribute no horizontal force on the balls. The zero-momentum (constant) mode does not oscillate, and as expected we see that there is no rest mass-energy for the massless wave equation Lagrangian.

It is worth noting that the solutions stay the same even if all the balls on all the discs are displaced by the same amount. This happens because the Lagrangian depends only the derivatives of ϕ and not on ϕ itself; adding a constant to ϕ does not alter the motion. This is an example of what is called "gauge invariance".

You may wish to think of gauge invariance as meaning "not caring about a shift of a field's values".

Massive wave equation (bowl potential)

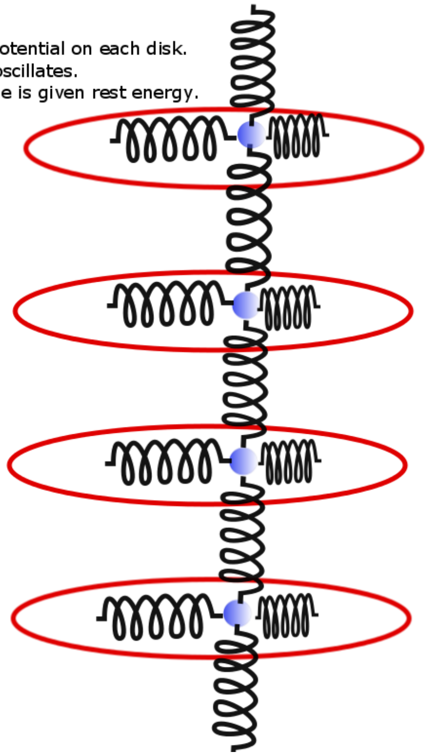
We introduce a bowl shaped potential



Massive wave equation		Potential
$\frac{d^2\phi}{dt^2} = \frac{d^2\phi}{dx^2} - m^2\phi$	$\Longleftrightarrow$	$V = \frac{1}{2}m^2\phi^2$

The Lagrangian and equations of motion are the same for both the massless spring and for the Higgs field (in one dimension, using our trick of aligning that dimension with the direction of the wave).

We will now consider introducing a bowl shaped, or quadratic, potential. We will see that we can do this by anchoring the balls on each disk to the centre ($\phi_1 = \phi_2 = 0$) with light "potential" springs.



In quantum field theory this gives spacetime an intrinsic energy density associated with non-zero field. The energy density grows as $|\phi|^2$.

In our mechanical model the "potential" springs store energy and give each mass a potential

$$V(\phi) = \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) = \frac{1}{2}m^2|\phi|^2$$

.

We find out what the rest mass-energy is by again considering the motion of a constant displacement of the slinky in this potential. The whole slinky will now oscillate due to the force from the potential springs. Since the slinky is never curved, the wave springs never influence the motion. The motion is simple harmonic motion as described by Luigi in Lecture 1.3.

This oscillation (energy) at zero momentum is what we call rest mass.

Note that by introducing the anchor springs, we no longer get the same motion if the whole slinky is translated sideways. We say *the mass term has broken gauge invariance*.

It is a general feature that introducing a potential, such as a mass term, for a field in the Lagrangian breaks shift invariance for that field. For this reason there is a deep connection between the masslessness of a photon, and the fact that electrostatic potential is defined only up to a constant: we can measure potential differences but not absolute potential.

Exercise:
Remember the plane wave quanta have energy $E = h\nu$ and momentum $p = \frac{h}{\lambda}$.

Subsitute the plane wave

$$\phi(x, t) = \cos(Et - px)$$

into the massive wave equation above and differentiate to show that these fields have the massive energy-momentum relation

$$E^2 = p^2 + m^2$$

Conclude that for zero momentum (and recognising we artificially set $c = 1$) we have found that $E = mc^2$ for scalar quanta!

5 Spontaneous symmetry breaking

We now think what might happen if the minimum of the potential is not at $\phi = 0$

The origins of the Higgs mechanism lie in the understanding of spontaneous symmetry breaking in superconductors. The works of physicists such as Ginzburg, Landau, Bardeen, Cooper, Schrieffer and Anderson are relevant.

Spontaneous symmetry breaking is quite familiar where it happens in magnets.

The magnetic potential energy is minimised if the spins of unpaired electrons in a cold ferromagnet align. With no background field the energy is the same no matter how which direction they align in. This can be thought of as a rotational invariance of the potential.

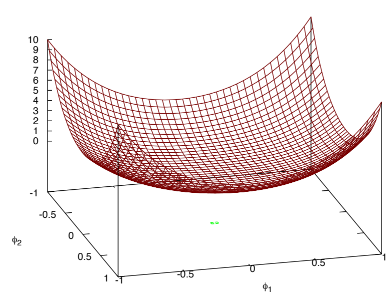
However, a direction can be *spontaneously chosen* and domains of magnetisation are formed. This is an example of spontaneous symmetry breaking.

In superconductors, electrons form "Cooper pairs" (one spin-up, one spin-down) that are weakly bound together by their interactions with vibrations in the crystal. This weak binding can only survive at low temperature, but can reduce the energy of the system.

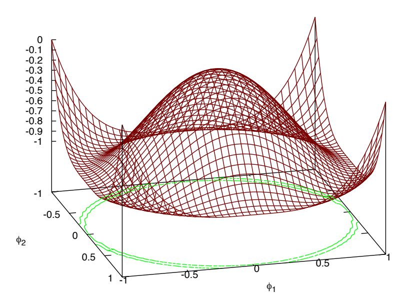
A two component (complex) wavefunction describes the Cooper pair density. The energy of the superconductor depends on the Cooper pair density in a complicated way. The Ginsburg-Landau wrote three dimensional field theory with a Mexican Hat potential for ϕ when $T < T_c$ as

$$V(\phi) = \alpha(T - T_c)|\phi|^2 + \beta|\phi|^4$$

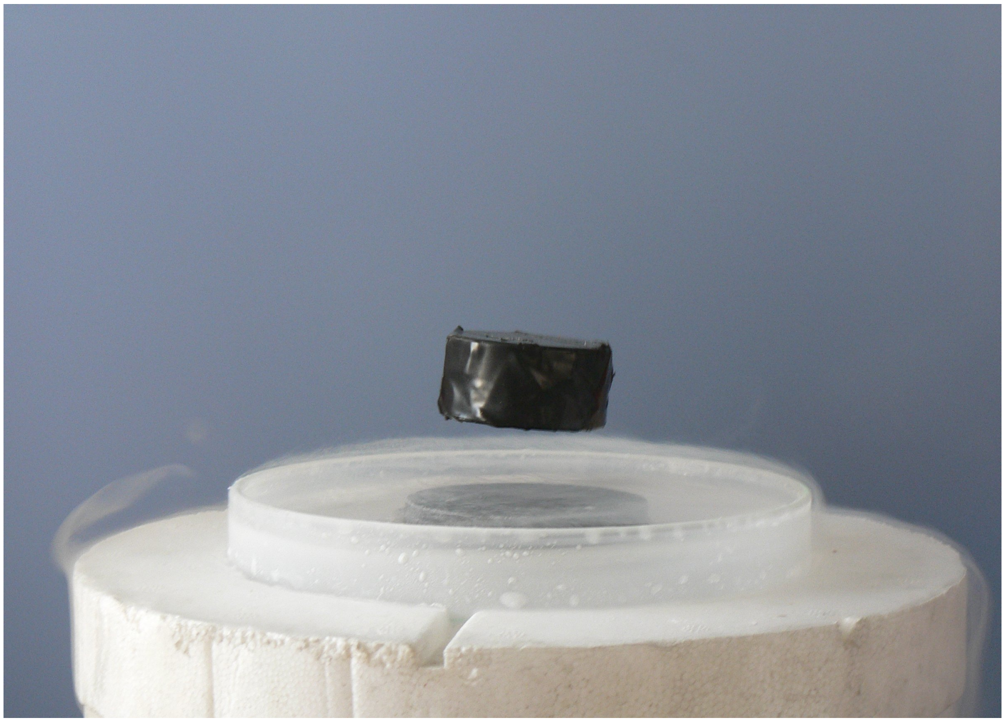
Above a critical temperature the energy is minimised if there are no Cooper pairs, and the material is a resistor. Below the critical temperature Coopers pairs condense and the material superconducts.



$T > T_c$: $\phi = 0$ is preferred



$T < T_c$: $|\phi| = v \neq 0$ is preferred
The minimum of $V(\phi)$ is a circle, and a point v must be spontaneously chosen.
We will call this the chosen *expectation value*.



Cooper pairs are charged, and respond to magnetism in a curious way: their motion modifies to eject magnetic field from the superconductor. This enables magnets to levitate on superconductors, and is used in some Maglev trains.

The magnetic field behaves as if it has mass in the superconductor, with mass related to a characteristic penetration depth known as the London penetration depth.

Some questions immediately sprang into particle physicists minds:

- can a scalar particle ϕ give mass to gauge bosons in a relativistic field theory?

Unfortunately this appeared to be ruled out by work by Yoichiro Nambu and Jeffrey Goldstone.

The barrier became known as Goldstone's theorem.

6 Goldstone's theorem

Goldstone and Nambu observed that if we break a symmetry of the Lagrangian spontaneously in a quantum field theory then massless particle species appear.

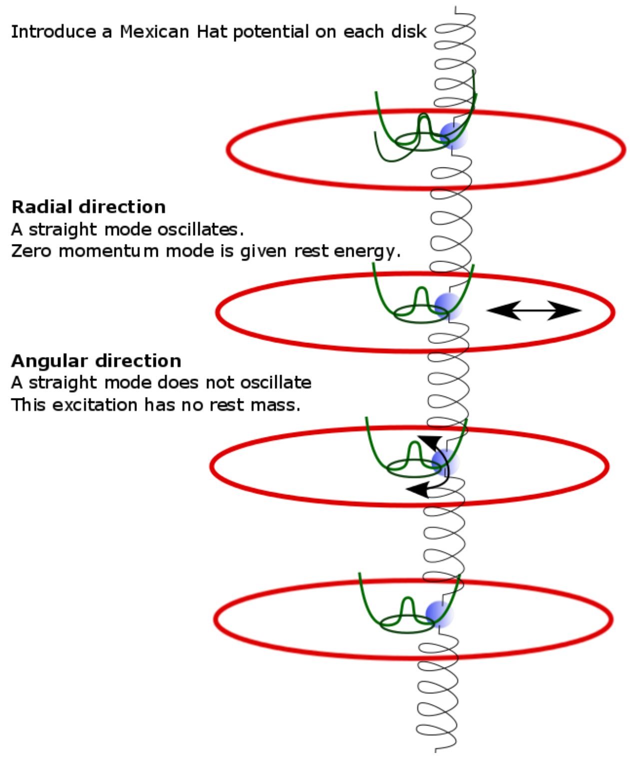
The number of massless particles is the dimension of the minimum of the potential.

This is a killer problem for spontaneous symmetry breaking, because massless scalar particles cost no energy to produce. They would be simple to detect and would have been seen in nature. Light bulbs should then radiate scalar bosons in addition to light and this would look like breakage of the conservation energy, for example.

We consider a Mexican hat potential and think about it in terms of our masses on discs.
This corresponds to the scalar part of the Lagrangian in Prof. Higg's 1964 model.

What do the wave solutions look like for the Mexican Hat $V(\phi)$?

We take $\phi = v$ as the spontaneously chosen *expectation value*, and try to analyse the rest mass. The key point here is that the restoring force from the potential depends on the direction in which the field is displaced from v.



- **Radial modes are massive:** Taking a zero momentum, constant mode and displacing it in the radial direction moves it up the sides of the potential. The potential pushes back and we have oscillation ($\equiv$ rest mass-energy) determined by the steepness of the sides.
- **Rotational modes are massless:** Taking a zero momentum, constant mode and displacing it around the circumference of the minimin receives no restoring force from the potential. These modes are massless.

This is the simplest example of Goldstone's theorem. There was a one dimensional minimum in the potential of the Lagrangian, and there is a single massless Nambu-Goldstone boson. This two component scalar field enters the Higgs model, and the counting of massless bosons is correct for that model.

We can consider the effect constraints on $|\phi|$ as a function of the number of degrees of freedom $\phi_1 \ldots \phi_N$.

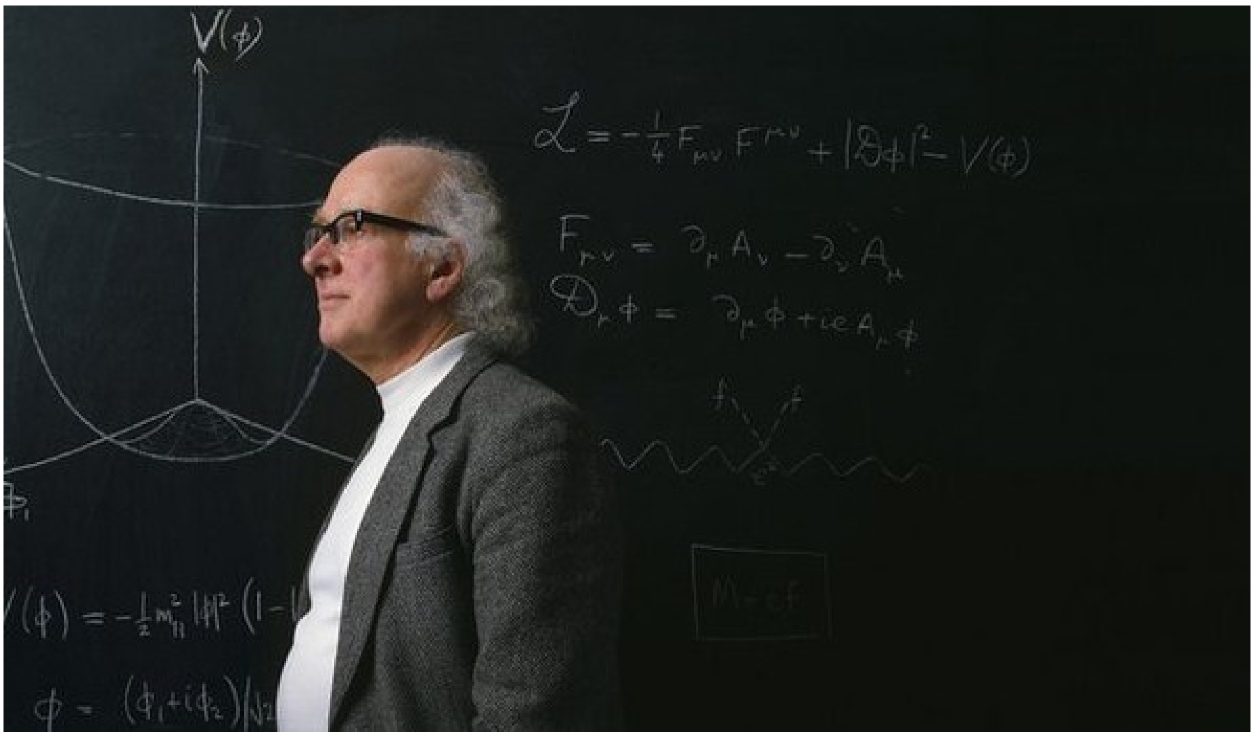
Components	Constraint	Dimension of minimum	Shape of minimum	Goldstone Bosons
1	$\phi_1^2 = v^2$	0	point	0
2	$\phi_1^2 + \phi_2^2 = v^2$	1	circle	1
3	$\phi_1^2 + \phi_2^2 + \phi_3^2 = v^2$	2	sphere	2
4	$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = v^2$	3	difficult![1]	3

The Higgs field that enters the standard model of particle physics has four components, and there are three Goldstone Bosons that enter the Higgs mechanism to give mass to three Weak bosons.

We will discuss the Higgs mechanism in the next section.

7 Higgs mechanism

The 1964 Higgs model



consisted of

$$\boxed{\text{Electromagnetism}} + \boxed{\text{Charged Klein Gordon field}} + \boxed{\text{Mexican Hat potential}}$$

The Higgs Lagrangian couples the Higgs field ϕ to electromagnetism through

$$D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$$

Here

$$A_0 \equiv \text{Voltage} \equiv \text{electrostatic potential}$$

Professor Higgs recognised that if the scalar fields interact with electromagnetism there is no absolute defintion of the electrostatic potential.

Introducing the electrostatic potential therefore messes with our lovely energy-momentum relations because we can shift the zero of energy by redefining the zero of potential. It is energy *relative* to potential that now matters.

The full treatment of this is complicated and given in supplementary notes for the brave.

A simpler route is taken in the explanation below - there is a subset of the symmetry used in the Higgs mechanism that is very familiar in high school physics: redefining the zero of electrostatic potential. The full symmetry also redefines the magnetic potential but is more complicated to think about.

We can make use of this simpler subset of the symmetry to show how the answer works by ignoring space-directions and considering the Higgs field only as a function of time.

Otherwise, the explanation is accurate.

Simple case

We ignore space entirely, and consider only a time varying oscillation of some energy E

$$\phi(t) = v(\cos Et, \sin Et)$$

The energy can be shifted by redefining the zero of electrostatic potential

$$V \to V + V_0$$

This makes the energy shift by a corresponding amount

$$E \to E + eV_0$$

After choosing $V_0 = -\frac{E}{e}$, the Higgs field loses all rotation in the angular direction

$$\phi(t) = (v, 0)$$

Full case

However, there is *even more freedom* in the choice of zero of potential. We let $V_0 \to V_0(t)$ *becoming any function of time* we want.

This is because the Maxwell's equations don't care about the time derivative of the potential (you can check this on Wikipedia!)

$$\frac{dV}{dt} \qquad \leftarrow \text{doesn't enter Maxwell's equations,}$$

but the electric field is the slope (or spatial derivative) of the potential

$$E = -\frac{dV}{dx} \qquad \leftarrow \text{enters Maxwell's equations,}$$

If we let the Higgs field be any *fully general function of time*

$$\phi(t) = R(t)(\cos\theta(t), \sin\theta(t))$$

We can make a special choice of the zero of potential

$$eV_0(t) = -\theta(t)$$

that leaves $\phi(t)$ with no angular rotation

$$\phi(t) = R(t)(1, 0)$$

When we include three space dimensions, the three components of the *magnetic* vector potential can also be used in the same way, and the same conclusion is reached.

7.1 Higgs mechanism conclusions

Escape to Goldstone's theorem

The angular oscillations, or Goldstone particles, can be wiped out with a choice of the zero of electrostatic potential! This choice does not alter the observable physics. Some theorists say that the gauge field has "eaten" the Goldstone boson.

Mass for gauge boson

If we take chosen expectation value $\phi = (v, 0)$ the kinetic energy term in Prof Higgs' Lagrangian becomes

$$T = |D_\mu \phi|^2 \to v^2 e^2 |A|^2$$

This is a *quadratic potential* for A and looks like a mass term for the gauge field. It was formed without breaking the gauge symmetry of A.

The mass generated is proportional to the Higgs field's squared expectation value v^2, and its coupling to the photon field e^2.

Giving a gauge field a mass in this way suggests the Higgs field permeating free space will expel those gauge bosons. Just like the Meissner effect in superconductors, we have a way to make the weak bosons weak. By excluding these weak bosons the Higgs field saves the universe from their nasty, radioactive effects.

Remnant neutral Higgs boson

Massive $R(t)$ oscillations *remain*. When quantized these are the neutral Higgs boson found by the LHC. Of the 1964 papers relevant to the Higgs mechanism, the paper by P.W. Higgs was unique in pointing out this remnant of the scalar field.

Degrees of freedom are not lost

In case you worry about where the degrees of freedom went, they reappeared as follows.

Massless photons have two degrees of freedom. That's why light is a transverse wave. Two perpendicular polaroid filters go dark because they remove both polarisations of light.

The disappearance of the angular degree of freedom for ϕ is not a problem. The missing degree of freedom reappears as an extra longitudinal polarisation of the now massive photon.

The extra degree of freedom must be there because if the photon is massive, we can always "boost" to the rest frame. In the rest frame there are no special directions corresponding to a direction of propagation: x,y, and z must be equivalent, and there are x, y, and z polarisations.

8 Summary

In this lecture you have understood

- Quantum field theories can be described by a Lagrangian density
- Having no potential leads to massless particles
- Having a quadratic potential leads to massive particles
- Superconductors develop a Mexican Hat potential for Cooper pairs of electrons in their superconducting phase. This leads to the expulsion of magnetic fields.
- Goldstone's theorem; introducing a Mexican Hat potential for charged scalars led to massless bosons that are not seen in nature.
- The Higgs mechanism introduced electromagnetism and exchanged the massless Goldstone degrees of freedom for an extra massive polarisation of the electromagnetic field
- The remnant, massive radial oscillations are the Higgs Boson discovered by the LHC.

9 Lenz's law approach

We're going to think about another way to approach the Higgs mechanism. This approach isn't as faithful to the treatment of the 1964 papers, and actually doesn't capture all the relevant features. It is, however, a bit more intuitive and gives in some ways a nice picture of what is going on.

In the video lectures we will accompany the explanation with some nice footage of the effects discussed.

Firstly, we remind you - verbally - of the meaning of some of Maxwell's equations.

1. Electric currents create a magnetic field that curls around the current

This is known as the Biot-Savart Law.

- This can be seen quite easily; we can stretch out a long wire vertically, driven by a 1.5V battery in a dead short.
- *Don't do this for long, and bin the battery afterwards - leaking battery acid in your favourite electronic device isn't fun.*
- Place a magnetic compass close to the wire and pretend to be an arctic explorer following "due north".
- Watch in dismay as your arctic explorer simply goes in circles around the wire!

If we coil the wire around in a circle, the "sausage" shape of magnetic field surrounding the wire gets curled round on itself like a "doughnut". In the middle the magnetic field created by all bits of the wire adds together, and gives us a strong(ish) electromagnet.

The video shows footage of a AA battery driving current through a small loop of wire to pick up a neodymium magnet, possible because the loop of wire has become an electromagnet.

In fact such electromagnets drive most powerful motors such as in a washing machine, or a car starter motor.

2. A changing Magnetic field causes the Electric field to curl around it

This is Faraday's law.

- Curl around twenty tight loops of copper wire and bundle them together in a coil (sellotape will do).
- Connect a voltmeter to the coil.
- Buy a neodymium magnet and try moving the magnet in and out of the coil. Set the meter to the most sensitive scale. You should see a voltage appear and disappear as you move the magnet in and out.
- Compare move the magnet in and out slowly. There should be no voltage created
- The measured voltage means that there is an electric field curling around the coil proportional to the rate of change of the magnetic field going through the coil. The faster the magnetic field changes, the bigger this electric field. In a conductor this drives electrons in circles around the magnetic field.

3. Lenz's law

So far we have seen

- Driving current around a loop creates a magnetic field.
- Changing the magnetic field will drive current around a loop.

There is a non-free lunch theorem.

When we bring a magnet up to a loop of wire, the induced current crates a magnetic field that *opposes* the incoming magnet.

This is known as Lenz's law, but it is really a statement of conservation of energy. Without this, we could generate electricity for free without doing any work! Shame.

4. Lenz's law demonstration

In the video we bring a strong magnet up to a copper pipe. Copper is non-magnetic, but a very good conductor.

When the magnet is dropped into the pipe, just like the wire coil, it must induce currents to flow around the pipe. These currents create an opposing magnetic field, and push back on the magnet.

This can be summarised as a good conductor fights back against changes to the magnetic field.

But, alas, the copper has non-zero resistance and the currents collapse away over time, and in the video we see the magnet floating down gently on a cushion of induced currents.

5. What if the conductor were a superconductor?

We can get a partial explanation for Meissner effect as follows.

If a magnet is brought up to a superconductor (zero resistance), then we can induce as much current as is required to resist the change in the magnetic field, because any non-cancelled electromotive force will drive an arbitrarily large current.

This effect is very similar to the rearrangement of charge in a conductor that excludes externally applied electric fields; the only stable configuration excludes the field and the surface charges and current densities are developed in whatever way is required to do so.

The difference with a superconductor is that the conductor can fight back "super hard".

That is to say, it can develop whatever current is required to exclude the magnetic field, and that current will not die away because in a superconductor the resistance is zero.

6. What does this mean for the Higgs mechanism?

In the Higgs mechanism, the Goldstone modes carry the charges of the Maxwell field are energetically "free" to excite because they are massless.

If we try to introduce gauge fields it will induce equal amounts of charged and anti-charged Goldstone modes rushing around trying to push the the gauge field back out.

In superconductor scenario, we have a lattice of positive nuclei and negative cooper pairs of electrons, and the cooper pairs can rush around pushing out magnetic (and electric) fields.

In the Higgs mechanism we have a charged scalar field with a net neutral vacuum. There is no energy cost to creating low momentum pairs of the Goldstone modes, and you could think of these as playing the role of charge carriers that exlude the electromagnetic fields.

Of course, in the standard model, the Higgs mechanism applies to the W and Z weak bosons, and not to the photon.