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Sequence selection principles for quasi-normal convergence ☆

Lev Bukovský *, Jaroslav Šupina

Institute of Mathematics, Faculty of Science, P.J. Šafárik University, Jesenná 5, 040 01 Košice, Slovakia

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ABSTRACT

In Bukovský et al. (2001) [5] the authors proved Theorem 5.10 saying that eleven seemingly different properties of a perfectly normal space X are equivalent. One of the properties says that every Borel image of X into ${}^{\omega}\omega$ is eventually bounded. B. Tsaban and L. Zdomskyy (in press) [17] have proved that any perfectly normal topological QN-space (for the definition see Bukovský et al. (1991) [4]) possesses this property, therefore all properties of the theorem. In this paper we simply prove that every perfectly normal topological QN-space possesses another property of that theorem – see Theorems 1 and 3. The main tools of our proof are sequence selection principles for quasi-normal convergence introduced in the paper. Some of introduced principles are worth studying in their own right and we initiate their study. Moreover, one of our main results immediately implies Recław's Theorem (Recław, 1997) [14].

1. Definitions and notation

J.E. Jayne and C.A. Rogers [10] have shown that any $\mathbf{\Delta}^0_2$-measurable mapping of an analytic subset of a complete metric space into another metric space is a discrete limit of continuous functions. The main result of the paper is a simple proof of a similar assertion for a normal topological space satisfying QSQ (for notations and terminology see below in this section and in Section 3).

Theorem 1. *If X is a normal topological space satisfying* QSQ*, then every $\mathbf{\Delta}^0_2$-measurable function from X into* $[0,1]$ *is a quasi-normal limit of a sequence of continuous functions, therefore also a discrete limit of such sequence, i.e. $\mathcal{D}_1(X) = \mathcal{A}_1(X)$.*

By Bukovský–Recław–Repický Theorem 2, Theorem 1 follows from Tsaban–Zdomskyy's Theorem 10, and vice versa. However the original proof [17] to Tsaban–Zdomskyy's Theorem is technically difficult and hard to fully grasp. Our proof of Theorem 1 employs a sequence selection principle introduced and investigated in the paper and is easy. Using ideas of the measure theory, for any topological space of the large inductive dimension zero, we construct sequences of functions satisfying conditions (a)–(c) of Section 3 with $A = \mathbb{Q}$. Then, using a corresponding selection principle, which satisfies every QN-space, we obtain the result. Moreover, this selection principle provides us with an alternative proof of Recław's Theorem saying that any QN-space is a σ-space.

Any considered topological space X is supposed to be Hausdorff and infinite. If $A \subseteq X$, then *the characteristic function* χ_A of A is defined as $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X \setminus A$. X has *the large inductive dimension* 0, written $\mathrm{Ind}(X) = 0$, if for any disjoint closed subsets $A, B \subseteq X$ there exists a clopen set U such that $A \subseteq U$ and $B \cap U = \emptyset$. By $\mathrm{C}_p(X)$ we denote

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* Corresponding author.
E-mail addresses: bukovsky@kosice.upjs.sk (L. Bukovský), jaroslav.supina@student.upjs.sk (J. Šupina).

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the set of all continuous functions from X to $\mathbb{R}$ with the subspace topology of the topological product ${}^X\mathbb{R}$. The Baire space ${}^\omega\omega$ is quasi-ordered by the *eventual dominating relation*

$$\alpha \leqslant^* \beta \equiv \{n \in \omega;\ \neg\alpha(n) \leqslant \beta(n)\} \quad \text{is finite.}$$

A set $\mathcal{F} \subseteq {}^\omega\omega$ is *eventually bounded* if there exists a sequence $\alpha \in {}^\omega\omega$ such that $\beta \leqslant^* \alpha$ for every $\beta \in \mathcal{F}$.

The notation "$f_n \xrightarrow{\mathrm{P}} f$ on X" means that the sequence of real valued functions $\{f_n\}_{n=0}^\infty$ converges on X to f pointwise. "$f_n \rightrightarrows f$ on X" means that the sequence of functions $\{f_n\}_{n=0}^\infty$ converges on X to f uniformly. A sequence of functions $\{f_n\}_{n=0}^\infty$ *converges quasi-normally* on X to f, written $f_n \xrightarrow{\mathrm{QN}} f$ on X, if there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^\infty$ (a control) converging to zero such that for every $x \in X$ and for all but finitely many $n \in \omega$ we have $|f_n(x) - f(x)| < \varepsilon_n$. If X is a topological space and $\{f_n\}_{n=0}^\infty$ is a sequence of continuous real valued functions, then $f_n \xrightarrow{\mathrm{QN}} f$ on X if and only if there exist closed sets X_m, $X_m \subseteq X_{m+1}$ for $m \in \omega$ such that $X = \bigcup_{m=0}^\infty X_m$ and $f_n \rightrightarrows f$ on each X_m, see e.g., [2] or [6]. A sequence of functions $\{f_n\}_{n=0}^\infty$ *converges discretely* on X to f, written $f_n \xrightarrow{\mathrm{D}} f$ on X, if for every $x \in X$ we have $f_n(x) = f(x)$ for all but finitely many $n \in \omega$.

A topological space X is a QN-*space* if each sequence of continuous functions converging to zero on X converges to zero quasi-normally as well. X is a wQN-*space* if each sequence of continuous functions converging to zero on X contains a subsequence that converges to zero quasi-normally.

$\mathbf{\Delta}_2^0(X)$ denotes the family of subsets of a topological space X which are both F_σ and G_δ. The following notations of sets of functions will be helpful (compare [5]):

$$\mathcal{A}_1(X) = \{f \in {}^X\mathbb{R};\ f \text{ is } \mathbf{\Delta}_2^0\text{-measurable}\},$$
$$\mathcal{D}_1(X) = \{f \in {}^X\mathbb{R};\ f \text{ is a discrete limit of continuous functions}\}.$$

It is well known that $\mathcal{D}_1(X) \subseteq \mathcal{A}_1(X)$. As an easy consequence of the above characterization of the quasi-normal convergence by the decomposition $X = \bigcup_n X_n$ and the uniform convergence, we obtain the following. If X is a normal topological space, then every quasi-normal limit of a sequence of continuous functions is also a discrete limit of a sequence of continuous functions, therefore it belongs to $\mathcal{D}_1(X)$. Hence, in a normal topological space a quasi-normal limit of a sequence of continuous functions is $\mathbf{\Delta}_2^0$-measurable. Pointwise limits of continuous functions are F_σ-measurable.

A family $\mathcal{U} \subseteq \mathcal{P}(X)$ is a *cover* of a topological space X if $X = \bigcup\mathcal{U}$ and $X \notin \mathcal{U}$. An infinite cover $\mathcal{U}$ is a γ-*cover* if every $x \in X$ lies in all but finitely many members of $\mathcal{U}$. $\Gamma(X)$ (or simply Γ) denotes the set of all countable open γ-covers. Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of X. The topological space X possesses the property $\mathrm{S}_1(\mathcal{A}, \mathcal{B})$ if for every sequence $\{\mathcal{U}_n\}_{n=0}^\infty$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.

2. Equivalent conditions

In Theorem 5.10 of [5], the authors established the equivalence of eleven properties of a perfectly normal topological space. We present three of them which are the most interesting for our next investigations.

Theorem 2 *(Bukovský–Recław–Repický). Let X be a perfectly normal space. The following conditions are equivalent.*

(1) *For each Borel measurable function $f : X \to {}^\omega\omega$ the set $f(X)$ is eventually bounded.*
(2) *A sequence of Borel measurable functions converging on X to zero converges quasi-normally.*
(3) *$\mathcal{D}_1(X) = \mathcal{A}_1(X)$ and X is a* QN-*space.*

B. Tsaban and L. Zdomskyy [17] have proved that any perfectly normal[1] topological QN-space satisfies the condition (1). Hence all those conditions are equivalent to "X is a QN-space". We shall show the same result by proving that $\mathcal{D}_1(X) = \mathcal{A}_1(X)$ for any normal QN-space X.

3. Selection principles for quasi-normal and discrete convergence

Let A, B denote one of the following types of convergence: P pointwise, Q quasi-normal, D discrete. We say that a topological space X satisfies ASB *selection principle*, if for any functions $f, f_n, f_m^n : X \to \mathbb{R}$, $n, m \in \omega$, such that

(a) $f_n \xrightarrow{\mathrm{A}} f$ on X,
(b) $f_m^n \xrightarrow{\mathrm{A}} f_n$ on X for every $n \in \omega$,
(c) every f_m^n is continuous,

[1] Note that the property "any open subset of X is a countable union of clopen subsets" considered in [17] is equivalent to "X is perfectly normal and $\mathrm{Ind}(X) = 0$".

there exists an unbounded $\beta \in {}^{\omega}\omega$ such that

$$f^n_{\beta(n)} \xrightarrow{\mathrm{B}} f \quad \text{on } X.$$

We shall mainly study the *quasi-normal–quasi-normal sequence selection principle* QSQ and the *discrete-pointwise sequence selection principle* DSP.

Note that all introduced properties are preserved by a continuous image.

We recall a much stronger well known sequence selection property introduced by A.V. Arkhangel'skiĭ [1]. A topological space X *has the property* (α_1) if for any $x \in X$ and for any sequence $\{\{x_{n,m}\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of sequences converging to x there exists a sequence $\{x_n\}_{n=0}^{\infty}$ converging to x such that the sequence $\{x_n\}_{n=0}^{\infty}$ contains all but finitely many members of the sequence $\{x_{n,m}\}_{m=0}^{\infty}$ for every n, equivalently, there exists a function $\beta \in {}^{\omega}\omega$ such that the countable set $\{x_{n,m};\ m \geqslant \beta(n) \wedge n \in \omega\}$ converges to x. By the results of M. Scheepers [16], L. Bukovský and J. Haleš [3], and M. Sakai [15], we have:

$$\text{A topological space } X \text{ is a QN-space} \equiv \mathrm{C}_p(X) \text{ has the property } (\alpha_1). \tag{1}$$

An important result of the paper is a direct proof of the following theorem. Note that the theorem can be deduced from the Tsaban–Zdomskyy Theorem by almost the same argument as the proof of the implication (1) → (2) of Theorem 16. But the point of our paper is to get a new proof of the Tsaban–Zdomskyy Theorem.

Theorem 3. *Any* QN-*space satisfies the quasi-normal–quasi-normal sequence selection principle* QSQ*.*

Proof. Assume that $f, f_n, f^n_m : X \to \mathbb{R}$, $n, m \in \omega$ are such that the corresponding conditions (a)–(c) of the definition of the principle QSQ hold true. We may assume[2] that the control of the quasi-normal convergences in (b) is $\{2^{-2m}\}_{m=0}^{\infty}$ and the control of the quasi-normal convergence in (a) is $\{\varepsilon_n\}_{n=0}^{\infty}$.

We denote

$$g^n_m(x) = \min\{\left|f^n_m(x) - f^n_{m+1}(x)\right| \cdot 2^{m+1}, 1\}.$$

Evidently every g^n_m is continuous and for a fixed $n \in \omega$ we have $g^n_m \xrightarrow{\mathrm{P}} 0$ on X. Since the space $\mathrm{C}_p(X)$ satisfies the condition (α_1), there exists an increasing function $\beta \in {}^{\omega}\omega$ such that the set $\{g^n_m; m \geqslant \beta(n) \wedge n \in \omega\}$ converges to 0.

We claim that $f^n_{\beta(n)} \xrightarrow{\mathrm{QN}} f$ with the control $\{2^{-\beta(n)} + \varepsilon_n\}_{n=0}^{\infty}$. Indeed, let $x \in X$. Then there exists an n_0 such that $g^n_m(x) < 1$ for any $n \geqslant n_0$ and any $m \geqslant \beta(n)$. Moreover, we may assume that $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geqslant n_0$. Hence $|f^n_m(x) - f^n_{m+1}(x)| < 2^{-m-1}$ and therefore $|f^n_{\beta(n)}(x) - f^n_m(x)| < 2^{-\beta(n)}$ for any $n \geqslant n_0$ and any $m \geqslant \beta(n)$. Thus $|f^n_{\beta(n)}(x) - f_n(x)| \leqslant 2^{-\beta(n)}$. Therefore, for $n \geqslant n_0$ we obtain $|f^n_{\beta(n)}(x) - f(x)| < 2^{-\beta(n)} + \varepsilon_n$. □

Hence, the relations of the introduced sequence selection principles with QN are as follows:

$$\mathrm{QN} \to \mathrm{QSQ} \to \mathrm{DSP}. \tag{2}$$

One can easily show that the unit interval [0, 1] does not possess the property DSP and therefore none of the considered properties (consider the sequences showing that the Dirichlet function $\chi_{\mathbb{Q}\cap[0,1]}$ is of the second Baire class).

4. Proof of Theorem 1

We give an elementary proof of Theorem 1. As a consequence we obtain Tsaban–Zdomskyy's Theorem 10. The only nontrivial result used in the proof of Theorem 10 not proved in the paper is the equivalence (1) used in the proof of Theorem 3.

We begin with three auxiliary results. In the second and the third lemmas we follow the common reasoning of measure theory or real function theory.

Lemma 4. *If a normal topological space X satisfies* DSP*, then* $\mathrm{Ind}(X) = 0$.

Proof. If $F \subseteq U \subseteq X$, F closed, U open, then there exists a continuous function f equal to 1 on F and to 0 on $X \setminus U$. Since [0, 1] does not satisfy DSP, F is not a surjection. If $a \in [0, 1] \setminus f(X)$, then the clopen set $f^{-1}((-1, a))$ separates F and U. □

A function f is called *simple*, if the range of f is a finite set of non-negative reals.

[2] If $\{\varepsilon_n\}_{n=0}^{\infty}$ is a control of quasi-normal convergence of some sequence of functions $\{f_n\}_{n=0}^{\infty}$, then for any control $\{\delta_n\}_{n=0}^{\infty}$ there exists a subsequence $\{f_{n_m}\}_{m=0}^{\infty}$ converging with this control.

Lemma 5. *If* $\mathrm{Ind}(X) = 0$*, then any simple* $\mathbf{\Delta}^0_2$*-measurable function* $g : X \to [0,1]$ *is a discrete limit of a sequence* $\{g_n\}_{n=0}^{\infty}$ *of simple continuous functions.*

Proof. Assume that $g = \sum_{i=0}^{k} a_i \chi_{A_i}$, where $A_i \in \mathbf{\Delta}^0_2$ are pairwise disjoint, $\bigcup_{i=0}^{k} A_i = X$ and $0 \leqslant a_0 < a_1 < \cdots < a_k \leqslant 1$. Then for every $i \leqslant k$ there exist a non-decreasing and a non-increasing sequence $\{F_n^i\}_{n=0}^{\infty}$ and $\{G_n^i\}_{n=0}^{\infty}$ of F_σ and G_δ sets, respectively, such that $A_i = \bigcup_n F_n^i = \bigcap_n G_n^i$. Since $\mathrm{Ind}(X) = 0$, for every $i \leqslant k$ and every $n \in \omega$ there exists a clopen set C_n^i such that $F_n^i \subseteq C_n^i \subseteq G_n^i$. Replacing eventually C_n^0 by $C_n^0 \cup (X \setminus \bigcup_{0<i\leqslant k} C_n^i)$ we may assume that $\bigcup_{i\leqslant k} C_n^i = X$. Let $D_n^i = C_n^i \setminus \bigcup_{j<i} C_n^j$. Then D_n^i are pairwise disjoint and $\bigcup_{i\leqslant k} D_n^i = X$. Set $g_n = \sum_{i=0}^{k} a_i \chi_{D_n^i}$. Since each D_n^i is clopen, g_n is continuous.

Let $x \in X$ and $g(x) = a_i$. Then there exists an n_0 such that for every $n \geqslant n_0$ we have $x \in F_n^i$ and $x \notin G_n^j$ for $j < i$. Then $x \in D_n^i$ and therefore $g_n(x) = a_i$. □

Lemma 6. *Assume that* X *is a topological space with* $\mathrm{Ind}(X) = 0$*. Then for any* $\mathbf{\Delta}^0_2$*-measurable function* $f : X \to [0,1]$ *there exist simple* $\mathbf{\Delta}^0_2$*-measurable functions* $f_n : X \to [0,1]$*,* $n \in \omega$ *and simple continuous functions* $f_m^n : X \to [0,1]$*,* $n, m \in \omega$ *such that* $f_n \rightrightarrows f$ *on* X *and* $f_m^n \xrightarrow{\mathrm{D}} f_n$ *on* X *for each fixed* n*.*

Proof. For any n and any $i < 2^n - 1$, we denote

$$A_n^i = \left\{x \in X;\ \frac{i}{2^n} \leqslant f(x) < \frac{i+1}{2^n}\right\}, \qquad A_n^{2^n-1} = \left\{x \in X;\ \frac{2^n-1}{2^n} \leqslant f(x)\right\}.$$

Then the sequence of simple $\mathbf{\Delta}^0_2$-measurable functions

$$f_n = \sum_{i=0}^{2^n-1} \frac{i}{2^n} \chi_{A_n^i}$$

converges uniformly to f with the control 2^{-n}. By Lemma 5, for every n there exists a sequence $\{f_m^n\}_{m=0}^{\infty}$ of simple continuous functions such that $f_m^n \xrightarrow{\mathrm{D}} f_n$ on X. □

Proof of Theorem 1. By Lemma 4 any normal topological space satisfying QSQ has the large inductive dimension zero and therefore the approximation functions f_n, f_m^n of Lemma 6 can be constructed. Since a uniform convergence is a quasi-normal convergence, by QSQ the function f is a quasi-normal limit of a sequence of continuous functions. □

By Theorems 3 and 1 the condition (5) of Theorem 2 can be simplified as "X is a QN-space".

We can easily prove a strengthening of Recław's Theorem [14].

Theorem 7. *If a perfectly normal topological space* X *satisfies* DSP*, then* X *is a* σ*-space.*

Proof. Assume that $F = \bigcup_n F_n$ is an F_σ set, F_n is closed and $F_n \subseteq F_{n+1}$ for any $n \in \omega$. We want to show that the function χ_F of the set F is G_δ-measurable.

Since X is perfectly normal, there exist closed sets $F_{n,m}$ such that $F_{n,m} \subseteq F_{n,m+1}$ and $X \setminus F_n = \bigcup_k F_{n,k}$ for any n and m. For any n and m, there exists a continuous function $f_{n,m} : X \to [0,1]$ such that $f_{n,m}(x) = 1$ for $x \in F_n$ and $f_{n,m}(x) = 0$ for $x \in F_{n,m}$. Then $f_{n,m} \xrightarrow{\mathrm{D}} \chi_{F_n}$ on X. Moreover, $\chi_{F_n} \xrightarrow{\mathrm{D}} \chi_F$.

Since X satisfies DSP, there exists a β such that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} \chi_F$. Then

$$X \setminus F = \bigcup_m \bigcap_{n \geqslant m} \{x \in X:\ f_{n,\beta(n)}(x) \leqslant 1/2\}$$

is an F_σ-set. □

Corollary 8 *(Recław). A perfectly normal* QN*-space is a* σ*-space.*

Lemma 9. *If a perfectly normal topological space* X *satisfies* QSQ *and* $f : X \to {}^{\omega}\omega$ *is Borel measurable, then there exist closed sets* X_m*,* $m \in \omega$ *such that* $X = \bigcup_m X_m$ *and* $f|X_m$ *is continuous for each* m*.*

Proof. If $f : X \to {}^{\omega}\omega$ is Borel measurable, then by Theorem 7 f is $\mathbf{\Delta}^0_2$-measurable, therefore by Theorem 1 there exist continuous $f_n : X \to [0,1]$ such that $f_n \xrightarrow{\mathrm{D}} f$ on X. Then there exist a sequence of closed sets X_m, $m \in \omega$ such that $\bigcup_m X_m = X$ and $f|X_m = f_n|X_m$ for sufficiently large n. □

Theorem 10 *(Tsaban–Zdomskyy). Any Borel image of a perfectly normal* QN*-space X into ${}^{\omega}\omega$ is eventually bounded.*

Proof. If $f : X \to {}^{\omega}\omega$ is Borel measurable, then by Theorem 3 and by Lemma 9 there are closed sets X_m such that $X = \bigcup_m X_m$ and $f|X_m$ are continuous. By Corollary 2.2 of [4] a wQN-subset of ${}^{\omega}\omega$ is eventually bounded. Therefore $f(X_m)$ is an eventually bounded subset of ${}^{\omega}\omega$. Since $f(X) = \bigcup_m f(X_m)$, the set $f(X)$ is eventually bounded as well. □

Due to Theorems 3, 1 and 7 we obtain

Theorem 11. *If X is a perfectly normal* QN*-space, then any Borel measurable function $f : X \to [0,1]$ is a quasi-normal limit of a sequence of simple continuous functions.*

Theorem 4.8 of [4] says that an image of a QN-space into $[0,1]$ by a quasi-normal limit of continuous functions is a QN-space. By Tsaban–Zdomskyy's Theorem one immediately obtains a more stronger result.

Corollary 12. *Any Borel image of a perfectly normal* QN*-space is a* QN*-space.*

5. Other selection principles

By considering the three introduced convergences, namely pointwise, quasi-normal and discrete, in the definition of ASB selection principles at the beginning of Section 3, one obtains nine such principles. Two of the principles, namely PSD and QSD, are too strong and no topological space satisfies them. The trivial relations among the rest of properties could be described by the following diagram.

$$\begin{array}{ccccc} & & & & \mathrm{DSD} \\ & & & & \downarrow \\ \mathrm{PSQ} & \longrightarrow & \mathrm{QSQ} & \longrightarrow & \mathrm{DSQ} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{PSP} & \longrightarrow & \mathrm{QSP} & \longrightarrow & \mathrm{DSP} \end{array}$$

The authors conjecture that DSP is weaker than DSQ or QSP. Therefore it worths presenting the following results.

In [3] the authors say that a space X has the *discrete sequence selection property*[3] if for every sequence $\{\{f^n_m\}_{m=0}^{\infty}\}_{n=0}^{\infty}$ of functions from $\mathrm{C}_p(X)$ such that $f^n_m \xrightarrow{\mathrm{D}} 0$ on X for every $n \in \omega$, there exists a sequence of natural numbers $\{m_n\}_{n=0}^{\infty}$ such that $f^n_{m_n} \xrightarrow{\mathrm{P}} 0$ on X. Let DSP* be a principle with almost the same definition as DSP but with the exception that we ask the function f to be continuous. Evidently DSP implies DSP* and DSP* implies the discrete sequence selection property.

Theorem 13. *A normal topological space X satisfying* DSP* *is a* wQN*-space.*

Proof. By Theorem 10 of [3] a normal topological space X has discrete sequence selection property if and only if X is a wQN-space. □

M. Sakai in [15] defined the property (γ,γ)-shrinkable[4] and showed that:

A perfectly normal topological space X is (γ,γ)-shrinkable $\equiv$ X is a σ-space.

Theorem 14. *A perfectly normal topological space satisfying* DSP* *is a σ-space.*

Proof. Let $\{U_n\}_{n=0}^{\infty}$ be an open γ-cover of X. Then $\chi_{U_n} \xrightarrow{\mathrm{D}} 1$. Since X is a perfectly normal space there is a non-decreasing sequence $\{F_{n,m}\}_{m=0}^{\infty}$ of closed sets such that $\bigcup_{m=0}^{\infty} F_{n,m} = U_n$ and continuous functions $f_{n,m} : X \to [0,1]$ such that $f_{n,m}(x) = 1$ for $x \in F_{n,m}$ and $f_{n,m}(x) = 0$ for $x \in X \setminus U_n$. Then $f_{n,m} \xrightarrow{\mathrm{D}} \chi_{U_n}$. By DSP* there is $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{P}} 1$. If we denote $F_n = f^{-1}_{n,\beta(n)}([\frac{1}{2},1])$, then $F_n \subseteq U_n$, $n \in \omega$ are closed sets and $\{F_n\}_{n=0}^{\infty}$ is a γ-cover. □

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[3] Actually they define that "a space $\mathrm{C}_p(X)$ has the discrete sequence selection property". We prefer to speak about the space X instead of $\mathrm{C}_p(X)$, since the studied selection principles of this paper are not related only to continuous functions.

[4] A space X is (γ,γ)-shrinkable if for every open γ-cover $\{U_n\}_{n=0}^{\infty}$ of X there exists a closed γ-cover $\{F_n\}_{n=0}^{\infty}$ of X such that $F_n \subseteq U_n$ for $n \in \omega$.

Hence, a perfectly normal topological space satisfying DSP* is a wQN-space and a σ-space. Thus by Theorems 4 and 6 of [9] we obtain

Corollary 15. *A perfectly normal topological space satisfying* DSP* *is a hereditary* $\mathrm{S}_1(\Gamma, \Gamma)$*-space.*

Theorem 16. *If X is a perfectly normal space then the following are equivalent.*

(1) *X is a* QN*-space.*
(2) *X satisfies* PSQ.
(3) *X satisfies* QSQ.
(4) *X satisfies* DSD.

Proof. (1) $\to$ (2) Assume that $f, f_n, f^n_m : X \to \mathbb{R}$, $n, m \in \omega$ satisfy the conditions (a)–(c) of the definition of the property PSQ. Since f^n_m are continuous, the functions f_n and f are Borel measurable. Let us define Borel measurable functions $\varphi, \psi : X \to {}^{\omega}\omega$ as

$$\psi(x)(n) = \min\left\{k \in \omega;\ (\forall m \geqslant k)\ \left|f_m(x) - f(x)\right| < \frac{1}{n+1}\right\}$$

and

$$\varphi(x)(n) = \min\left\{k \in \omega;\ (\forall m \geqslant k)\ \left|f^n_m(x) - f_n(x)\right| < \frac{1}{n+1}\right\}$$

for any $x \in X$ and any $n \in \omega$. By Tsaban–Zdomskyy's Theorem both functions have eventually bounded images and therefore there are increasing functions $\gamma, \beta \in {}^{\omega}\omega$ bounding $\psi(X)$ and $\varphi(X)$, respectively. We may assume that $\gamma(0) = 0$. Let $\varepsilon_n = \frac{1}{m+1}$ for $\gamma(m) \leqslant n < \gamma(m+1)$. Then the sequence $\{f^n_{\beta(n)}\}^{\infty}_{n=0}$ converges quasi-normally to f with the control $\{\varepsilon_n + \frac{1}{n+1}\}^{\infty}_{n=0}$.

(2) $\to$ (3) Trivial implication.

(3) $\to$ (1) If $f : X \to {}^{\omega}\omega$ is Borel measurable, then by Theorem 13 and similar arguments as in the proof of Theorem 10, $f(X)$ is eventually bounded. The assertion (1) follows by Theorem 2.

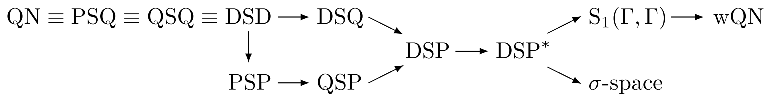
(1) $\to$ (4) Assume that X is a perfectly normal QN-space and $f, f_n, f^n_m : X \to \mathbb{R}$, $n, m \in \omega$ are such that the assumptions in the definition of the property DSD hold true. Since f^n_m are continuous, the functions f_n and f are Borel measurable. Let us define Borel measurable function $\varphi : X \to {}^{\omega}\omega$ as

$$\varphi(x)(n) = \min\left\{k \in \omega;\ (\forall m \geqslant k)\ f^n_m(x) = f_n(x)\right\}$$

for any $x \in X$ and $n \in \omega$. By Tsaban–Zdomskyy's Theorem the function φ has eventually bounded image and therefore there is an increasing $\beta \in {}^{\omega}\omega$ eventually bounding $\varphi(X)$. Then $f^n_{\beta(n)} \xrightarrow{\mathrm{D}} f$.

(4) $\to$ (1) Let $\mathcal{F}_\Gamma$ denote the family of all countable closed γ-covers. By Corollary 20 of [17] for a perfectly normal space $\mathrm{QN} \equiv \mathrm{S}_1(\mathcal{F}_\Gamma, \mathcal{F}_\Gamma)$. Let $\mathcal{F}_n = \{F_{n,m};\ m \in \omega\}$ be a closed γ-cover for any $n \in \omega$. Since X is perfectly normal, for any $n, m \in \omega$ there exists a continuous function $f_{n,m} : X \to [0, 1]$ such that $f_{n,m}(x) = 0$ for $x \in F_{n,m}$ and $f_{n,m}(x) \neq 0$ for $x \notin F_{n,m}$. Then $f_{n,m} \xrightarrow{\mathrm{D}} 0$ and by DSD there is $\beta \in {}^{\omega}\omega$ such that $f_{n,\beta(n)} \xrightarrow{\mathrm{D}} 0$. Then $\{F_{n,\beta(n)};\ n \in \omega\}$ is a γ-cover.[5] □

We can summarize the relationships between considered notions in a diagram.



Note that by A. Dow [7] there is a model of **ZFC** (Laver's model [12]) in which $\mathrm{QN} \equiv \mathrm{wQN}$. Therefore almost all these principles are equivalent in this model. Moreover, A.W. Miller and B. Tsaban [13] proved that these properties are trivial in this model: any set with any of those properties has cardinality $< \mathfrak{b}$. In [4] the authors by modifying a construction of Galvin–Miller [8] obtained a γ-set of cardinality $\mathfrak{c}$ that is not QN-set, provided that $\mathfrak{p} = \mathfrak{c}$. In [11] it was essentially shown that $\mathrm{QN} \not\equiv \mathrm{S}_1(\Gamma, \Gamma)$ in any model of $\mathbf{ZFC} + \mathfrak{t} = \mathfrak{b}$. A.W. Miller and B. Tsaban [13] show that $\mathrm{QN} \not\equiv \mathrm{S}_1(\Gamma, \Gamma)$ provided that there exists an unbounded tower; in particular provided that $\mathfrak{t} = \mathfrak{b}$ or $\mathfrak{b} < \mathfrak{d}$. However we do not know anything about the remaining implications.

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[5] Note that a weaker version of principle DSD is sufficient: For every sequence $\{\{f^n_m\}^{\infty}_{m=0}\}^{\infty}_{n=0}$ of functions from $\mathrm{C}_p(X)$ such that $f^n_m \xrightarrow{\mathrm{D}} 0$ on X for every $n \in \omega$, there exists a sequence of natural numbers $\{m_n\}^{\infty}_{n=0}$ such that $f^n_{m_n} \xrightarrow{\mathrm{D}} 0$ on X. Compare this with the discrete sequence selection property.

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