Range Sidelobe Suppression in a Desired Doppler Interval

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***Abstract*—We present a novel method of constructing a *Doppler resilient* pulse train of Golay complementary waveforms, for which the range sidelobes of the pulse train ambiguity function vanish inside a desired Doppler interval. This is accomplished by coordinating the transmission of a Golay pair of phase coded waveforms in time according to the 1's and -1's in a biphase sequence. The magnitude of the range sidelobes of the pulse train ambiguity function is shown to be proportional to the magnitude spectrum of the biphase sequence. Range sidelobes inside a desired Doppler interval are suppressed by selecting a sequence whose spectrum has a high-order null at a Doppler frequency inside the desired interval. We show that the spectrum of the biphase sequence obtained by oversampling the length-2^M *Prouhet-Thue-Morse (PTM) sequence* by a factor m has an Mth-order null at all rational Doppler shifts $\theta_0 = 2\pi l/m$, where $l \neq 0$ and $m \neq 1$ are co-prime integers. This spectrum also has an $(M-1)$th-order null at zero Doppler and $(M-h-1)$th-order nulls at all Doppler shifts $\theta_0 = 2\pi l/(2^h m)$, where $l \neq 0$ and $m \neq 1$ are again co-prime and $1 \leq h \leq M-1$.**

I. Introduction

Phase coding is a common technique in radar for constructing waveforms with impulse-like autocorrelation functions [1],[2]. In this technique, a long pulse is phase coded with a unimodular sequence and the autocorrelation function of the coded waveform is controlled via the autocorrelation function of the unimodular sequence. A key issue in phase coding is the presence of range sidelobes in the ambiguity function of the coded waveforms. Range sidelobes due to a strong reflector can result in masking of nearby weak targets. It is however impossible to design a single unimodular sequence for which the aperiodic autocorrelation function has no range sidelobes. This has led to the idea of using complementary sets of unimodular sequences [3]–[7] for phase coding.

The most famous class of complementary sequences are Golay complementary sequences or Golay pairs introduced by Marcel Golay [3],[8], which have the property that the sum of their autocorrelation functions vanishes at all delays other than zero. Thus, if each sequence is transmitted separately and the autocorrelation functions are added together the output will be free of range sidelobes. In other words, the effective ambiguity function of a Golay pair of phase coded waveforms is free of range sidelobes along the zero-Doppler axis. However, this ideal property is sensitive to Doppler effect. Off the zero-Doppler axis the ambiguity function of Golay pairs of phase coded waveforms has large range sidelobes, e.g., see [1],[2] and [9]. The ambiguity function of a pulse train of Golay complementary waveforms, in which the two waveforms are transmitted alternatively in time over several pulse repetition intervals (PRIs), suffers from the same problem. The sensitivity of Golay complementary waveforms to Doppler has been a major barrier in adoption of these waveforms for radar pulse compression. Various generalizations of complementary waveforms, including multiple complementary waveforms [7], multiphase (or polyphase) complementary waveforms [10],[11], and near-complementary waveforms [12] suffer from a similar problem.

A natural question to ask is whether or not it is possible to construct a *Doppler resilient* pulse train of Golay complementary waveforms, for which the range sidelobes of the ambiguity function vanish inside a desired Doppler interval. This question was recently considered in [13],[14], where it is shown that by carefully choosing the order in which a Golay pair of phase coded waveforms is transmitted over time we can clear out the range sidelobes of the pulse train ambiguity function along modest (close to zero) Doppler shifts. The developments in [13],[14] led to the discovery that if the transmission of a Golay pair of phase coded waveforms is coordinated in time according to the entries in a biphase sequence then the magnitude of the range sidelobes can be controlled by shaping the spectrum of the biphase sequence [15].

In this paper, we extend the result of [13]–[15] to construct pulse trains of Golay complementary waveforms, for which the range sidelobes of the ambiguity function vanish inside a desired Doppler interval away from zero. This is accomplished by coordinating the transmission of a Golay pair of phase coded waveforms in time according to the 1's and -1's in a $(2^M, m)$*-PTM sequence*. The $(2^M, m)$-PTM sequence has length $2^M \times m$ and is obtained by repeating each 1 and -1 in the length-2^M PTM sequence m times, that is by oversampling a length-2^M PTM sequence by a factor m. We show that the spectrum of the $(2^M, m)$-PTM sequence has Mth-order nulls at all rational Doppler shifts $\theta_0 = 2\pi l/m$, where $l \neq 0$ and $m \neq 1$ are co-prime integers. This spectrum also has an $(M-1)$th-order null at zero Doppler and $(M-h-1)$th-order nulls at all $\theta_0 = 2\pi l/(2^h m)$, where $l \neq 0$ and $m \neq 1$ are

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again co-prime and $1 \leq h \leq M-1$. These high-order nulls suppress the range sidelobes. Numerical examples demonstrate the annihilation of range sidelobes in the ambiguity function of $(2^M, m)$-PTM pulse trains.

II. Golay Complementary Waveforms

Definition 1: Two length L unimodular sequences of complex numbers $x(\ell)$ and $y(\ell)$ are Golay complementary if for $k = -(L-1), \ldots, (L-1)$ the sum of their autocorrelation functions satisfies

$$C_x(k) + C_y(k) = 2L\delta(k), \quad (1)$$

where $C_x(k)$ is the autocorrelation of $x(\ell)$ at lag k and $\delta(k)$ is the Kronecker delta function. Henceforth we may drop the discrete time index ℓ from $x(\ell)$ and $y(\ell)$ and simply use x and y. Each member of the pair (x, y) is called a Golay sequence.

The baseband waveform $s_x(t)$ phase coded by the Golay sequence x is given by

$$s_x(t) = \sum_{\ell=0}^{L-1} x(\ell)\Omega(t - \ell T_c) \quad (2)$$

where $\Omega(t)$ is a unit energy pulse shape of duration T_c and T_c is the chip length. The ambiguity function $\chi_{s_x}(\tau, \nu)$ of $s_x(t)$ is given by

$$\chi_{s_x}(\tau, \nu) = \int_{-\infty}^{\infty} s_x(t)\overline{s_x(t-\tau)}e^{-j\nu t}dt \quad (3)$$

where $\overline{s(t)}$ is the complex conjugate of $s(t)$, and $\chi_\Omega(\tau, \nu)$ is the ambiguity function of the pulse shape $\Omega(t)$.

If the complementary waveforms $s_x(t)$ and $s_y(t)$ are transmitted separately in time, with a T sec time interval between the two transmissions, then the effective ambiguity function of the radar waveform $S(t) = s_x(t) + s_y(t - T)$ is given by[1]

$$\chi_S(\tau, \nu) = \chi_{s_x}(\tau, \nu) + e^{j\nu T}\chi_{s_y}(\tau, \nu). \quad (4)$$

Since the chip length T_c is typically very small, the relative Doppler shift over the duration LT_c of a single waveform is negligible compared to the relative Doppler shift over the PRI duration T, and the ambiguity function $\chi_S(\tau, \nu)$ can be approximated by

$$\chi_S(\tau, \nu) = \sum_{k=-(L-1)}^{L-1} [C_x(k) + e^{j\nu T}C_y(k)]\chi_\Omega(\tau + kT_c, \nu). \quad (5)$$

Along the zero-Doppler axis ($\nu = 0$), the ambiguity function $\chi_S(\tau, \nu)$ reduces to

$$\chi_S(\tau, 0) = 2L\chi_\Omega(\tau, 0). \quad (6)$$

[1]The ambiguity function of $S(t)$ has two range aliases (cross terms) which are offset from the zero-delay axis by $\pm T$. In this paper, we ignore the range aliasing effects and only focus on the *mainlobe* of the ambiguity function, which corresponds to $\chi_S(\tau, \nu)$ given in (4). Range aliasing effects can be accounted for using standard techniques devised for this purpose (e.g. see [1]) and hence will not be further discussed.

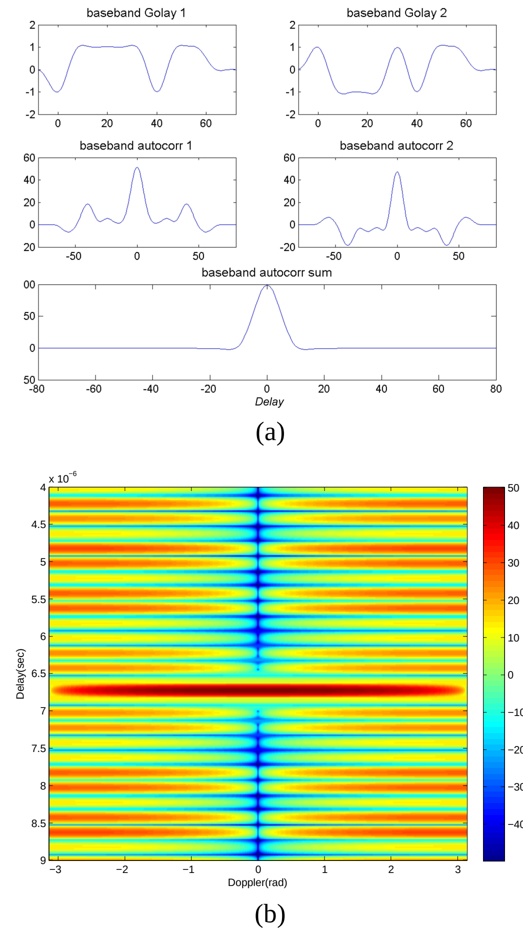


Fig. 1. (a) The perfect autocorrelation property of a Golay pair of phase coded waveforms (b) The ambiguity function of a Golay pair of phase coded waveforms seperated in time.

This means that along the zero-Doppler axis the ambiguity function $\chi_S(\tau, \nu)$ is "free" of range sidelobes.[2] This perfect autocorrelation property (ambiguity response along zero-Doppler axis) is illustrated in Fig. 1(a) for a Golay pair of phase coded waveforms.

Off the zero-Doppler axis however, the ambiguity function has large sidelobes in delay (range) as Fig. 1(b) shows. The color bar values are in dB. The range sidelobes persist for a conventional pulse train of Golay complementary waveforms, where the transmitter alternates between $s_x(t)$ and $s_y(t)$ during several PRIs. The range sidelobes in the ambiguity function can cause masking of a weak target that is situated near a strong reflector. Figure 2 shows the delay-Doppler map at the output of a matched filter, when a conventional pulse train of Golay complementary waveforms is transmitted over 256 PRIs. The horizontal axis shows Doppler and the vertical axis depicts delay. Color bar values are in dB. The radar scene contains three stationary reflectors at different ranges and two slow-moving targets, which are 30dB weaker than the stationary reflectors. This value has been chosen to make the

[2]The shape of the autocorrelation function depends on the autocorrelation function $\chi_\Omega(\tau, 0)$ for the pulse shape $\Omega(t)$. The Golay complementary property eliminates range sidelobes caused by replicas of $\chi_\Omega(\tau, 0)$ at nonzero integer delays.

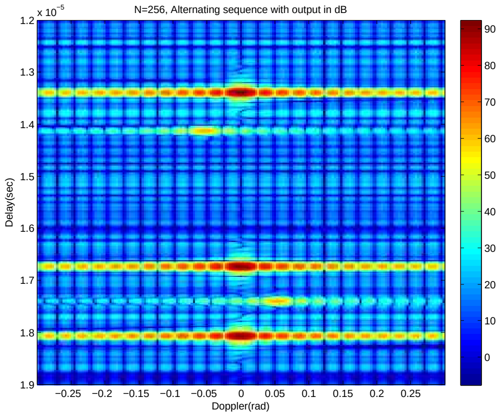


Fig. 2. Doppler induced range sidelobes in the ambiguity function of a conventional (alternating) pulse train of Golay complementary waveforms result in masking of weak targets by strong reflectors that are close by.

slow-moving targets barely visible, and if they were slightly weaker, it would not be possible to resolve them because of the range sidelobes from the strong stationary reflectors. The Golay complementary sequences used in this example are of size $L = 64$ and the pulse shape is a raised cosine. The chip length is $T_c = 100$ *n*sec, the carrier frequency is 17 GHz, and the PRI is $T = 50$ μsec.

III. PTM Pulse Trains of Golay Complementary Waveforms

It is natural to ask whether or not it is possible to construct a *Doppler resilient* pulse train of Golay complementary waveforms, for which the range sidelobes of the pulse train ambiguity function vanish inside a desired Doppler interval. In this section, we consider the annihilation of range sidelobes along modest Doppler shifts and show that by carefully choosing the order in which the Golay waveforms $s_x(t)$ and $s_y(t)$ are transmitted in a pulse train we can clear out the range sidelobes of the ambiguity function in an interval along the zero-Doppler axis.

Definition 2: Consider a biphase sequence $\mathcal{P} = \{p_n\}_{n=0}^{N-1}$, $p_n \in \{-1, 1\}$ of length N, where N is even. Let 1 represent $s_x(t)$ and let -1 represent $s_y(t)$. We define the *$\mathcal{P}$-pulse train* $Z_{\mathcal{P}}(t)$ of $(s_x(t), s_y(t))$ as

$$Z_{\mathcal{P}}(t) = \frac{1}{2}\sum_{n=0}^{N-1}\left[(1+p_n)s_x(t-nT) + (1-p_n)s_y(t-nT)\right]. \tag{7}$$

The nth entry in $S_{\mathcal{P}}(t)$ is $s_x(t)$ if $p_n = 1$ and $s_y(t)$ if $p_n = -1$. Consecutive entries are separated in time by a PRI T.

The ambiguity function of the $\mathcal{P}$-pulse train $Z_{\mathcal{P}}(t)$, after ignoring the pulse shape ambiguity function and discretizing in delay, can be written as [15]

$$\chi_{Z_{\mathcal{P}}}(k,\theta) = \frac{1}{2}[C_x(k) + C_y(k)]\sum_{n=0}^{N-1} e^{jn\theta} + \frac{1}{2}[C_x(k) - C_y(k)]\sum_{n=0}^{N-1} p_n e^{jn\theta} \tag{8}$$

where $\theta = \nu T$ is the relative Doppler shift over a PRI. The first term on the right-hand-side of (8) is free of range sidelobes due to the complementary property of Golay sequences x and y. The second term represents the range sidelobes, as $C_x(k) - C_y(k)$ is not an impulse. The magnitude of the range sidelobes is proportional to the magnitude of the spectrum $S_{\mathcal{P}}(\theta)$ of the sequence $\mathcal{P}$, which is given by

$$S_{\mathcal{P}}(\theta) = \sum_{n=0}^{N-1} p_n e^{jn\theta}. \tag{9}$$

The question is how to design the sequence $\mathcal{P}$ to suppress the range sidelobes along a desired Doppler interval. One way to accomplish this is to design the sequence $\mathcal{P}$ so that its spectrum $S_{\mathcal{P}}(\theta)$ has a high-order null at a Doppler frequency inside the desired interval. This idea has been explored in [13]–[15], where it is shown that the spectrum of a PTM sequence of length 2^{M+1} has an Mth-order null at $\theta = 0$.

Definition 3: [16],[17] The Prouhet-Thue-Morse (PTM) sequence $\mathcal{P} = (p_k)_{k\geq 0}$ over $\{-1, 1\}$ is defined by the following recursions:

1) $p_0 = 1$
2) $p_{2k} = p_k$
3) $p_{2k+1} = \overline{p}_k = -p_k$

for all $k > 0$.

Example: The PTM sequence of length 8 is

$$\mathcal{P} = (p_k)_{k=0}^{7} \;=\; +1\; -1\; -1\; +1\; -1\; +1\; +1\; -1.$$

The corresponding pulse train of Golay complementary waveforms is given by

$$Z_{\mathcal{P}}(t) = s_x(t) + s_y(t-T) + s_y(t-2T) + s_x(t-3T) + s_y(t-4T) + s_x(t-5T) + s_x(t-6T) + s_y(t-7T).$$

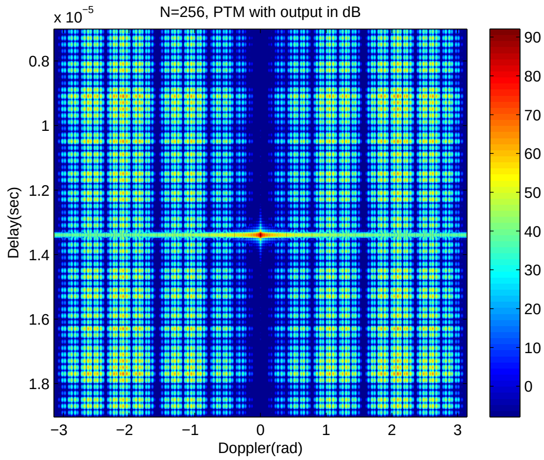
The ambiguity function of $Z_{\mathcal{P}}(t)$ has a second-order null along the zero-Doppler axis.

Figure 3(a) shows the ambiguity function of a length-($N = 2^8$) PTM pulse train of Golay complementary waveforms, which has a seventh-order null at zero-Doppler. The horizonal axis is Doppler shift in rad and the vertical axis is delay in sec. The magnitude of the pulse train ambiguity function is color coded and presented in dB scale. A zoom-in around zero-Doppler is shown in Fig. 3(b), demonstrating that the range sidelobes inside the Doppler interval $[-0.1, 0.1]$ rad have been cleared out. They are at least 80 dB below the peak of the ambiguity function. Figure 4 shows the effect of range sidelobe suppression in bringing out weak targets in the presence of strong reflectors for the five target scenario discussed earlier in Fig. 2. This example demonstrates the value of PTM pulse trains for radar imaging.

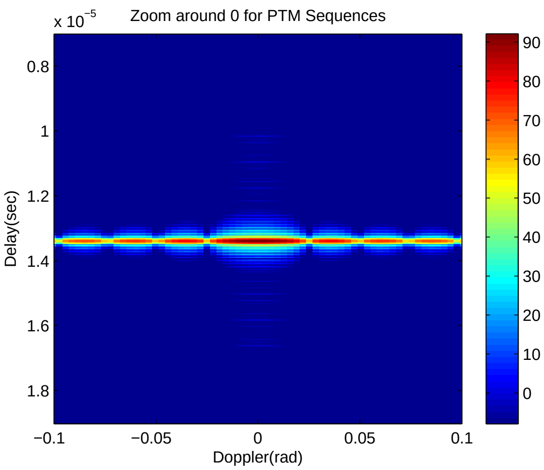
IV. Oversampled PTM Pulse Trains

We now consider the design of biphase sequences whose spectra have high-order nulls at Doppler frequencies other than zero. Consider the Taylor expansion of the spectrum $S_{\mathcal{P}}(\theta)$ around $\theta = \theta_0 \neq 0$:

$$S_{\mathcal{P}}(\theta) = \sum_{t=0}^{\infty}\frac{1}{n!} f_{\mathcal{P}}^{(t)}(\theta_0)(\theta-\theta_0)^t \tag{10}$$



(a)



(b)

Fig. 3. Ambiguity function of a length-($N = 2^8$) PTM pulse train of Golay complementary waveforms: (a) the entire Doppler band (b) Doppler band $[-0.1, 0.1]$ rad.

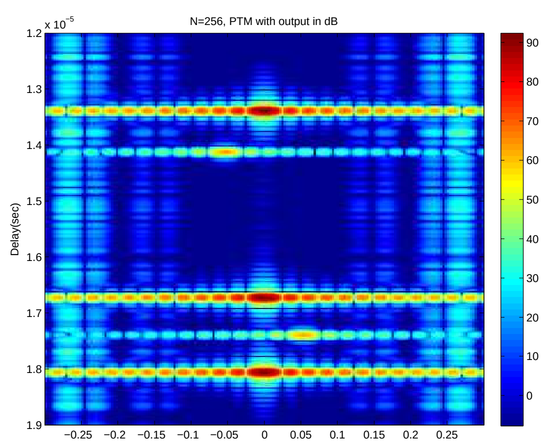


Fig. 4. The PTM pulse train clears the Doppler induced range sidelobes along modest Doppler shifts, and brings out the weak targets.

where the coefficients $f_{\mathcal{P}}^{(t)}(\theta_0)$ are given by

$$f_{\mathcal{P}}^{(t)}(\theta_0) = \left[\frac{d^t}{d\theta^t} S_{\mathcal{P}}(\theta)\right]_{\theta=\theta_0} = j^t \sum_{n=0}^{N-1} n^t p_n e^{jn\theta_0}, \quad (11)$$

for $t = 0, 1, 2, \cdots$. We wish to zero-force all the derivatives $f_{\mathcal{P}}^{(t)}(\theta_0)$ up to order M, that is we wish to design the sequence $\mathcal{P}$ so that

$$f_{\mathcal{P}}^{(t)}(\theta_0) = 0, \quad \text{for all} \quad t = 0, 1, \cdots, M \quad (12)$$

We consider rational Doppler shifts $\theta_0 = 2\pi l/m$, where $l \neq 0$ and $m \neq 1$ are co-prime integers. We assume the length of $\mathcal{P}$ is $N = mq$ for some integer q. If we express $0 \leq n \leq N-1$ as $n = rm + i$, where $0 \leq r \leq q-1$ and $0 \leq i \leq m-1$, then using the binomial expansion for $n^t = (rm+i)^t$ we can write $f_{\mathcal{P}}^{(t)}(\theta_0)$ as

$$f_{\mathcal{P}}^{(t)}(\theta_0) = j^t \sum_{u=0}^{t} \binom{t}{u} m^u \sum_{i=0}^{m-1} i^{t-u} e^{j\frac{2\pi l i}{m}} \left[\sum_{r=0}^{q-1} r^u p_{rm+i}\right]$$

Define a length-q sequence $\{b_r\}_{r=0}^{q-1}$ as $b_r = p_{rm+i}$, $0 \leq r \leq q-1$. If $\{b_r\}_{r=0}^{q-1}$ satisfies

$$\sum_{r=0}^{q-1} r^u b_r = 0, \quad \text{for all } 0 \leq u < t \quad (13)$$

then the coefficient $f_{\mathcal{P}}^{(t)}(\theta_0)$ will be zero. It follows that the zero-forcing condition in (13) will be satisfied if $\{b_r\}_{r=0}^{q-1}$ is the PTM sequence of length 2^t. We note that $f_{\mathcal{P}}^{(M)}(\theta_0)$ is automatically zero as $\sum_{i=0}^{m-1} e^{j\frac{2\pi l i}{m}} = 0$. Therefore, to zero-force the derivatives $f_{\mathcal{P}}^{(t)}(\theta_0)$ for all $t \leq M$, it is sufficient to select $\mathcal{P} = \{p_n\}_{n=0}^{2^M m-1}$ such that each $\{p_{rm+i}\}_{r=0}^{q-1}$, $i = 0, \cdots, m-1$ is the length-2^M PTM sequence. We call such a sequence a $(2^M, m)$-*PTM sequence*. The $(2^M, m)$-PTM sequence has length $2^M \times m$ and is constructed from the length-2^M PTM sequence by repeating each 1 and -1 in the PTM sequence m times, that is by oversampling the PTM sequence by a factor m.

Let $\mathcal{P} = \{p_n\}_{n=0}^{2^M m-1}$ be the $(2^M, m)$-PTM sequence, that is $\{p_{rm+i}\}_{r=0}^{2^M-1}$, $i = 0, \cdots, m-1$ is a PTM sequence of length 2^M. Then the spectrum $S_{\mathcal{P}}(\theta)$ of $\mathcal{P}$ has Mth-order nulls at all $\theta_0 = 2\pi l/m$ where $l \neq 0$ and $m \neq 1$ are co-prime integers.

Corollary: Let $\mathcal{P}$ be the $(2^M, m)$-PTM sequence. Then the spectrum $S_{\mathcal{P}}(\theta)$ of $\mathcal{P}$ has

1) an $(M-1)$th-order null at $\theta_0 = 0$.
2) $(M-h-1)$th-order nulls at all $\theta_0 = 2\pi l/(2^h m)$, where l and $m \neq 1$ are co-prime, and $1 \leq h \leq M-1$.

Example: The spectrum of the $(2^3, 2)$-PTM sequence, shown in solid line in Fig. 5, has a third-order null at $\theta_0 = \pi$ rad, a second-order null at $\theta_0 = 0$ rad, first-order nulls at $\theta_0 = \pi/2$ rad and $\theta_0 = 3\pi/2$ rad, and zeroth-order nulls at $\theta_0 = (2k+1)\pi/4$ rad for $k = 0, 1, 2, 3$. Fig. 5 also shows the spectrum of the $(2^2, 2)$-PTM sequence (dashed line), which has a second-order null at $\theta_0 = \pi$ rad, a first-order null at $\theta_0 = 0$ rad, and zeroth-order nulls at $\theta_0 = \pi/2$ rad and $\theta_0 = 3\pi/2$ rad.

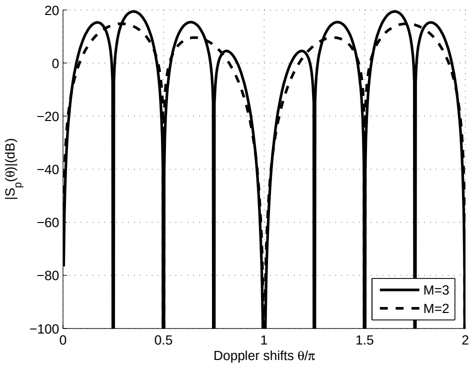
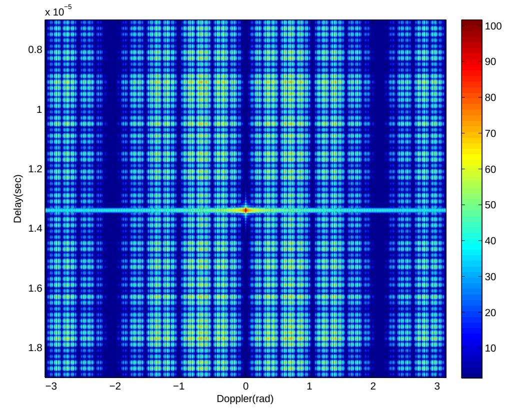


Fig. 5. The spectra of $(2^3, 2)$- and $(2^2, 2)$-PTM sequences.

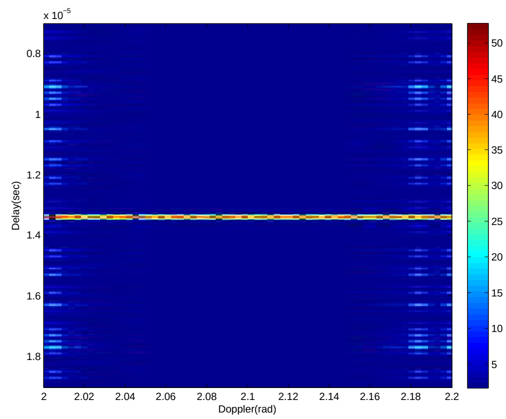
Figure 6(a) shows the ambiguity function of a $(2^8, 3)$-PTM sequence of Golay complementary waveforms. The color bar values are in dB. This ambiguity function has an eighth-order null at $\theta_0 = \pm 2\pi/3$, a seventh-order null at zero Doppler, sixth-order nulls at $\theta_0 = \pm\pi/3$, and so on. A zoom-in around $\theta_0 = 2\pi/3$ is provided in Fig. 6(b) to demonstrate that range sidelobes in this Doppler region are significantly suppressed. The range sidelobes in this region are at least 80 dB below the peak of the ambiguity function.

V. Conclusions

Doppler resilient pulse train of Golay complementary waveforms are constructed by coordinating the transmission of a Golay pair of phase coded waveforms in time according to the 1's and -1's in a PTM sequence or its oversampled versions. The magnitude of the range sidelobes of the pulse train ambiguity function of the constructed pulse trains are proportional to the magnitude spectra of $(2^M, m)$-PTM sequences, which have high-order nulls at rational Doppler shifts $\theta_0 = 2\pi l/m$, where $l \neq 0$ and $m \neq 1$ are co-prime integers. These high-order nulls suppress the range sidelobes of the pulse train ambiguity function inside Doppler intervals where Doppler shifts $\theta_0 = 2\pi l/m$ lie in. Numerical examples demonstrate the annihilation of range sidelobes in the ambiguity functions of $(2^M, m)$-PTM pulse trains.



(a)



(b)

Fig. 6. Ambiguity function of the $(2^8, 3)$-PTM pulse train of Golay complementary waveforms: (a) the entire Doppler band (b) Zoom in around $\theta_0 = 2\pi/3$.

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