Combinatorial Enumeration, Spring 2010

Homework 1 - solutions

1. (a) Find all groups G for which the action of G on itself by left conjugation is transitive.

   (b) Group actions $\alpha : G \to S(X)$ and $\beta : H \to S(Y)$ are *isomorphic* ($\alpha \simeq \beta$) iff there is a group isomorphism $\varphi : G \rightarrowtail\!\!\!\!\twoheadrightarrow H$ and a bijection $f : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$ such that $f(g \cdot x) = \varphi(g) \cdot f(x)$ for all $g \in G$, $x \in X$.

   Let $\alpha : G \to S(X)$ be a transitive action, $x \in X$, G_x the stabilizer of x under α, and $\beta : G \to S(G/G_x)$ the action of G on the set of left cosets of G_x by left multiplication. Prove that $\alpha \simeq \beta$.

   Solution:

   (a) Let $g \in G$ be arbitrary. If this action is transitive, there is $h \in G$ such that $h \cdot e = g$. But $h \cdot e = heh^{-1} = e$, so $g = e$. Hence the only such group is the trivial group $G = \{e\}$.

   (b) Let $y \in X$ be arbitrary. Since α is transitive, there is $g \in G$ such that $g \cdot_\alpha x = y$. Define $f : X \to G/G_x$ by $f(y) = g\,G_x$.

      (i) We have $g_1 \cdot_\alpha x = g_2 \cdot_\alpha x \iff g_1^{-1} g_2 \cdot_\alpha x = x \iff g_1^{-1} g_2 \in G_x \iff g_1 G_x = g_2 G_x$, so f is unambigously defined and injective. It is also surjective, since $g\,G_x = f(g \cdot_\alpha x)$ for any $g \in G$.

      (ii) Let $g \in G$ and $y \in X$ be arbitrary, and let $h \in G$ be such that $h \cdot_\alpha x = y$. Then $f(g \cdot_\alpha y) = f(g \cdot_\alpha (h \cdot_\alpha x)) = f(gh \cdot_\alpha x) = gh\,G_x = g \cdot_\beta h\,G_x = \varphi(g) \cdot_\beta f(y)$ where $\varphi = \mathrm{id}_G$, so $\alpha \simeq \beta$.

2. (a) Show that the cycle index polynomial for the standard action of the cyclic group C_n is given by

   $$P_{C_n}(y_1, \ldots, y_n) = \frac{1}{n} \sum_{k \,|\, n} \varphi(k) y_k^{n/k}$$

   where $\varphi(k)$ denotes Euler's totient function.

   (b) Compute the cycle index polynomial $P_{D_n}(y_1, \ldots, y_n)$ for the standard action of the dihedral group D_n.

   Solution:

   (a) We need to determine the cycle type of r^k where $0 \le k < n$. Let x be a vertex of the n-gon. The length of the cycle of r^k containing x equals the order of $r^k \in C_n$, which is the least $\ell \ge 1$ such that n divides $k\ell$. Hence $k\ell = \mathrm{lcm}(n, k)$, and $\ell = \mathrm{lcm}(n, k)/k = n/d$ where $d = \gcd(n, k)$. So r^k contains d cycles of length n/d, and contributes $y_{n/d}^d$ to the sum in the cycle index polynomial. For each divisor d of n, there are $\varphi(n/d)$ elements $k \in \{0, 1, \ldots, n-1\}$ such that $\gcd(n, k) = d$. Therefore

   $$P_{C_n}(y_1, \ldots, y_n) \;=\; \frac{1}{n} \sum_{d \,|\, n} \varphi\left(\frac{n}{d}\right) y_{n/d}^d \;=\; \frac{1}{n} \sum_{d \,|\, n} \varphi\left(d\right) y_d^{n/d}.$$

(b) Since $|D_n| = 2|C_n|$, the contribution of rotations to $P_{D_n}(y_1, \ldots, y_n)$ is $(1/2)P_{C_n}(y_1, \ldots, y_n)$. To analyze reflections, distinguish two cases.

i. n even: There are $n/2$ reflections across median, each having $n/2$ cycles of length 2. There are also $n/2$ reflections across main diagonal, each having $(n-2)/2$ cycles of length 2 and 2 cycles of length 1. Hence the total contribution of reflections to the sum in the cycle index polynomial is $(n/2)(y_2^{n/2} + y_1^2 y_2^{(n-2)/2})$.

ii. n odd: There are n reflections, each having $(n-1)/2$ cycles of length 2 and 1 cycle of length 1, contributing $n y_1 y_2^{(n-1)/2}$ to the sum in the cycle index polynomial.

Since $|D_n| = 2n$, we thus obtain

$$P_{D_n}(y_1, \ldots, y_n) = \frac{1}{2n} \sum_{d \mid n} \varphi(d)\, y_d^{n/d} + \begin{cases} \frac{1}{4}(y_2^{n/2} + y_1^2 y_2^{(n-2)/2}), & n \text{ even}, \\ \frac{1}{2} y_1 y_2^{(n-1)/2}, & n \text{ odd}. \end{cases}$$

3. (a) A roulette wheel has 37 sectors. In how many different ways can the sectors be colored if we have 3 colors at our disposal? We consider two colorings different if one cannot be turned into the other by spinning the wheel.

(b) Same as above, but now we count only those colorings in which each of the 3 colors is actually used.

(c) How many different necklaces containing 30 glass beads can be made if we have beads of 2 different colors? Two necklaces are considered different if one cannot be turned into the other by rotating it and/or turning it over.

Solution:

(a) We are counting colorings under the standard action of C_{37}. The cycle index polynomial is (see 2(a))

$$P_{C_{37}}(y_1, y_2, \ldots, y_{37}) = \frac{1}{37}\left(y_1^{37} + 36\, y_{37}\right),$$

so by the RPT the answer is

$$P_{C_{37}}(3, 3, \ldots, 3) = \frac{3^{37} + 108}{37} = 12\,169\,835\,294\,351\,283.$$

(b) Denote by A_i the set of colorings which are missing color i. Then the set of colorings not missing any of the m colors is $\bigcap_{i=1}^m A_i^c$, where A_i^c is the complement of A_i. By the inclusion-exclusion principle and the RPT, we obtain

$$|\bigcap_{i=1}^{m} A_i^c| = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} P_\alpha(k, k, \ldots, k).$$

So the answer is

$$\sum_{k=0}^{3} (-1)^{3-k} \binom{3}{k} P_{C_{37}}(k, k, \ldots, k) = 12\,169\,824\,150\,652\,350.$$

(c) We are counting colorings under the standard action of D_{30}. The cycle index polynomial is (see 2(b))

$$P_{D_{30}}(y_1, y_2, \ldots, y_{30}) = \frac{1}{60}\left(y_1^{30} + 15y_1^2y_2^{14} + 16y_2^{15} + 2y_3^{10} + 4y_5^6 + 2y_6^5 + 4y_{10}^3 + 8y_{15}^2 + 8y_{30}\right),$$

so by the RPT the answer is

$$P_{D_{30}}(2, 2, \ldots, 2) = 17\,920\,860.$$

4. A *matching* in a graph is a set of mutually nonadjacent edges. Determine the number m_n of matchings in the complete graph K_n embedded in the plane as a regular n-gon with all its diagonals, inequivalent under rotations (i.e., the standard action of the cyclic group C_n). For example, K_4 (drawn as a square with both diagonals) has five inequivalent matchings: the empty matching, a single side, a single diagonal, two parallel sides, both diagonals. The first few numbers m_n are:

n	1	2	3	4	5	6	7	8	9	10
m_n	1	2	2	5	6	18	34	108	294	984

Solution:

By the CFL,

$$m_n = \frac{1}{n}\sum_{g\in C_n} |\mathrm{Fix}\, g|$$

where $\mathrm{Fix}\, g$ is the set of matchings in K_n fixed by g. In how many ways can we construct a matching M fixed by r^k? Denote by G the cyclic subgroup of C_n generated by r^k. As we saw in 2(a), the action of G partitions the vertices of K_n into $d = \gcd(n, k)$ orbits of size n/d. Denote by R a set of d consecutive vertices. Since R contains one representative from each orbit, it suffices to define M on R, then extend it by rotational symmetry. A vertex in R can be safely matched to any vertex in another orbit. It cannot be matched to a vertex in its own orbit, unless it is its antipode, which can only happen if n is even and d divides $n/2$. Hence we can construct M in three stages:

1. Select $2j$ of the orbits and match them in pairs. This can be done in $\binom{d}{2j}(2j-1)!!$ ways.

2. For each pair of matched orbits $\{\mathcal{O}_1, \mathcal{O}_2\}$, select one of the n/d vertices in $\mathcal{O}_2$ that will be matched with the vertex in $\mathcal{O}_1 \cap R$. Since there are j pairs of orbits, this can be done in $(n/d)^j$ ways.

3. For each of the unmatched $d - 2j$ orbits $\mathcal{O}$, there are

$$t(n,d) = \begin{cases} 2, & \text{if } 2d \mid n, \\ 1, & \text{otherwise} \end{cases} \tag{1}$$

options of either matching the vertex in $\mathcal{O} \cap R$ to its antipodal vertex in $\mathcal{O}$, or not.

Here $2j$ can have any even value between 0 and d, so

$$|\mathrm{Fix}\, r^k| \;=\; \sum_{0\le 2j\le d} \binom{d}{2j} (2j-1)!! \left(\frac{n}{d}\right)^j t(n,d)^{d-2j}$$

where $d = \gcd(n,k)$. As in 2(a), for each divisor d of n there are $\varphi(n/d)$ values $k \in \{0,1,\dots,n-1\}$ such that $\gcd(n,k) = d$. Hence the answer is

$$m_n \;=\; \frac{1}{n}\sum_{d|n} \varphi\left(\frac{n}{d}\right) \sum_{0\le 2j\le d} \binom{d}{2j} (2j-1)!! \left(\frac{n}{d}\right)^j t(n,d)^{d-2j}$$

with $t(n,d)$ as given in (1).