A question

We saw in class that species having identical cycle index series are not necessarily isomorphic, but the species in that counterexample were not molecular. The question arose:

If M_1 and M_2 are <u>*molecular*</u> *species having identical cycle index series, does it follow that they are isomorphic?*

The answer

From Theorems 3 and 4 of *Section 1.5: Molecular species* (see Lesson 9) we see that this question is equivalent to the following:

If H_1 and H_2 are subgroups of S_n having identical cycle index polynomials, does it follow that they are conjugate?

Professors P. Potočnik and M. Conder kindly showed that the answer to this question is again negative. The smallest counterexample is furnished by the subgroups

$$\begin{aligned} H_1 &= \{id, (1,2)(4,5), (1,4)(2,5), (1,5)(2,4)\}, \\ H_2 &= \{id, (1,5)(4,6), (2,3)(4,6), (1,5)(2,3)\} \end{aligned}$$

of S_6. Obviously these two groups have identical cycle index polynomial

$$P_{H_1:[n]}(y_1, \ldots, y_6) = P_{H_2:[n]}(y_1, \ldots, y_6) = \frac{1}{4}\left(y_1^6 + 3y_1^2 y_2^2\right),$$

but they are not conjugate: If they were, H_1 could be obtained from H_2 by a suitable relabeling of $[n]$. This, however, is impossible since H_2 contains a pair of elements sharing a transposition (namely $(4,6)$), whereas H_1 contains no such pair.

The above groups H_1 and H_2 are obviously isomorphic since they are both isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. It seems that for $n \geq 8$ there exist subgroups of S_n having identical cycle index polynomials which not only are not conjugate, but are also non-isomorphic.