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# Compact open topology and CW homotopy type

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**Abstract**

The class $\mathcal{W}$ of spaces having the homotopy type of a CW complex is not closed under formation of function spaces. In 1959, Milnor proved the fundamental theorem that, given a space $Y \in \mathcal{W}$ and a compact Hausdorff space $X$, the space $Y^X$ of continuous functions $X \to Y$, endowed with the compact open topology, belongs to $\mathcal{W}$. P.J. Kahn extended this in 1982, showing that $Y^X \in \mathcal{W}$ if $X$ has finite $n$-skeleton and $\pi_k(Y) = 0$, $k > n$.

Using a different approach, we obtain a further generalization and give interesting examples of function spaces $Y^X \in \mathcal{W}$ where $X \in \mathcal{W}$ is not homotopy equivalent to a finite complex, and $Y \in \mathcal{W}$ has infinitely many nontrivial homotopy groups. We also obtain information about some topological properties that are intimately related to CW homotopy type.

As an application we solve a related problem concerning towers of fibrations between spaces of CW homotopy type.


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*Keywords:* Homotopy type; CW complex; Function space; Tower of fibrations

## 1. Introduction

It is well known that the class $\mathcal{W}$ of topological spaces having the homotopy type of a CW complex is not closed under formation of function spaces. In [13], Milnor has shown that if a space $Y$ belongs to $\mathcal{W}$ and $X$ is a compact Hausdorff space, the space of continuous functions $Y^X$ endowed with the compact open topology also belongs to $\mathcal{W}$.

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In a generalization in [10], Kahn has weakened the compactness assumption for the domain $X$, thereby imposing restrictions on the target $Y$. He proved that if $X$ and $Y$ are connected CW complexes such that $\pi_k(Y) = 0$ for $k \geqslant n+1$ and $X$ has finite $n$-skeleton, then the space $Y^X$ has the homotopy type of a CW complex. He uses an inverse limit approach, reducing the problem to the fact that for a tower of fibrations between contractible spaces, the limit space is also contractible.

In Section 2 we extend the result of Kahn, obtaining a sufficient condition on connected CW complexes $X$ and $Y$ for the function space $Y^X$ to have CW homotopy type with no *a priori* restrictions on homotopy groups of $Y$. We present an example of a space $Y^X$ of CW homotopy type, where $X$ is not homotopy equivalent to a finite complex, and $Y$ has infinitely many nontrivial homotopy groups.

In Section 3 we restate the result of Section 2 in a sharper version for the case when $X$ is a countable complex, obtaining also some information about topological properties of $Y^X$ that are intimately related to CW homotopy type. We show that, roughly, if $X$ is a skeleton-finite complex, and $H^i(X; \pi_j(Y)) = 0$ for all large enough $j$ and all large enough $i$ such that $i \leqslant j$, then $Y^X$ has CW homotopy type, and is well-pointed for any choice of base-point.

We give an interesting generalization of the main theorem, which implies that for any countable complex $X$, and a complex $Y$ with $\pi_k(Y) = 0$ for $k \geqslant n+1$, $\Omega^n Y^X$ has CW homotopy type.

As an application of our results we present in Section 4 an explicit counterexample that clarifies a problem regarding a small error in Theorem B of [5] (as noted in [6]).

The proofs of the main theorems of Sections 2 and 3 are based on an inverse limit argument, viewing $Y^X$ as the limit of inverse system $\{Y^L\}$, where $L$ ranges over finite subcomplexes of $X$ and the bonding maps are restriction fibrations $Y^{L_2} \to Y^{L_1}$. Milnor's theorem implies that every space in this system has CW homotopy type, and we investigate conditions under which the limit space has CW type as well. This approach was inspired by the paper of Dydak and Geoghegan [5]. In the proof of Theorem 2.1 we employ a technique used by Geoghegan in [8].

Our approach differs from that of Kahn in two aspects. First, he views $Y^X$ as the inverse limit of $\{Y^{X^{(n)}} \mid n\}$, where $X^{(n)}$ denotes the $n$-skeleton of $X$. Next, he works in the category of compactly generated Hausdorff spaces. In particular, function spaces are endowed with the compactly generated refinement of the compact open topology. We use throughout the compact open topology, and observe in Section 3 that if a Hausdorff space $Z$ has CW type, so has $k(Z)$, where $k$ denotes the "compact generation" functor, and that for a countable complex $X$, the space $Y^X$ has CW type if and only if $k(Y^X)$ has CW type. Hence our Corollary 2.4 implies Theorem 1.1. of [10], and in the general case not vice versa.

CW COMPLEXES, SIMPLICIAL COMPLEXES AND ANRS. For a topological space $Z$ the following are equivalent:

(i) $Z$ has the homotopy type of a CW complex.
(ii) $Z$ has the homotopy type of a simplicial complex (with the CW topology).
(iii) $Z$ has the homotopy type of a simplicial complex with the metric topology.
(iv) $Z$ has the homotopy type of an absolute neighbourhood retract (ANR) for the class of metric spaces.

(See [13], Theorem 2, for details on equivalence of (i), (ii), (iii), and [11], Theorem I.4.1, for details on equivalence of (ii) and (iv).) For our purposes (as for homotopy theory in general) the class of CW complexes is the most appropriate, but we will also find use for the equivalence of (i) and (iii).

On the other hand, the class of ANRs enjoys some nice topological properties that the class of CW complexes does not; for example, it is closed under formation of Cartesian products and loop spaces.

Indeed, as has been pointed out by the referee (see also Milnor [13], Corollary 2), in the "ANR version" the following weaker form of Milnor's theorem is an immediate corollary of the exponential law (see Mardešić and Segal [11], Theorem I.3.4): Let $Y$ be an ANR and $X$ a compact metric space. Then $Y^X$ is an ANR.

**Conventions and notation.** We work with specific CW decompositions of the domain complex $X$, and formulate the theorems accordingly. Since homotopy equivalences $X \simeq X'$ and $Y \simeq Y'$ induce a homotopy equivalence $Y^X \simeq Y'^{X'}$ (see Theorem 6.2.25 of [12] for a proof), everything can be interpreted in a homotopy invariant form. The standard conditions on the domain complex $X$ are finiteness of each skeleton $X^{(k)}$ and finiteness of a single skeleton $X^{(n)}$. We refer the reader to the papers of Wall [18,19] for an algebraic characterization of spaces having the homotopy type of such complexes.

Since we are discussing spaces of CW homotopy type, we distinguish strictly between homotopy equivalent and *weakly* homotopy equivalent, shortly just equivalent and weakly equivalent, respectively. Occasionaly we shorten 'homotopy type' to 'type'.

The term fibration is used for a Hurewicz fibration, that is a (not necessarily surjective) map with the homotopy lifting property with respect to all spaces.

We use the following notation.

- $Y^X$ the space of continuous functions $X \to Y$, endowed with the compact open topology,
- $C_g(L)$ the path component of $Y^L$ that contains $g|_L : L \to Y$, where $L$ is a subcomplex of $Y$ and $g : X \to Y$ is a map,
- $X(K)$ the smallest subcomplex of the CW complex $X$ that contains $K$.

## 2. A sufficient condition for the function space $Y^X$ to have CW homotopy type

The main result of this section is the following

**Theorem 2.1.** *Let $X$ and $Y$ be connected CW complexes, and let $g : X \to Y$ be a map. Suppose there exists a finite subcomplex $T$ of $X$ with the following properties*

(i) $X^{(1)} \subset T$,
(ii) *every subcomplex $L \leqslant X$ containing $T$ is contained in a subcomplex $L'$ such that $L'/L$ is finite and*

$$H^i\big(L', T; (g|_{L'})^{\#}\pi_j(Y)\big) = 0 \quad \textit{for all } j \geqslant 2 \textit{ and all } i \leqslant j,$$

*where the cohomology is taken with local coefficients, $(g|_{L'})^{\#}\pi_j(Y)$ being the $\mathbb{Z}\pi_1(L')$-module obtained by restriction $(g|_{L'})_{\#}:\pi_1(L') \to \pi_1(Y)$.*

*Then the path component $C$ of $g$ in $Y^X$ is open and closed in $Y^X$, and is homotopy equivalent to the path component of $g|_T$ in $Y^T$, hence has the homotopy type of a CW complex.*

**Addendum.** *If $Y$ is simply connected, then condition* (i) *of Theorem* 2.1 *may be omitted.*

**Corollary 2.2.** *If the conditions of the theorem are satisfied for all maps $g: X \to Y$, then $Y^X$ is homotopy equivalent to the union of some path components of $Y^T$, hence has the type of a CW complex.*

**Corollary 2.3.** *If $X$ has no* 1*-cells* (*or $T$ can be so chosen that $\pi_1(T)$ is trivial*), *or if $Y$ is a simple space, the cohomology in property* (ii) *of the theorem is the usual cohomology $H^i(L', T; \pi_j(Y))$, independently of the map $g: X \to Y$. If the condition of the theorem is satisfied, the conclusion applies to all path components of $Y^X$ which is again homotopy equivalent to the union of some path components of $Y^T$.*

**Corollary 2.4.** *If $X$ is homotopy equivalent to a connected CW complex with finite $n$-skeleton, and $Y$ is homotopy equivalent to a connected CW complex with $\pi_k(Y)$ trivial for $k \geqslant n+1$, then $Y^X$ belongs to $\mathcal{W}$.*

**Remark 2.5.** Corollary 2.4 corresponds to the result of Kahn [10, Theorem 1.1] for the compact open topology. Note that Corollary 2.4 implies Theorem 1.1 of [10], since for Hausdorff spaces $Z_1$, $Z_2$, $Z_1 \simeq Z_2$ implies $k(Z_1) \simeq k(Z_2)$. See also Corollary 3.5.

We shall make use of the following “strong form” of the exponential law for the compact open topology (see [4]).

**Lemma 2.6.** *The exponential law $Y^{L\times B} \approx (Y^L)^B$ holds true if the product $L \times B$ is compactly generated and $L$ is Hausdorff.*

For the sake of brevity, we adopt the following

**Notation.**

(i) *For $L_1 \leqslant L_2$, we denote the restriction fibration $R_{L_1,L_2}: Y^{L_2} \to Y^{L_1}$. Note that $R(C_g(L_2)) = C_g(L_1)$.*

(ii) *We denote the cohomological condition of Theorem* 2.1 *as*

$$H^i\big(L', T; (g|_{L'})^{\#}\pi_j(Y)\big) = 0, \quad \forall j \geqslant 2,\ \forall i \leqslant j. \qquad (\mathcal{WE}_g(L'))$$

(*The subcomplex $T$ of $X$ is assumed fixed.*) *The letters $\mathcal{WE}$ stand for* “*weak equivalence*”; *see Lemma* 2.7.

**Lemma 2.7.** *Consider the restriction fibration $Y^{L'} \xrightarrow{R} Y^T$. The property $\mathcal{WE}_g(L')$ implies that $R|_{C_g(L')} : C_g(L') \to C_g(T)$ is a weak homotopy equivalence and $R^{-1}(C_g(T)) = C_g(L')$. The latter implies that $C_g(L')$ is both open and closed in $Y^{L'}$.*

**Proof.** Denote by $F$ the fibre over $g|_T$ of $Y^{L'} \to Y^T$. We claim that $\pi_k(F, g|_{L'})$ is trivial, for all $k \geqslant 0$. By Lemma 2.6 a loop $(S^k, *) \to (F, g|_{L'})$ corresponds to a map $\phi : S^k \times L' \to Y$. To prove the claim we have to extend $\phi$ over $B^{k+1} \times L'$, the restriction to $B^{k+1} \times T$ being $g \circ \mathrm{pr}_T$. If $T$ contains the 1-skeleton of $X$, then the subcomplex $B^{k+1} \times T \cup S^k \times L'$ contains the 2-skeleton of $B^{k+1} \times L'$, for any $k \geqslant 0$, so in this case $\phi$ is already 2-extended. All further obstructions vanish. As regards the addendum to Theorem 2.1, we note that if $\pi_1(Y) = 0$, then $\phi$ can be 2-extended with no conditions on $T$. □

We begin with a technical lemma.

**Lemma 2.8.** *Assume $p : E \to B$ is a fibration and $\pi : E \to B$ is a homotopy equivalence. Let $M(\pi)$ denote the mapping cylinder of $\pi$. If $r : M(\pi) \to B$ is a map such that $r|_{E\times 1} = p$, then there exist a map $s : B \to E$ such that $p \circ s = r|_B$ and a homotopy $h : E \times I \to E$ between the composite $s \circ \pi$ and $\mathrm{id}_E$ such that $p \circ h = r \circ q$ where $q : E \times I + B \to M(\pi)$ is the quotient map.*

*If, in addition, $\pi(e_0) = b_0$ and $r(b_0) = p(e_0)$ where $\{e_0\} \hookrightarrow E$ and $\{b_0\} \hookrightarrow B$ are closed cofibrations, then it can be arranged that $s(b_0) = e_0$ with the homotopy being* $\mathrm{rel}\{e_0\}$.

**Remark.** If $p : E \to B$ is a fibration and a homotopy equivalence, then taking $\pi$ to be $p$ and $r$ the standard retraction $M(p) \to B$, we obtain a right inverse $s$ for $p$ and a fibre-preserving homotopy between $s \circ p$ and the identity map.

**Proof.** Consider the following diagram

$$\begin{array}{ccc} E \times 1 & \xrightarrow{\mathrm{id}_E} & E \\ \simeq\downarrow & \nearrow & \downarrow p \\ M(\pi) & \xrightarrow{r} & B \end{array}$$

The inclusion $E \times 1 \hookrightarrow M(\pi)$ is both a closed cofibration and a homotopy equivalence, hence Theorem 3 of [17] may be applied to obtain the filling which yields both a map $s$ and a homotopy $h$ as claimed.

For the second part of the lemma, one quickly checks that the inclusion $q(E \times 1 \cup \{e_0\} \times I) \hookrightarrow M(\pi)$ is a closed cofibration (and a homotopy equivalence). Making an obvious replacement in the above diagram, we conclude the proof. □

We are now ready to prove the theorem.

**Proof of Theorem 2.1.** Keeping the map $g$ fixed, write $C(L)$ instead of $C_g(L)$ and $\mathcal{WE}(L)$ instead of $\mathcal{WE}_g(L)$. Since $T$ is finite, $Y^T \in \mathcal{W}$ by Milnor's theorem. Hence $C(T)$

is both open and closed in $Y^T$, and $C(T) \in \mathcal{W}$. The assumptions imply $\mathcal{WE}(X)$, hence $R_{T,X}^{-1}(C(T)) = C(X)$, and $C(X)$ is open and closed in $Y^X$. Define

$$\Lambda = \bigl\{(L, E, h) \mid T \leqslant L \leqslant X, E : C(T) \to C(L); \\ R_{T,L} \circ E = \mathrm{id}_{C(T)}, h : E \circ R_{T,L} \simeq \mathrm{id}_{C(L)}\bigr\}.$$

Note that by definition, for $(L, E, h) \in \Lambda$, $C(T) \simeq C(L) \in \mathcal{W}$.

Order $\Lambda$ by setting $(L_1, E_1, h_1) \prec (L_2, E_2, h_2)$ if $L_1 \leqslant L_2$, and the following coherence conditions hold:

$$R_{L_1,L_2} \circ E_2 = E_1, \tag{$*$}$$

$$R_{L_1,L_2} \circ h_2 = h_1 \circ (R_{L_1,L_2} \times \mathrm{id}_I). \tag{$**$}$$

This is evidently a partial ordering.

Let $S = \{(L_\alpha, E_\alpha, h_\alpha) \mid \alpha\}$ be a nonempty totally ordered subset of $\Lambda$. We claim that there exists an upper bound for $S$. Let $L = \bigcup_\alpha L_\alpha$. Form an inverse system $\{C(L_\alpha) \mid \alpha\}$ where the bond for $L_\alpha \leqslant L_\beta$ is the restriction map $R_{L_\alpha,L_\beta}$. One verifies readily that the inverse limit equals $C = \{f : L \to Y \mid f|_{L_\alpha} = R_{L_\alpha,L}(f) \in C(L_\alpha), \forall\alpha\}$ with the natural projections $R_{L_\alpha,L} : C \to C(L_\alpha)$.

The system of maps $E_\alpha : C(T) \to C(L_\alpha)$ defines a map $E : C(T) \to C$ with $E_\alpha = R_{L_\alpha,L}E$ by virtue of $(*)$. In particular, $R_{T,L}E = \mathrm{id}_{C(T)}$. Furthermore, by virtue of $(**)$ the system of homotopies $h_\alpha \circ (R_{L_\alpha,L} \times \mathrm{id}_I) : C \times I \to C(L_\alpha)$ defines a homotopy $h : C \times I \to C$ such that $h : ER_{T,L} \simeq \mathrm{id}_C$ and $R_{L_\alpha,L} \circ h = h_\alpha \circ (R_{L_\alpha,L} \times \mathrm{id}_I)$. This exhibits $C$ as homotopy equivalent to $C(T)$. Therefore $C$ must be path connected and since it contains $C(L)$, it follows that $C = C(L)$. Hence $(L, E, h)$ is an upper bound for $S$.

Let now $(L, E, h)$ be a maximal element of $\Lambda$. If $L \neq X$, then there exists a cell $e$ of $X - L$. By assumption (ii) of Theorem 2.1 there exists a subcomplex $L'$ containing $L \cup X(e)$ such that $L'/(L \cup X(e))$ is finite and $\mathcal{WE}(L')$ holds. Then $C(L') \to C(T)$ is a weak equivalence, by Lemma 2.7. Write $L' = L \cup M$ where $M$ is a finite subcomplex. Since $C(L) \to C(T)$ is an equivalence, $C(L \cup M) \to C(L)$ must be a weak equivalence. Since $R_{T,L\cup M}^{-1}(C(T)) = C(L \cup M)$, it follows that $R_{L,L\cup M}^{-1}(C(L)) = C(L \cup M)$. This implies that $C(L \cup M)$ is a topological pull-back of $C(M) \to C(L \cap M) \leftarrow C(L)$. Therefore $C(M) \to C(L \cap M)$ is a weak equivalence, hence by Whitehead's theorem an equivalence, since by Milnor's theorem, $C(M), C(L \cap M) \in \mathcal{W}$. We apply (the remark after) Lemma 2.8 to obtain a right inverse $\varepsilon$ for $R_{L\cap M,M} : C(M) \to C(L \cap M)$ and a fibre-preserving homotopy $\chi : \varepsilon \circ R_{L\cap M,M} \simeq \mathrm{id}_{C(M)}$. Then the map $e : C(L) \to C(L \cup M)$, $\varphi \mapsto \varphi \sqcup \varepsilon(\varphi|_{L\cap M})$, is a right inverse for $R_{L,L\cup M}$, and $C(L \cup M) \times I \to C(L \cup M)$, $(\phi, t) \mapsto \phi|_L \sqcup \chi(\phi|_M, t)$, is a (fibre-preserving) homotopy between $e \circ R_{L,L\cup M}$ and $\mathrm{id}_{C(L\cup M)}$. Hence $C(L \cup M) \to C(L)$ is a homotopy equivalence.

Denote by $M(E \circ R_{T,L\cup M})$ and $M(R_{T,L})$ the mapping cylinders of $E \circ R_{T,L\cup M} : C(L \cup M) \to C(L)$ and $R_{T,L} : C(L) \to C(T)$, respectively. Note that $E \circ R_{T,L\cup M}$ is a homotopy equivalence. Let $r : M(R_{L,L\cup M}) \to C(L)$ be the composite of the map $M(E \circ R_{T,L\cup M}) \to M(R_{T,L})$ induced by $R_{L,L\cup M} \times \mathrm{id}_I + R_{T,L}$ and the map $M(R_{T,L}) \to C(L)$ induced by $h + E$. Since $R_{T,L} \circ E = \mathrm{id}$, $r$ is well defined. Since $r|_{C(L\cup M)\times 1} = h_1 \circ R_{L,L\cup M} = R_{L,L\cup M}$, we may appeal again to Lemma 2.8 to obtain a map $u : C(L) \to C(L \cup M)$ such that $R_{L,L\cup M} \circ u = E \circ R_{T,L}$ and a homotopy $H : C(L \cup M) \times I \to C(L \cup M)$ between $u \circ E \circ R_{T,L\cup M}$ and the identity such that $R_{L,L\cup M} \circ H = h \circ (R_{L,L\cup M} \times \mathrm{id}_I)$.

The equality $R_{L,L\cup M} \circ u = E \circ R_{T,L}$ implies also that $R_{T,L\cup M} \circ (u \circ E) = \mathrm{id}_{C(T)}$. This shows $(L, E, h) \prec (L \cup M, u \circ E, H) \in \Lambda$ which implies $X(e) \leqslant M \leqslant L$, by maximality. The contradiction completes the proof of Theorem 2.1. □

**Example 1.** Here we illustrate the reason for the complicated statement of Theorem 2.1. Let $X = K(\mathbb{Z}_2, 1) = \mathbb{R}P^\infty$ and let $Z$ be a simply connected CW complex of finite type. Set $Y := Z_{(\mathfrak{l})}$, the localization of $Z$ with respect to a nonempty set $\mathfrak{l}$ of odd primes. Note that $Y$ may have infinitely many nontrivial homotopy groups (for instance if $Z$ is any simply connected finite complex that is not contractible).

By Theorem 2.1, $Y^X$ has the homotopy type of a CW complex (namely, $Y$). To see this, take $T = \{*\}$ and for each finite $L \leqslant \mathbb{R}P^\infty$ let $L'$ be $\mathbb{R}P^{2m}$ for some $m$ such that $L \leqslant \mathbb{R}P^{2m}$. Since $Y$ is $\mathfrak{l}$-local, and $\mathbb{R}P^{2m}$ is 2-local while $2 \notin \mathfrak{l}$, evidently $H^i(\mathbb{R}P^{2m}, T; \pi_j(Y)) = \widetilde{H}^i(\mathbb{R}P^{2m}; \pi_j(Y)) = 0$ for all $i, j$. This implies $\mathcal{WE}(\mathbb{R}P^{2m})$.

On the other hand, it is *not* true that $\mathcal{WE}(\mathbb{R}P^{2m+1})$; since $\widetilde{H}_{2m+1}(\mathbb{R}P^{2m+1}) \cong \mathbb{Z}$, and $\pi_j(Y) \neq 0$ for infinitely many $j$, also $H^{2m+1}(\mathbb{R}P^{2m+1}, *; \pi_j(Y)) \neq 0$ for infinitely many $j$. Even more, while it follows from Theorem 2.1 that the evaluation fibration $Y^X \to Y$, and the fibrations $Y^X \to Y^{\mathbb{R}P^{2m}}$ are homotopy equivalences, $Y^{\mathbb{R}P^{2m+1}}$ is not equivalent to $Y^{\mathbb{R}P^{2m}} \simeq Y$. This follows easily by considering the fibration $Y^{\mathbb{R}P^{2m+1}} \to Y^{\mathbb{R}P^{2m}}$. The fibre over the constant map is homeomorphic to the space of pointed maps $(\mathbb{R}P^{2m+1}/\mathbb{R}P^{2m}, *) \to (Y, *)$, that is $(Y, *)^{(S^{2m+1}, *)}$, and this is clearly not contractible if $Y$ has infinitely many nontrivial homotopy groups.

Therefore it is essential that the appropriate complex $L'$ be chosen so as to meet the requirement (ii) of Theorem 2.1. We exploit this in Example 3.

## 3. Countable domain

If $X$ is a countable complex, then the assumptions of Theorem 2.1 may be slightly relaxed, and the conclusions somewhat strengthened. Moreover, the proof may be simplified to a large extent. The improved theorem reads as follows.

**Theorem 3.1.** *Assume $X$ is a connected countable CW complex. Let $Y$ be any connected CW complex, and let $g : X \to Y$ be a map. Suppose there exists a filtration $T = L_0 \leqslant L_1 \leqslant L_2 \leqslant \cdots$ for $X$ consisting of finite subcomplexes of $X$ such that $X^{(1)} \subset T$ and*

$$H^i\big(L_n, L_{n-1}; (g|_{L_n})^\# \pi_j(Y)\big) = 0$$

*for all $j \geqslant 2$ and all $i \leqslant j$. Then the path component $C$ of $g$ is open and closed in $Y^X$, and is homotopy equivalent to the path component of $g|_T$ in $Y^T$.*

*Furthermore, if $f$ is any map in $C$, then the inclusions $\{f\} \hookrightarrow C$ and $\{f\} \hookrightarrow Y^X$ are cofibrations.*

**Addendum.** *If $Y$ is simply connected, it is not necessary that $T$ contain the* 1*-skeleton of $X$.*

We begin with some lemmas.

Recall that a topological space $Z$ is called locally equiconnected if there exist a neighbourhood $V$ of the diagonal $\Delta$ in $Z \times Z$ and a homotopy $\lambda : V \times I \to Z$ such that

$$\lambda(z, w, 0) = z, \quad \lambda(z, w, 1) = w, \quad \lambda(z, z, t) = z, \quad \forall z, w, t.$$

**Lemma 3.2.** *If $T$ is a finite, and $Y$ an arbitrary CW complex, then $Y^T$ is locally equiconnected.*

**Proof.** See Milnor [13, Proof of Lemma 3]. □

**Lemma 3.3.** *If $Z$ is a perfectly normal locally equiconnected space, then, for every point $z \in Z$, the inclusion $\{z\} \hookrightarrow Z$ is a closed cofibration.*

**Proof.** By Strøm [17], a closed inclusion $A \hookrightarrow Z$ is a cofibration if and only if there exist a neighbourhood $U$ of $A$ which deforms rel $A$ in $Z$ to $A$, and a function $\alpha : Z \to I$ such that $A = \alpha^{-1}(0)$ and $\alpha|_{Z-U} = 1$.

It follows immediately from the definition of local equiconnectedness that each point $z$ has a neighbourhood that deforms rel $z$ in $Z$ to $z$. Since in a perfectly normal space every closed set is a $G_\delta$, we obtain the function $\alpha : Z \to I$ with the desired properties using the Urysohn lemma. □

**Lemma 3.4.** *Let $X$ be a countable, and $Y$ an arbitrary CW complex. Then*

(i) *the space $Y^X$ is paracompact and perfectly normal, and*
(ii) *$Y^X$ is homotopy equivalent to a first countable space, from which it follows that the natural map $k(Y^X) \to Y^X$ is a homotopy equivalence.*

**Proof.** For (i), we note that $X$ may be formed as an expanding union of compact subcomplexes. Therefore by Cauty [3], $Y^X$ is stratifiable. This implies that it is paracompact and perfectly normal.

We turn to (ii) by taking first a simplicial complex $Z$, equipped with the metric topology, that is homotopy equivalent to $Y$. Then $Z^X$ is homotopy equivalent to $Y^X$. Since $Z$ is metrizable, $Z^X$ has a basis consisting of sets of type $U(f, K, \varepsilon) = \{g \mid d_K(f, g) < \varepsilon\}$ where $d$ is a metric for $Z$ and $K$ is a compact subset of $X$. Note that for $f \in Z^X$ the sets $U(f, L, \frac{1}{n})$ where $L$ is a finite subcomplex of $X$, and $n \in \mathbb{N}$, form a countable local basis for $f$. Hence $Z^X$ is first countable. (It can be easily shown that $Z^X$ is metrizable but we will not use this.) In particular, it is compactly generated (see [16]). Because the function spaces involved are Hausdorff, $Y^X \simeq Z^X$ implies $k(Y^X) \simeq k(Z^X)$. Since $k(Z^X) = Z^X$, it follows that $k(Y^X) \simeq Y^X$. □

**Corollary 3.5** (of Lemma 3.4(ii)). *If $X$ has the homotopy type of a connected countable CW complex, and $Y \in \mathcal{W}$ is a Hausdorff space, then $Y^X$ has CW type if and only if $k(Y^X)$ has CW type.*

**Proof of Theorem 3.1.** From the long exact sequence of the triple for cohomology with local coefficients (see, for example, [15]) it follows by induction that

$$H^i\big(L_n, T; (g|_{L_n})^{\#}\pi_j(Y)\big) = 0 \quad \text{for all } j \geqslant 2 \text{ and all } i \leqslant j.$$

The generalization of Milnor's theorem on the cohomology of an expanding union to local coefficients (see [20], Theorem 2.10*) yields an exact sequence

$$0 \to \varprojlim{}^1 H^{i-1}(L_n, T; G_n) \to H^i(X, T; G) \to \varprojlim H^i(L_n, T; G_n) \to 0,$$

where $G_n = (g|_{L_n})^{\#}\pi_j(Y)$ and $G = g^{\#}\pi_j(Y)$. This shows that $H^i(X, T; g^{\#}\pi_j(Y)) = 0$ for all $j \geqslant 2$ and all $i \leqslant j$. Hence the path component $C$ of $g$ is open and closed in $Y^X$ by Lemma 2.7.

We associate to the filtration $T \leqslant L_1 \leqslant L_2 \leqslant \cdots$ an inverse sequence of fibrations $\cdots \to C(L_2) \to C(L_1) \to C(T)$ in which every space has the homotopy type of a CW complex by Milnor's theorem. By Lemma 2.7 and Whitehead's theorem, every fibration in the sequence is also a homotopy equivalence. Assume the notation of the proof of Theorem 2.1. Using the same technique as in the last two paragraphs of the mentioned proof (the "induction step" in the Zorn lemma argument), we can inductively construct sections $E_n : C(T) \to C(L_n)$ and homotopies $h_n : C(L_n) \times I \to C(L_n)$ between $E_n \circ R_{T,L_n}$ and $\mathrm{id}_{C(L_n)}$ that satisfy the coherence conditions $(*)$ and $(**)$. The coherence conditions imply the existence of a map $E : C(T) \to C(X)$ and a homotopy $h : C(X) \times I \to C(X)$ by the existence part of the universal property of inverse limit. By uniqueness, $E$ is a section of the restriction fibration $R_{T,X} : C(X) \to C(T)$, and $h$ is a homotopy between $E \circ R_{T,X}$ and $\mathrm{id}_{C(X)}$.

To prove the second part of the theorem, fix a function $f \in C(X)$. Since $Y$ is a CW complex, it is locally equiconnected (see [7]). Because the $L_n$ are finite, the spaces $Y^{L_n}$ are also locally equiconnected, by Lemma 3.2. By Lemma 3.4, the spaces $Y^{L_n}$ are perfectly normal, hence by Lemma 3.3, the inclusions $\{f|_{L_n}\} \hookrightarrow Y^{L_n}$ and therefore $\{f|_{L_n}\} \hookrightarrow C(L_n)$ are closed cofibrations.

If in the above inductive construction we apply Lemma 2.8 (which gives the maps $E_n$, and homotopies $h_n$) in full, we may assume that $E_n$ maps $f|_T$ to $f|_{L_n}$ and the homotopy $h_n$ is rel$\{f|_{L_n}\}$. Therefore $E(f|_T) = f$ and the homotopy $h$ is rel$\{f\}$. Since $\{f|_T\} \hookrightarrow Y^T$ is a cofibration, there exists a neighbourhood $U$ of $f|_T$ which deforms in $C(T)$ to $\{f|_T\}$ rel$\{f|_T\}$. Denote this deformation by $k$ and define $\widetilde{U} = R_{T,X}^{-1}(U) \subset C(X)$. The composite $E \circ k \circ (R_{T,X} \times \mathrm{id}_I) : \widetilde{U} \times I \to C(X)$ is a homotopy rel$\{f\}$ between $E \circ R_{T,X}|_{\widetilde{U}}$ and the constant map to $f$. If we concatenate this homotopy with $h|_{\widetilde{U} \times I}$ we get a deformation in $C(X)$ of $\widetilde{U}$ to $\{f\}$ rel$\{f\}$. We conclude as in the proof of Lemma 3.3, noting that $Y^X$ is perfectly normal by Lemma 3.4.

Hence $\{f\} \hookrightarrow C(X)$ and $\{f\} \hookrightarrow Y^X$ are closed cofibrations. □

**Example 2** (*Example* 1 *enhanced*). Let $X$ be a skeleton-finite complex with a minimal cell decomposition in the sense of Hatcher [9, 4.C]. This implies the existence of a filtration $L_1 \leqslant L_2 \leqslant \cdots$ for $X$ consisting of finite subcomplexes, such that for each $n \in \mathbb{N}$, the inclusion induced morphisms $H_k(L_n) \to H_k(X)$ are bijective for $k \leqslant n$ and zero for $k > n$.

Assume that $Y$ is a simple space, and that there exist an $i_0 \in \mathbb{N}$ and a $j_0 \geqslant 2$, so that for all $j \geqslant j_0$ and all $i$ with $i_0 \leqslant i \leqslant j$, $H^i(X; \pi_j(Y)) = 0$. Let $N = \max\{i_0, j_0\}$.

Then the restriction fibration $Y^X \to Y^{L_N}$ is a homotopy equivalence onto the union of some path components. The filtration that satisfies the conditions of Theorem 3.1 is $T = L_N \leqslant L_{N+1} \leqslant \cdots$.

Note that this example includes the case where $X = K(G, n)$, for a finite group $G$, and $G \otimes \pi_k(Y) = 0$ for all large enough $k$.

In particular, let $X = K(G, n)$ where $G$ is a finite $p$-group, and let $Y$ be any (simply connected) finite complex. Then, for any subset $\mathfrak{l}$ of primes different from $p$, the function space $Y^X_{(\mathfrak{l})}$ of maps from $X$ to the $\mathfrak{l}$-localization of $Y$, has the homotopy type of a CW complex.

In the following proposition we show that if the cohomological conditions of Theorem 3.1 are suitably weakened, then an iterated loop space of the function space in question has CW homotopy type.

**Proposition 3.6.** *Let $X$ and $Y$ be connected CW complexes with $X$ countable and $X^{(0)}$ a point, and let $g : X \to Y$ be a map. Let $r \geqslant 1$. Suppose there exists a filtration $L_0 \leqslant L_1 \leqslant L_2 \leqslant \cdots$ for $X$ consisting of finite subcomplexes of $X$ such that*

$$H^i\big(L_n, L_{n-1}; (g|_{L_n})^{\#}\pi_j(Y)\big) = 0$$

*for all $j \geqslant 2$, all $i \leqslant j - r$, and all $n$. Then for each path component $E$ of the iterated loop space $\Omega^{r-1}(Y^X, g)$ there exists a fibration $D \to E$ with discrete fibre, and the total space $D$ having the homotopy type of a CW complex.*

*In particular, $\Omega^r(Y^X, g)$ has CW type.*

**Proof.** Fix a number $n$ and denote by $F$ the fibre of the restriction fibration $Y^{L_n} \to Y^{L_{n-1}}$. The hypotheses imply $\pi_k(F, g) = 0$ for $k \geqslant r$ (see the proof of Lemma 2.7). Hence the morphism $\pi_i(\Omega^{r-1}(C(L_n), g)) \to \pi_i(\Omega^{r-1}(C(L_{n-1}), g))$ is bijective for $i \geqslant 2$, and injective for $i = 1$.

(Note that if $r = 1$, $\Omega^{r-1}(C(L_n), g) = C(L_n) = C_g(L_n)$ in the notation of Theorem 2.1. If $r > 1$, then since all path-components of $\Omega(C(L_n), g)$ are homotopy equivalent, we may assume that all further loop space iterations are taken with respect to the constant map of the previous one.)

The space $\Omega^{r-1}(C(L_0), g)$ has the homotopy type of a CW complex $W$. Let $Z$ be a universal covering space of $W$. Denote by $Z_0$ the pull-back of $\Omega^{r-1}(C(L_0), g) \to W \leftarrow Z$. Then $q_0 : Z_0 \to \Omega^{r-1}(C(L_0), g)$ is a covering space and $\pi_1(Z_0) = 0$ (for each path component).

If the covering space $Z_{n-1} \to \Omega^{r-1}(C(L_{n-1}), g)$ has already been constructed, form a pull-back

$$\begin{array}{ccc} Z_n & \longrightarrow & Z_{n-1} \\ \downarrow & & \downarrow \\ \Omega^{r-1}(C(L_n), g) & \longrightarrow & \Omega^{r-1}(C(L_{n-1}), g) \end{array}$$

Note that since $Z_n \to \Omega^{r-1}(C(L_{n-1}), g)$ is a covering space with CW type base, $Z_n$ has CW type as well, and since the fibre of the bottom row fibration has trivial fundamental group (on each path-component), it follows that $\pi_1(Z_n)$ is trivial, and $\pi_k(Z_n) \to \pi_k(Z_{n-1})$ is an isomorphism for all $k \geqslant 1$.

Proceeding inductively we obtain a commutative ladder

$$\begin{array}{ccccccccc} Z_\infty & \longrightarrow & \cdots & \longrightarrow & Z_2 & \longrightarrow & Z_1 & \longrightarrow & Z_0 \\ \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\ W_\infty & \longrightarrow & \cdots & \longrightarrow & \Omega^{r-1}C(L_2) & \longrightarrow & \Omega^{r-1}C(L_1) & \longrightarrow & \Omega^{r-1}C(L_0) \end{array}$$

where each square is a pull-back square, all horizontal arrows are fibrations, and $Z_\infty$, $W_\infty$ denote the inverse limits of the respective rows. Of course $W_\infty$ equals $\Omega^{r-1}C(X)$.

By construction, the following is also a pull-back square.

$$\begin{array}{ccc} Z_\infty & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ W_\infty & \longrightarrow & \Omega^{r-1}C(L_0) \end{array}$$

Pick a path component $E$ of $W_\infty$. There exists a path-component $D$ of $Z_\infty$ covering $E$. If we denote by $D_n$ the image of $D$ in $Z_n$, then $D_{n+1} \to D_n$ is a fibration and a homotopy equivalence for each $n$. Hence the limit space of $\{D_n\}$ is homotopy equivalent to $D_0$ (see also Theorem B of [5], and [8]), therefore is path connected, and since it contains $D$, it must equal $D$. This shows that $D$ has the homotopy type of a CW complex.

Since $D \to E$ is a fibration with discrete fibre and CW type total space, $\Omega E = \Omega^r(Y^X, g)$ has CW type (see Schön [14]). $\square$

**Corollary 3.7.** *If the skeleton $X^{(n-r)}$ of a countable complex $X$ is compact, and $\pi_k(Y) = 0$ for $k \geqslant n+1$, then $\Omega^r Y^X$ has the homotopy type of a CW complex* (*for any path component of $Y^X$*).

We note that the above corollary holds also without the countability assumption on $X$, by a modification of the proof of Theorem 2.1.

We apply Proposition 3.6 and the corollary to the following

**Proposition 3.8.** *Let $X$ be a connected countable CW complex, and $Y$ a connected CW complex with $\pi_k(Y)$ nontrivial for at most finitely many $k$. Then the following are equivalent.*

(i) *The space $Y^X$ has the type of a locally equiconnected space,*
(ii) *the space $Y^X$ has the type of a CW complex.*

**Proof.** We need only prove that (i) implies (ii). Let $Y^X$ be homotopy equivalent to the locally equiconnected space $Z$. Then $\Omega^k Z$ is locally equiconnected for each $k \geqslant 0$ and each path component of $Z$ (see Lemma 3.2). In particular, every space $\Omega^k Z$ is semilocally contractible. Since $\Omega^k Z$ is homotopy equivalent to $\Omega^k Y^X$, it follows that $\Omega^k Y^X$ is semilocally contractible, for each $k$. In particular, all path components of every $\Omega^k Y^X$ are open.

Furthermore, note that $\Omega^k Y^X$ may be viewed as a certain (closed) subset of the space $Y^{X\times I^k}$, which is paracompact and perfectly normal, by Lemma 3.2. Hence so is $\Omega^k Y^X$.

If $n$ is the largest integer for which $\pi_n(Y) \neq 0$, it follows from Proposition 3.6 that $\Omega^n(Y^X, g)$ has CW homotopy type, for all $g$. Since $\Omega^{n-1}(Y^X, g)$ is paracompact and semilocally contractible, it has the homotopy type of a CW complex by Allaud [1]. Proceeding inductively we conclude that $Y^X$ has the homotopy type of a CW complex. □

**Remark.** If $Y$ has the homotopy type of $\Omega W$ where $W$ is a countable complex, then $Y^X$ has the type of a locally equiconnected space if it is semilocally contractible. To see this, note that by a theorem of Milnor, $\Omega W$ has the type of a topological group $G$. Then $G^X$ is also a topological group. If $Y^X$ is semilocally contractible, so is $G^X$, and it follows by [2], Theorem 2.3. that $G^X$ is actually locally equiconnected.

## 4. An application

We recall the following Theorem B of [5]. Let $\cdots \to Z_3 \xrightarrow{p_3} Z_2 \xrightarrow{p_2} Z_1$ be an inverse sequence of Hurewicz fibrations, where each space $Z_n$ has the homotopy type of a CW complex, and for all $N$, all but finitely many of the fibrations $p_n$ have $N$-connected fibres. Then the inverse limit $Z_\infty$ has the homotopy type of a CW complex if and only if all but finitely many of the $p_n$ are homotopy equivalences.

The authors of [5] have remarked in [6] that the proof of the above theorem (as given in [5]) has a gap, and that it is not clear whether the theorem is true. We give here a counterexample.

**Lemma 4.1.** *Let $d, n$ be natural numbers. There exists a finite CW pair $(D, S)$ with $D$ contractible, $S$ a Moore complex of type $(\mathbb{Z}_d, n)$, and $D/S$ a Moore complex of type $(\mathbb{Z}_d, n+1)$.*

**Proof.** Form $S$ as a sphere $S^n$ with a disk $B^{n+1}$ attached along a map of degree $d$, $S = B^{n+1} \sqcup_d S^n$. Let $S \cup B^{n+1}$ denote $S$ with another disk attached along $\mathrm{id}: S^n \to S^n$. If $\beta$ represents the generator of $\pi_{n+1}(S \cup B^{n+1})$, $D$ may be constructed as $D = B^{n+2} \sqcup_\beta (S \cup B^{n+1})$. □

**Example 3.** Let $\alpha: \mathbb{N} \to \mathbb{P}$ denote an increasing enumeration of the primes. Let $Y$ be a CW complex with $\pi_{n^2}(Y) = \mathbb{Z}_{\alpha(n)}$ for all $n \geqslant 2$, and $\pi_k(Y) = 0$ otherwise. Let $(D_{n+1}, S_n)$ be the CW pair from Lemma 4.1 for which $S_n$ is a Moore complex $M(\mathbb{Z}_{\alpha(n)}, n)$. For $n \geqslant 2$ define $L_n = \mathbb{R}P^{2n} \vee (\bigvee_{i=2}^n D_i) \vee S_n$. Then $L_{n-1} = \mathbb{R}P^{2n-2} \vee (\bigvee_{i=2}^{n-1} D_i) \vee S_{n-1}$, and this is a subcomplex of $L_n$ in a natural way. Observe that $L_n/L_{n-1} = (\mathbb{R}P^{2n}/\mathbb{R}P^{2n-2}) \vee (D_n/S_{n-1}) \vee S_n$. It follows that

$$H_k(L_n, L_{n-1}) = \begin{cases} \mathbb{Z}_2, & k = 2n-1, \\ \mathbb{Z}_{\alpha(n-1)} \oplus \mathbb{Z}_{\alpha(n)}, & k = n, \end{cases}$$

and $H_k(L_n, L_{n-1}) = 0$ otherwise. Define

$$X = \bigcup_{n=2}^{\infty} L_n = \mathbb{R}P^\infty \vee \bigvee_{n=2}^{\infty} D_n \xrightarrow[\simeq]{\text{retraction}} \mathbb{R}P^\infty.$$

Since $X \simeq \mathbb{R}P^\infty$, also $Y^X \simeq Y^{\mathbb{R}P^\infty}$, which has the homotopy type of $Y$ by Example 1.

We consider the restriction fibration $F = F_n \hookrightarrow Y^{L_n} \to Y^{L_{n-1}}$. If we represent a loop $(S^k, *) \to (F, *)$ by the adjoint $S^k \times L_n \to Y$ as in the proof of Lemma 2.7, the obstructions for $\pi_k(F) = 0$ (that is the obstructions for extending the adjoint loop to a map $B^{k+1} \times L_n \to Y$) lie in the groups $H^{j-k}(L_n, L_{n-1}; \pi_j(Y))$. Note that the only nontrivial possibilities are

$$H^n\big(L_n, L_{n-1}; \pi_{(n-1)^2}(Y)\big) \cong \mathbb{Z}_{\alpha(n-1)},$$
$$H^{n+1}\big(L_n, L_{n-1}; \pi_{(n-1)^2}(Y)\big) \cong \mathbb{Z}_{\alpha(n-1)},$$
$$H^n\big(L_n, L_{n-1}; \pi_{n^2}(Y)\big) \cong \mathbb{Z}_{\alpha(n)},$$
$$H^{n+1}\big(L_n, L_{n-1}; \pi_{n^2}(Y)\big) \cong \mathbb{Z}_{\alpha(n)}.$$

This shows that $\pi_k(F_n) = 0$ for $k \leqslant (n-1)^2 - (n+2) = n^2 - 3n - 1$. Hence $\cdots \to Y^{L_6} \to Y^{L_5} \to Y^{L_4}$ is an inverse system of fibrations between spaces of CW type, and the connectivity of the fibres increases to infinity. The limit space $Y^X$ has the homotopy type of a CW complex, but none of the fibrations is a homotopy equivalence. To see this, note that the fibre $F_n$ (over the constant map) is homeomorphic to the space of pointed maps $(Y, *)^{(L_n/L_{n-1}, *)} = (Y, *)^{((\mathbb{R}P^{2n}/\mathbb{R}P^{2n-2}) \vee (D_n/S_{n-1}) \vee S_n, *)}$. Hence $(Y, *)^{(S_n, *)}$ is dominated by $F_n$. But

$$\pi_{n^2-n}\big((Y, *)^{(S_n, *)}, \text{const}\big) \cong \big[S^{n^2-n} \wedge S_n, Y\big]_* \cong \mathbb{Z}_{\alpha(n)},$$

by obstruction theory. Therefore $F_n$ is not contractible, and this concludes the counterexample.

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