8. The Gelfand-Naimark-Segal (GNS) Theorem

Preview of Lecture: In lecture, we won't discuss the proofs of the technical results we'll need about states for this lecture (eg Lemmas 8.8 and 8.11). However, these are important both for von Neumann algebraic applications and for C*-algebras, so you should read the proofs carefully and ask questions in office hours if you're confused.

We will prove Theorem 8.9 in lecture as well as Theorem 8.1. We'll discuss irreducible representations but, depending on time, perhaps not the proof of Proposition 6.1. We will, however, discuss the proof of Proposition 5.10.

There are a lot of exercises in this section! If there's time, we'll discuss a few in lecture (so please let us know if there are any that you'd particularly like to see).

The main goal of this section is the following theorem:

Theorem 8.1 (Gelfand-Naimark)**.** *Every* C*-*algebra* A *admits a faithful nondegenerate representation* $\pi : A \to B(\mathcal{H})$. *If* A *is separable,* π *can be chosen to be separable.*

As an immediate corollary, every C*-algebra A is isomorphic to a norm-closed $*$-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. (It can be useful to take this as the definition of a C*-algebra, which justified our using the term "C*-algebra" for abstract (not concretely represented) C*-algebras.)

Throughout this section, we generally assume A is unital, for simplicity; the arguments can all be made in general, by taking some care with approximate identities. See Exercise 8.15 below (and Remark 8.16 for why we can't just unitize our way out of this one).

Our first step on the road to proving Theorem 8.1 has to do with *states*.

Definition 8.2. A *state* on a C*-algebra A is a linear functional $\phi : A \to \mathbb{C}$ which is *positive* in that $\phi(a) \geq 0$ whenever $a \geq 0$, and such that

$$\|\phi\| := \sup\{|\phi(a)| : \|a\| = 1\} = 1.$$

The closed convex (**exercise**) subset $\mathcal{S}(A) \subset A^*_{\leq 1}$ consisting of states is called the state space.

Example 8.3. If π is a representation of A on $\mathcal{H}$, and $h \in \mathcal{H}$ has norm 1, the function

$$\phi(a) := \langle \pi(a)h, h\rangle$$

is a state on A.

Exercise 8.4. If $\pi : A \to \mathbb{C}$ is a character, then it is a representation (by 2.15). What is the state corresponding to a character π?

Exercise 8.5. Show that any positive linear functional $\phi : A \to \mathbb{C}$ is $*$-preserving, i.e. $\phi(a^*) = \overline{\phi(a)}$ for all $a \in A$.

Exercise 8.6. Show that $\mathcal{S}(A)$ is a closed convex subset of $A^*_{\leq 1}$. It follows from Alaoglu's theorem that it is weak*-compact. What does the Krein-Milman theorem say about $\mathcal{S}(A)$?

Given a state[5] ϕ on A, if we define $[a, b]_\phi := \phi(b^*a)$, then this form on A is sesquilinear (linear in the first variable, conjugate linear in the second variable) and satisfies the Cauchy-Schwarz inequality:

Exercise 8.7. Show that $|[a, b]_\phi| \leq [a, a]_\phi\, [b, b]_\phi = \phi(a^*a)\phi(b^*b)$.

Here are a few facts about states that we will need later.

Lemma 8.8. *Let* A *be a unital* C*-*algebra.*

(1) If ϕ *is a state on* A*, then* $\phi(1) = 1$.
(2) If ϕ *is a bounded linear functional on* A *which satisfies* $1 = \|\phi\| = \phi(1)$*, then* ϕ *is a state.*

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[5] Actually, all you need is a positive linear functional for the following assertions and exercise.

Proof. If ϕ is a state, then $|\phi(1)| = \phi(1) \le \|\phi\| = 1$. For the other inequality, Exercise 8.7 tells us that

$$|\phi(a)|^2 \le \phi(1)\phi(a^*a).$$

Moreover, since $\|a^*a\| \ge a^*a$ by the functional calculus, we have $\phi(a^*a) \le \|a^*a\|\phi(1)$. It follows that for any $a \in A$ with $\|a\| = 1$,

$$|\phi(a)|^2 \le \phi(1)^2,$$

and hence $\|\phi\| \le \phi(1)$. We conclude that $1 = \|\phi\| = \phi(1)$.

If $1 = \phi(1) = \|\phi\|$, then once we know that ϕ is positive, ϕ must be a state. Pick $a \in A_+$ and write $\phi(a) = \alpha + i\beta$. If necessary, replace a with $-a$ to ensure that $\beta \ge 0$. We begin by showing that $\beta = 0$. Fix $n \in \mathbb{N}$ and observe that, since $a^* = a$,

$$\|n - ia\|^2 = \|(n + ia^*)(n - ia)\| = \|n^2 + in(a^* - a) + a^2\| \le n^2 + \|a\|^2.$$

On the other hand, $|\phi(n - ia)|^2 = |n\phi(1) - i\alpha + \beta|^2 = (n + \beta)^2 + \alpha^2$. So,

$$(n^2 + \|a\|^2) = \|\phi\|^2(n^2 + \|a\|^2) \ge \|\phi\|^2\, \|n - ia\|^2 \ge |\phi(n - ia)|^2 = n^2 + 2n\beta + \beta^2 + \alpha^2.$$

In order for this inequality to hold for all $n \in \mathbb{N}$ we must have $\beta = 0$, as claimed.

To complete the proof that ϕ is positive, fix a positive $a \in A_+$ with $\|a\| \le 1$. Then Proposition 3.6 implies that $\|1 - a\| \le 1$. Since $\|\phi\| = 1$ by hypothesis,

$$1 \ge \|1 - a\| \ge \phi(1 - a) = \phi(1) - \phi(a) = 1 - \phi(a).$$

It follows that $\phi(a) \ge 0$ for any positive a. □

The next theorem is the cornerstone of our proof of Theorem 8.1.

Theorem 8.9 (GNS construction)**.** *If ϕ is any state on a unital C^*-algebra A, there is a nondegenerate representation $\pi_\phi : A \to B(\mathcal{H})$ and a unit vector $h \in \mathcal{H}$ such that $\phi(a) = \langle \pi_\phi(a)h, h\rangle$ for any $a \in A$.*

Such a vector h is called a *cyclic vector* for the representation π.

Proof. We will build $\mathcal{H}$ out of A itself. Let $N_\phi = \{a \in A : [a, a]_\phi = 0\}$. Observe (check!) that N_ϕ is a vector subspace of A, which is closed in norm. (The fact that N_ϕ is closed under addition follows from Exercise 8.7. Proving that N_ϕ is closed in norm is also a good exercise.)

In fact, Exercise 8.7 actually proves that N_ϕ is a left ideal in A: if $x \in N_\phi$ and $a \in A$ then

$$|[ax, ax]_\phi| = |\phi(x^*(a^*ax))| \le \phi(x^*x)\phi(x^*(a^*a)^2x) = 0,$$

so $ax \in N_\phi$.

Therefore, let X be the vector space quotient $X = A/N_\phi$, and define an inner product on X by

$$\langle a + N_\phi, b + N_\phi\rangle_\phi := \phi(b^*a).$$

The fact that N_ϕ is a left ideal means that $\langle\cdot,\cdot\rangle_\phi$ is well defined.

Take $\mathcal{H}_\phi$ to be the completion of X with respect to the norm induced by $\langle\cdot,\cdot\rangle_\phi$. Then our representation $\pi_\phi : A \to B(\mathcal{H}_\phi)$ is given by left multiplication: $\pi_\phi(a)(b + N_\phi) = ab + N_\phi$.

To see that π_ϕ is actually a representation, we need to check that $\pi_\phi(a)$ is a bounded linear operator for all a, and also check that π_ϕ is linear, multiplicative and $*$-preserving. In checking that π_ϕ is $*$-preserving, you will see why we defined $\langle\cdot,\cdot\rangle_\phi$ as we did.

We use the functional calculus to show that $\pi_\phi(a)$ is a bounded operator. Since $a^*a \in A$ is positive, we have $\|a^*a\|1 - a^*a \ge 0$ is a positive element of A. Thus, for any $x \in A$, Exercise 3.11 tells us that

$$0 \le x^*(\|a^*a\|1 - a^*a)x = \|a^*a\|x^*x - x^*a^*ax,$$

so $(ax)^*ax \le \|a^*a\|x^*x$. In particular,

$$\begin{aligned}\|\pi_\phi(a)\|^2 &= \sup\{\langle \pi_\phi(a)(x + N_\phi), \pi_\phi(a)(x + N_\phi)\rangle_\phi : \phi(x^*x) = 1\}\\ &= \sup\{\phi((ax)^*(ax)) : \phi(x^*x) = 1\}\\ &\le \|a^*a\| = \|a\|^2.\end{aligned}$$

So, we conclude that π_ϕ is a representation of A on a Hilbert space $\mathcal{H}_\phi$. To see that π_ϕ is nondegenerate, suppose that $\pi_\phi(a)(x + N_\phi) = 0$ for all a. In particular, taking $a = 1$,

$$0 = \|\pi_\phi(1)(x + N_\phi)\|^2 = \langle x + N_\phi, x + N_\phi\rangle_\phi = \phi(x^*x),$$

so we must have $x \in N_\phi$.

Finally, note that the unit vector h such that $\phi(a) = \langle \pi(a)h, h\rangle$ is $h = 1 + N_\phi$. $\square$

Remark 8.10. If you ever see in some proof in the literature a representation unceremoniously associated to some state (in particular for a trace that is moreover a state), it's assumed to be the GNS representation constructed as above.

To prove Theorem 8.1, we will take the direct sum of a lot of the representations whose existence we have just established.

Lemma 8.11. *Let A be a C^*-algebra and a a nonzero normal element of A. Then there is a state τ on A such that $|\tau(a)| = \|a\|$.*

Proof. Let $B = C^*(\{a, 1\}) \subseteq \widetilde{A}$. Fix $\lambda \in \sigma(a)$ with $|\lambda| = r(a)$ maximal, and let $g_\lambda : C(\sigma(a)) \to \mathbb{C}$ be given by evaluation at λ. Observe that g_λ is a positive linear functional, and $g_\lambda(1) = 1$, so g_λ is a state on $C(\sigma(a))$.

Because g_λ can be viewed as a linear functional on the closed subspace B of $\widetilde{A}$, the Hahn-Banach Theorem tells us there is a norm one linear functional τ on A which extends g_λ. As $\tau(1) = g_\lambda(1) = 1$, Lemma 8.8 tells us that τ is also a state. Furthermore, as the Gelfand transform $\Gamma : B \overset{\cong}{\to} C(\sigma(a))$ takes a to the function $f(z) = z$, it follows that $|\tau(a)| = |\lambda| = r(a)$, which equals $\|a\|$ by the fact that the Gelfand transform is isometric. $\square$

Corollary 8.12. *If $F \subseteq \mathcal{S}(A)$ is a subset of the states of A which is dense in the weak-$*$ topology, then for any $a \in A$,*

$$\sup\{|\phi(a)| : \phi \in F\} = \|a\|.$$

We are finally ready to prove our main theorem.

Proof of Theorem 8.1. Choose a subset F of $\mathcal{S}(A)$ which is dense in the weak-$*$ topology on $\mathcal{S}(A) \subseteq A^*$. Define $\pi := \bigoplus_{\phi \in F} \pi_\phi$, where π_ϕ is the representation arising from the state ϕ as in the previous Theorem.

Fix $a \in A$. Since $\phi(1) = 1$,

$$\|\pi(a)\|^2 = \sup_{\phi \in F} \|\pi_\phi(a)\|^2 = \sup\{\langle \pi_\phi(a^*a)\xi, \xi\rangle : \phi \in F, \xi \in \mathcal{H}_\phi\} \geq \sup_{\phi \in F} \langle \pi_\phi(a^*a)1, 1\rangle = \sup_{\phi \in F} \phi(a^*a) = \|a\|^2.$$

As π is a $*$-homomorphism and therefore norm-decreasing, it follows that $\|\pi(a)\| = \|a\|$ for all $a \in A$. The fact that π is nondegenerate follows from the fact that each π_ϕ is nondegenerate, which in turn follows from our construction of $\mathcal{H}_\phi$ as a completion of (a quotient of) A.

If A is separable, then [4, Theorem V.5.1] implies that $A^* \supseteq \mathcal{S}(A)$ is too, so we can take the set F to be countable. The separability of A implies the separability of $\mathcal{H}_\phi$ for each ϕ,[6] so $\mathcal{H}$ is separable. $\square$

Definition 8.13. The representation

$$\pi_u := \bigoplus_{\phi \in \mathcal{S}(A)} \pi_\phi : A \to B\left(\bigoplus_{\phi \in \mathcal{S}(A)} \mathcal{H}_\phi\right) =: B(\mathcal{H}_u)$$

is called the *universal representation* of A.
Equivalently, we could form the direct sum over a weak*-dense subset of $\mathcal{S}(A)$.

Remark 8.14. This representation has a special extra property in that the associated von Neumann algebra $\pi_u(A)''$ is isometrically isomorphic to A^{**} (ask Roy). Both are often called the *enveloping von Neumann algebra* of A.

Exercise 8.15. Generalize the results in this section to non-unital C*-algebras. (In particular, you will have to show that an approximate unit $(e_\lambda)_\lambda$ becomes Cauchy in A/N_ϕ for any state ϕ, and hence gives rise to a cyclic vector in any GNS representation π_ϕ.)

[6]There is something to check here, since the norm on $\mathcal{H}_\phi$ is not the same as the norm on A. **Exercise:** How do they relate?

Remark 8.16. Why can't we just unitize in Exercise 8.15? Well, as easy as it was to always guarantee a unique extension of a $*$-homomorphism to the unitization, it is no longer true in general for positive linear maps. (We'll return to this in Proposition 9.20.) In fact, the non-unital version of Theorem 8.9 is required to prove this for states because it allows us to borrow from this fact for representations. The proof of this fact takes us a little off course, so we will state it here with reference:

[5, Corollary 1.9.7] Every state on a nonunital C*-algebra A extends uniquely to a state on $\tilde{A}$.

What does Theorem 8.1 say about Abelian C*-algebras? In this case, the Riesz-Markov-Kakutani representation theorem tells us that states on $C_0(X)$ are in bijection with probability measures on X, so that $\phi(f) = \int_X f\, d\mu_\phi$. Note that N_ϕ consists of the set of C_0 functions on X which are 0 off a μ_ϕ-null set. Thus, $\mathcal{H}_\phi = \overline{C_0(X)/N_\phi} \cong L^2(X, \mu_\phi)$, and π_ϕ represents $C_0(X)$ on $L^2(X, \mu_\phi)$ as multiplication operators:

$$\pi_\phi(f)\xi = x \mapsto f(x)\xi(x).$$

To me at least, this is reminiscent of the link between the continuous and the Borel functional calculus.

Exercise 8.17. What does the universal representation of an Abelian C*-algebra look like?

Exercise 8.18. Let A be a C*-algebra.

(1) Show that for any self-adjoint element $a \in A$, there exists a state ϕ on A so that $|\phi(a)| = \|a\|$. (Hint: Assume A is unital (or unitize). Then recall exercise 8.4 for the C*-algebra $C^*(a, 1)$. You'll need the Hahn-Banach theorem (specifically Corollary 4.4 from the Prerequisite Notes) to finish up.)
(2) Show that for any $b \in A$, there exists a representation $\pi : A \to B(\mathcal{H})$ and unit vector $h \in \mathcal{H}$ so that $\|\pi(b)x\| = \|b\|$. (Hint: Apply the first part to $a = b^*b$.)
(3) Use Exercise 4.16 to give a different argument for the last claim in Theorem 8.1, i.e. that any separable C*-algebra has a faithful separable representation.

8.1. **Applications.** We've already seen the GNS theorem invoked several times, for structural results about C*-algebras. Here are some of those delayed proofs.

Exercise 8.19. Show that if $0 \le a \le b$, then $\|a\| \le \|b\|$, without assuming a and b commute.

Exercise 8.20. Show that if the C*-algebra A is finite dimensional as a vector space, then we may take the Hilbert space $\mathcal{H}$ of Theorem 8.1 to be finite dimensional. *Hint:* Show that you only need finitely many states $\phi \in F$, and that H_ϕ is finite dimensional for all ϕ.

Exercise 8.21. Use the GNS theorem to give a very quick proof of Theorem 3.10.

Exercise 8.22. For a commutative C*-algebra A, what would a weak$*$-dense subspace of $\mathcal{S}(A)$ look like?

The following should rightfully be called a Definition/Theorem. The proof uses results that take us a little far afield, so we give it as a definition and refer you to [5, Lemma 1.9.1-Theorem 1.9.4] for a proof.

Definition 8.23. A $*$-representation $\pi : A \to B(\mathcal{H})$ of a C*-algebra A is *irreducible* if one of the following equivalent conditions hold:

(1) π has no proper invariant subspaces, i.e. no subspace $V \subsetneq \mathcal{H}$ so that $\pi(a)V \subset V$ for all $a \in A$.
(2) π has no proper invariant manifolds (i.e. subspaces which may or may not be closed).
(3) $\pi(A)' = \mathbb{C}1_{\mathcal{H}}$.

Under the additional assumption that π has a cyclic unit vector $h \in \mathcal{H}$, these are also equivalent to

(4) The state $a \mapsto \langle \pi(a)h, h\rangle$ is *pure*, i.e. it is an extreme point in the state space $\mathcal{S}(A)$.

Remark 8.24. We have a couple remarks on irreducible representations:

(1) First, it's sometimes helpful to see a non-example: Let $\pi_i : A \to B(\mathcal{H}_i)$, $i = 1, 2$ be two nondegenerate representations of A. Then $\pi_1 \oplus \pi_2 : A \to B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is not irreducible. (Evidently we don't bother with calling things "reducible".)
(2) Notice that a character on a C*-algebra is a pure state. (Indeed, for any states ϕ_1, ϕ_2 and $\alpha_1, \alpha_2 \in (0, 1)$ with $\alpha_1 + \alpha_2 = 1$, the map $\alpha_1\phi_1 + \alpha_2\phi_2$ will not be multiplicative.) It turns out ([5, Lemma 1.9.10]) that you can use a Krein-Milman argument to strengthen parts (1) and (2) of Exercise 8.18 to hold for pure states/irreducible representations. Then an argument like part (3) will allow you to prove the conclusion of Corollary 8.12 where F consists of all pure states of A.

(3) Not every C*-algebra has a faithful irreducibe representation. Such C*-algebras are called *primitive.*

Exercise 8.25. If $\pi : A \to B(\mathcal{H})$ is irreducible, what does that say about the von Neumann algebra $\pi(A)''$? (Looking for a one word answer.)

Proof of Proposition 6.1. Suppose that A is a finite dimensional C*-algebra. By GNS, view A as a subalgebra of $B(\mathcal{H})$, where $\mathcal{H}$ is finite dimensional. Thus, A is an algebra of *compact* operators.

It turns out [5, Corollary I.10.6] that every irreducible representation of $K(\mathcal{H})$ is unitarily equivalent to the identity representation. Thus, decompose the (identity) representation of A on $\mathcal{H}$ into a direct sum of irreducible representations $\pi_i : A \to B(\mathcal{H}_i)$ where $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Then each $\mathcal{H}_i$ must be finite dimensional, and we must have only finitely many terms in this direct sum decomposition, since $\mathcal{H}$ is finite dimensional. In other words, if $\mathcal{H}_i \cong \mathbb{C}^{n_i}$, then $\pi_i(A) \cong M_{n_i}$. Thus,

$$A = \bigoplus_i \pi_i(A) \cong \bigoplus_i M_{n_i}$$

as desired. □

The proof of Proposition 5.10 relies on *positive definite functions* on groups, and their connection with states on $C^*(G)$.

Definition 8.26. Let G be a discrete group. A function $\psi : G \to \mathbb{C}$ is *positive definite* if, for any finite subset $F \subseteq G$, the matrix M^ψ in $M_F(\mathbb{C})$ given by

$$M^\psi_{s,t} = \psi(s^{-1}t)$$

is positive.

Proposition 8.27. *If ϕ is a state on $C^*(G)$, then the function $\psi^\phi(g) = \langle \pi_\phi(u_g)1, 1\rangle_\phi$ is positive definite. Conversely, every positive definite function defines a state on $C^*(G)$.*

Proof. If ϕ is a state on $C^*(G)$, we compute that

$$M^{\psi^\phi}_{s,t} = \langle \pi_\phi(u_{s^{-1}t})1, 1\rangle = \langle \pi_\phi(u_t)1, \pi_\phi(u_s)1\rangle = \phi(u_t)\overline{\phi(u_s)}.$$

In other words, if T is the matrix with entries indexed by elements of G, such that the first column consists of the entry $\phi(u_s)$ in the sth row, and T is zero in all other columns, then $M^{\psi^\phi} = T^*T$ is positive. So ψ^ϕ is positive definite, as claimed.

For the converse, given a positive definite function ψ, define $\phi_\psi(\sum_g a_g u_g) := \frac{1}{\psi(e)} \sum_g a_g \psi(g)$. By construction, ϕ_ψ is a linear functional on $\mathbb{C}G$. Considering the set $F = \{e\}$ tells us that $\psi(e) > 0$, so ϕ_ψ is well defined, and moreover that

$$\phi_\psi(u_e) = \frac{\psi(e)}{\psi(e)} = 1. \tag{8.1}$$

Moreover, ϕ_ψ is bounded with respect to $\|\cdot\|_u$, because

$$|\phi_\psi(\sum_g a_g u_g)| = |\langle \pi_{\phi_\psi}(\sum_g a_g u_g)1, 1\rangle| \le \|\pi_{\pi_\psi}(\sum_g a_g u_g)\| \le \|\sum_g a_g u_g\|_u. \tag{8.2}$$

It now follows that $\|\phi_\psi\| = 1$: equation (8.1) implies that $\|\phi_\psi\| = \sup\{|\phi_\psi(f)| : \|f\|_u = 1\} \ge |\phi_\psi(u_e)| = 1$, and equation (8.2) implies that $\|\phi_\psi\| \le 1$. Thus, Lemma 8.8(2) tells us that ϕ_ψ extends to a state on $C^*(G)$. □

Proof of Proposition 5.10. We first address the case of the reduced C*-algebras. Suppose $G \le H$ are discrete groups, and decompose $\ell^2(H) = \bigoplus_h \ell^2(Gh)$ via the right cosets of G. Notice that the left regular representation of $\mathbb{C}G \subseteq \mathbb{C}H$ on $\ell^2(H)$ preserves this decomposition, and $\ell^2(Gh) \cong \ell^2(G)$ (via a canonical isomorphism) for any $h \in H$. As the operator norm of a direct sum satisfies

$$\|f \oplus g\| = \max\{\|f\|, \|g\|\},$$

it follows that the norm induced on $\mathbb{C}G$ by the left regular representation λ^H is the same as the norm induced by λ^G. In other words, the inclusion $\mathbb{C}G \subseteq \mathbb{C}H$ is isometric with respect to the reduced norm, so $C^*_r(G) \subseteq C^*_r(H)$.

Now, we show that if $G \le H$ (and G is countable) then $C^*(G) \le C^*(H)$. The fact that G countable implies that $C^*(G)$ is separable. In this case,$C^*(G)^*$ is also separable, so there exists a faithful state ϕ on

$C^*(G)$: namely, for a weak-$*$ dense subset $\{\omega_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}(A)$, take $\phi = \sum_n 2^{-n}\omega_n$. It is straightforward to check that, thanks to the density of $\{\omega_n\}_n$, $\phi(a) = 0$ implies $a = 0$, so ϕ is indeed faithful.

Consider the positive definite function ψ^ϕ on G which Proposition 8.27 associates to ϕ. Extend it to ψ on H by setting $\psi(h) = 0$ whenever $h \notin G$. To see that ψ is positive definite, note first that if $s, t \in H$ and $sG \neq tG$, then $s^{-1}t \notin G$ and therefore $M^\psi_{s,t} = 0$. In other words, for any finite set F, M^ψ is block diagonal, where each block is indexed by $F \cap sG$ for a single *left* coset sG of G. Block diagonal matrices are positive precisely when each block is positive, so to see that ψ is positive definite it suffices to consider the matrices M^ψ associated to finite sets $F \subseteq sG$ which are contained in a single coset. For any $g, h \in G$,

$$M^\psi_{sg,sh} = \psi(g^{-1}h) = \psi^\phi(g^{-1}h) = M^{\psi^\phi}_{g,h},$$

so the fact that ψ^ϕ is positive definite implies that ψ is as well.

Now, consider the GNS representation π_ψ associated to ϕ_ψ. As ϕ_ψ and ϕ agree on $C^*(G)$, it follows that for any $f \in C^*(G)$,

$$\|\tilde{\iota}(f)\|_{u,H} \geq \|\pi_\psi(f)\| = \|\pi_\phi(f)\|.$$

The fact that ϕ is faithful means that π_ϕ is injective and therefore isometric, by Theorem 4.11: if $\pi_\phi(f) = 0$ then

$$\begin{aligned}0 = \|\pi_\phi(f)\|^2 &= \sup\{\|\pi_\phi(f)[a]\|^2 : [a] \in \mathcal{H}_\phi = \overline{C^*(G)}^{\|\cdot\|_\phi}, \|[a]\| = 1\} \\ &= \sup\{\phi(a^*f^*fa) : \phi(a^*a) = 1\} \geq |\phi(f^*f)| = |\phi(f)|^2\end{aligned}$$

by Exercise 8.5. The fact that ϕ is faithful then implies that $f = 0$. In other words, $\|\pi_\phi(f)\| = \|f\|_{u,G}$ for any $f \in C^*(G)$. So $\|\tilde{\iota}(f)\|_{u,H} \geq \|f\|_{u,G}$. As we saw in Monday's notes that $\tilde{\iota} : C^*(G) \to C^*(H)$ is norm-decreasing, it now follows that $\|\tilde{\iota}(f)\|_{u,H} = \|f\|_{u,G}$. Consequently, $\tilde{\iota}$ must be injective: if $\tilde{\iota}(f) = 0$ then $f = 0$. □

Remark 8.28. For those that are wondering whether all of this rigamarole about positive definite functions is really necessary: If you try to extend a state from $C^*(G)$ to $C^*(H)$ by just making it zero on all elements not coming from $C^*(G)$, it's hard to prove directly that this extension is still a state.