Limit Order Book Modeling

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Ergodicity of the Markovian Model

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I would like to thank Professor Kolokoltsov for all the support and encouragement he has given me throughout my project.

1 Introduction to Limit Order Books

The definition that Investopedia and [1] give for a Limit Order Book (abbr. LOB) is that it's a record of outstanding limit orders maintained by the security specialist who works at the exchange. A limit order is a type of order expressing the willingness of one to buy or sell a security at a specific price. A **buy limit order** is an order to **buy** a security at a specific price while a **sell limit order** is an order to **sell** a security at a specific price. Thus, Limit Order Books are essentially used to store the list of all the interests of market participants, and are utilized by the vast majority of electronic markets (all equity, futures and other listed derivatives markets). It is essentially a file in a computer that contains all the orders sent to the market, with their characteristics such as the **sign** of the order (buy or sell), the **price**, the **quantity**, a **timestamp** giving the time the order was recorded by the market, and often, other market-dependent information. Its evolution over time describes the way the market moves under the influence of its participants, where their *collective interaction* (their submissions of all different kinds of orders) leads to the finding of the best price of a security. Such financial markets where buyers and sellers meet via a limit order book are called **order-driven markets**. On the other hand, there exist also what we call **quote-driven markets**, where only the buy and sell offers for a security from designated market makers, dealers, or specialists are displayed. These "market makers" will post the bid and ask price that they are willing to accept at that time. Quote-driven markets are most commonly found in markets for bonds, currencies, and commodities. In the dissertation we are only interested in order-driven markets. In such markets, incoming limit orders are arranged in the LOB subject to some priority rules. In most stock exchanges in the world, the priority rule is *price/time* and is the rule we shall adhere to in this project. This means that buy and sell limit orders are arranged in **levels** in the LOB. Buy limit orders are sorted from the highest price to the lowest. This implies that the price of a *level 1* limit buy order represents the highest price a market participant/trader is willing to pay for a specific security. On the other hand, sell limit orders are sorted from the lowest price to the highest. This implies that the price of a *level 1* sell limit order represents the lowest price a market participant/trader is willing to sell a particular security for. Furthermore, in the case where limit orders (of the same sign) are submitted at the same price, they will be sorted according to what time they were submitted (the earlier the submission of the order, the lower the level of the order in the LOB).

An illustration of this is given in Figure 1. We can see on the bid side the first 13 levels of limit buy orders of IBM stock. In this case, the highest price that a trader is willing to pay to get an IBM stock is $187.31 (and this is precisely the *level 1* buy limit order) at that specific time. Moreover, the trader from DRCTEDGE is willing to pay this much for 200 lots of IBM stock. We can also see that there are quite a few buy limit orders at the same price. For example, traders from ARCA, NSDQ and NYSE are all willing to pay \$187.28 for an IBM stock (at different quantities). And by the price/time priority rule, as the trader from ARCA had submitted the buy limit order at that price earlier than NSDQ and NYSE, and NYSE had submitted it the latest out of the three traders, the ordering in the LOB on the bid side is in the order ARCA, NSDQ, NYSE. On the ask side of the LOB, we see the first 13 levels of limit sell orders of IBM stock. In this case, the lowest price that a trader is willing to sell an IBM stock for is \$187.35 (and this is precisely the *level 1* sell limit order) at that specific time. The mechanism of the ordering of the sell limit orders is the same as that of the buy limit orders with the only difference, as mentioned before, that they are ordered from lowest price to highest.

IBM 187.34 -0.13 (-0.07%) DU18007

MM Name	Bid Price	Size	MM Name	Ask Price	Size
DRCTEDGE	187.31	2	NYSE	187.35	3
BYX	187.29	1	NSDQ	187.38	1
EDGEA	187.29	1	NYSE	187.39	2
ARCA	187.28	2	NYSE	187.40	2
NSDQ	187.28	3	NSDQ	187.42	4
NYSE	187.28	2	NYSE	187.42	2
NSDQ	187.27	2	TMBR	187.42	2
ARCA	187.26	1	NSDQ	187.43	1
NSDQ	187.26	3	TMBR	187.43	2
NYSE	187.26	1	ARCA	187.44	2
BATS	187.25	1	BATS	187.44	1
NSDQ	187.25	4	DRCTEDGE	187.44	1
NSDQ	187.24	1	NSDQ	187.44	1

Figure 1: The first few levels of the LOB of IBM stock at a particular point in time (taken from the InteractiveBrokers trading platform) where the names under the *MM Name* column are the traders of IBM stock, and the numbers under the *Size* column are the quantities of IBM shares (in 100s) traders are willing to buy or sell at the price on the left.

Essentially, three types of orders can be submitted:

- *Limit order*: an order to specify a price at which one is willing to buy/sell a certain number of shares, with the corresponding price and quantity, at any point in time;
- *Market order*: an order to immediately buy/sell a certain quantity, at the best available opposite quote;
- *Cancellation order*: an order to cancel an existing limit order.

Limit orders are stored in the order book until a market or cancellation order arrives. If a market order arrives, then the best opposite limit order is executed immediately i.e. the market order will trade immediately with the best opposite limit order. So, for example, in Figure 1 if we submit a market buy order for 200 IBM stock, then we will trade immediately with the best sell limit order (the order at the top of of the ask side of the LOB i.e. the order at the *first level*), that is, we will trade with the trader with the lowest selling price for IBM stock at that particular point in time (in this case, this trader is from NYSE). This will result in the size of the best sell limit order to be reduced by 200 (so in this case, the size will reduce from 300 to 100). If we had submitted a buy market order of size greater than 300, then the NYSE ask limit order at the first level in the ask side of the LOB would get "annihilated", and this buy market order would also reduce (possibly even annihilate) the sizes of the sell limit orders at the preceding levels by the same mechanism. Thus, a buy market order, depending on its size, could cause a shift in the ordering of the sell limit orders. Therefore, buy market orders affect the *ask side of the book* i.e. the existing sell limit orders. Conversely, sell market orders affect the *bid side of the book* i.e. the existing buy limit orders. In the case where we have a cancellation of a limit order (say, a trader no longer wants to keep the limit order they had submitted earlier), then this limit order vanishes from the order book and causes the preceding limit order (i.e. the limit order one level below the limit order just canceled) to go up a level (i.e. take up the place of the canceled limit order). It follows that, a buy limit cancellation order affects the *bid side of the book* and a sell limit cancellation order affects the *ask side of the book*.

2 The Set-Up

2.1 Terminology and Some Notation

In this sub-section we introduce some terminology that will be used throughout the dissertation. Traders who submit limit orders will be called *liquidity providers* and traders who submit market orders will be called *liquidity takers*. We shall also use the word *bid* interchangeably with the word buy, and the word *ask* with the word sell.

We refer to the best ask limit order price (i.e. the lowest price that a trader is willing to sell a particular security - located at the top of the ask side of the LOB) as the *ask price* and denote it by P^A.

We refer to the best bid limit order price (i.e. the highest price that a trader is willing to pay to get a particular security - located at the top of the bid side of the LOB) as the *bid price* and denote it by P^B.

The gap between the ask price and the bid price is called the *spread* $S := P^A - P^B$ and is always positive since no rational trader would be willing to sell at a price lower than the highest price that one would be prepared to pay for the security.

Define the *mid-price* $P := \frac{P^A + P^B}{2}$ as the average between the ask and bid price.

Define the *tick size* ΔP to represent the smallest quantity by which a price can change. And so, for this project, the prices of orders will not be continuous, but rather, multiples of ΔP.

The *shape* of the order book refers to all the different quantities of shares that the LOB contains at each level (in both the bid and the ask side) from all the orders submitted by the liquidity providers.

We will call *aggressive orders* those that instantaneously change the mid-price P.

2.2 Aim of the Dissertation

The aim of this dissertation is, from the vast collection of models that have been built to model LOBs, to explore the Markovian zero-intelligence model of the order book and study (and prove rigorously) all the recent developments and results relating to its stability and ergodic behaviour. In particular, we will build upon the models presented in [4] and [20] in a similar way that has been done in [1] and [13]. More precisely, as was done in the preceding references, we will show that in these models of LOB the convergence to the stationary state happens exponentially fast (under a certain norm). Notably, original proofs (with accommodating diagrammatic illustrations in the Appendix) have been given for these results in this dissertation that give a lot of insight into the mechanism and inner workings of the SDE describing the evolution of the order book. Additionally, some interesting proofs for results from the Theory of Markov processes (cf. [14]) have been given in full detail in this dissertation, e.g., the family of the transition operators of a Markov process forming a Markov propagator (see Theorem 4.2.1).

Zero-intelligence models of LOB are those that assume that each trader does not have a particular trading strategy they adhere to. Basically, these are models where each trader submits orders in a purely random fashion and does not act on any particular trading strategy (i.e. the order book is purely stochastic). There is plenty of evidence for the predictable power of zero-intelligence models, such as in [9], which used data from the London Stock Exchange.

Another example of an interesting zero-intelligence model of LOB is one that was built by physicists (cf. [3]) where placing a limit order is equivalent to depositing a particle, withdrawing an order (cancelling it) is equivalent to evaporating the particle and a transaction is equivalent to the annihilation of an existing deposited particle in the order book, whereby a market order (modelled as a particle) collides with the best limit order of the opposite sign.

We will model the submissions of the different orders as *independent Poisson processes*, we will model the *shape* of the order book as a Markov process and we will impose some boundary conditions on it to ensure its Markovianity. Everything will be defined and explained in due course.

3 Evolution of the Order Book

In this section we will present the general framework of our model.

3.1 Notation and Expressions

We shall describe the order book in the following way: each side (bid and ask) will have a finite number $K > 0$ of levels where possible limit orders can be placed, ranging from 1 to K ticks away from the best available opposite quote. This means that the price of a limit order will be given in number of ticks (this means that the actual nominal value will just be some multiple of the tick ΔP).

We represent the order book (at time t) as a $2K$-dimensional vector $(\boldsymbol{a}(t); \boldsymbol{b}(t)) \coloneqq (a_1(t), ..., a_K(t); b_1(t), ..., b_K(t))$, with $t \geq 0$, where

- $\boldsymbol{a}(t) \coloneqq (a_1(t), ..., a_K(t))$ is a K-dimensional vector representing the ask side of the book at time t.
  - $a_i(t), i \in \{1, ..., K\}$ is the number of shares available at a price that is i ticks away from the best bid price P^B, from all the ask limit orders submitted at this price, at time t.
- $\boldsymbol{b}(t) \coloneqq (b_1(t), ..., b_K(t))$ is a K-dimensional vector representing the bid side of the book at time t.
  - $b_i(t), i \in \{1, ..., K\}$ is the number of shares available at a price that is i ticks away from the best ask price P^A, from all the bid limit orders submitted at this price, at time t.

Comment 3.1.1. By convention, $b_i \leq 0$ and $a_i \geq 0$ for all $i \in \{1, ..., K\}$.

- We will assume that each limit order (both ask and bid) is of a constant size $q \in \mathbb{N}$ i.e. each order is of a q number of shares. It follows that $a_i(t) \in \{0, q, 2q, 3q, ...\}$ and $b_i(t) \in$

$\{..., -3q, -2q, -q, 0\}$. We could very well take q to be the minimum order size on a specific market.

- We define the <u>*cumulative depth*</u> by

$$A_i(t) := \sum_{k=1}^{i} a_k(t), \text{ for the ask side of the book}$$

and

$$B_i(t) := \sum_{k=1}^{i} |b_k(t)|, \text{ for the bid side of the book}$$

and their respective inverses

$$A^{-1}(\gamma) := \inf\{p \in \{1, ..., K\} : A_p > \gamma\}$$

$$B^{-1}(\gamma) := \inf\{p \in \{1, ..., K\} : B_p > \gamma\}$$

for some γ number of shares.

- $i_S := A^{-1}(0) = B^{-1}(0) = \frac{S}{\Delta P}$ is the index referring to the first non-empty level on both the bid and ask side of the order book, where, recall that $S := P^A - P^B$ is the spread. It is easy to see that i_S is actually just the spread in number of ticks.

In this way we adopt a *finite moving frame* for the order book, where different events in the order book cause the K price levels to change. We will impose the following *boundary condition* on our order book $(\boldsymbol{a}(t); \boldsymbol{b}(t))$:

- Every time the moving frame leaves a certain price level, the number of shares at that level is set to a preset a_∞ or b_∞ depending on the side of the order book the moving frame leaves from, see Figure 5 for an illustration. a_∞ is some (non-zero) number from $\{q, 2q, 3q, ...\}$ and b_∞ is some (non-zero) number from $\{..., -3q, -2q, -q\}$.
  - This, first and foremost, ensures the Markovianity of our model, as the future state of our order book $(\boldsymbol{a}(t); \boldsymbol{b}(t))$ is not dependent on the price levels that have been visited and then left by the moving frame at some prior time, but is only dependent on the present state of the order book i.e. we don't need to keep track of the price levels that have been visited and then left by the moving frame at some prior time. Figure 5 makes this very clear.
  - If we did not have this boundary condition, then there would be the possibility that the finite moving frame leaves all the current price levels and therefore causes the order book

to become empty (unrealistic). The boundary condition ensures this doesn't happen.

– This, in turn, ensures that P^A and P^B are always defined, which will be important for our analysis.

One could argue that it's quite unrealistic for when there is a shift in the moving frame, for there to be price levels appearing with a preset a_∞ or b_∞ (depending on the side of the book the moving frame shifts away from) amount of shares. While this is true, one could very well replace the quantities a_∞ and b_∞ by stochastic processes taking into account both the time and the price level. The good thing is that, even if we did take these quantities to be random variables instead of constants, this wouldn't change our analysis and the results we will present in the dissertation, and so for simplicity, we just take these quantities to be constants.

It's actually a very realistic assumption that the placement of limit orders is restricted on a finite interval, because empirically, studies show that limit orders placed far from the *mid-price* P usually expire or are canceled before they are executed (cf. [24] and [7]). Moreover, we can just take K in our model to be very large.

As mentioned before, we assume prices of orders are multiples of the tick size ΔP (i.e. we are in a discrete trading environment). For a continuous trading environment analysis, we refer to [15], where it is actually assumed that the explicit incorporation of market orders is not necessary, as these may be represented by very high-priced limit buy orders or very low-priced limit sell orders (due to the fact that it's a continuous trading environment). In [15] the arrival of orders of any specified type are modeled as Poisson processes (as we will do in this dissertation). A key difference in that analysis to ours, however, is the unrealistic assumption that orders cannot be canceled there.

In simulations of order books (cf. [7]), it turns out that even if we choose to have variable-sized orders e.g. half-normal distributions with standard deviation $\sqrt{\frac{2}{\pi q}}$, we still get very similar results to simulations where orders are of constant size q (as the model considered in this dissertation assumes). This means that our work in this dissertation with our constant order size assumption has carry-over to the "more realistic" order book where the order sizes may vary.

One could also argue that the order book becoming completely empty at some point in time is not an impossibility. This would however, cause problems in our analysis as we would have both P^A and P^B be undefined. Logically speaking however, the order book becoming completely empty is extremely unlikely, and so it is safe to assume that it's an insignificant event to take under consideration.

We denote and represent all the possible events that affect our order book, $(\boldsymbol{a}(t); \boldsymbol{b}(t))$, by the following counting processes (which will be modeled mathematically in the next subsection as independent Poisson processes):

- $M^{\pm}(t)$ is the event of a buy(+)/sell(−) market order being submitted at time t.
- $L_i^{\pm}(t)$ is the event of a sell(+)/buy(−) limit order (of size q shares) at level $i \in \{1, ..., K\}$ being submitted at time t.
- $C_i^{\pm}(t)$ is the event of a cancellation at time t of a sell(+)/buy(−) limit order at level $i \in \{1, ..., K\}$.

Comment 3.1.2. Recall that $P^A(t) > P^B(t)$ for all $t \geq 0$. It thus follows that buy limit orders $L_i^-(t)$ always arrive below the ask price at time t, $P^A(t)$; and similarly, sell limit orders $L_i^+(t)$ always arrive above the bid price at time t, $P^B(t)$.

Next, we express the dynamics of the ask side, $\boldsymbol{a}(t)$, of our order book at time t by the following SDE, which is driven by Poisson processes (as said before, the exact details will be given in the next subsection):

$$da_i(t) = -\mathbb{1}_{\{a_i(t)\neq 0\}}(q - A_{i-1}(t))_+ dM^+(t) + q dL_i^+(t) - \mathbb{1}_{\{a_i(t)\neq 0\}} q dC_i^+(t) \tag{1}$$

$$+ \left(J^{M^-}\left(\boldsymbol{a}(t)\right) - \boldsymbol{a}(t)\right)_i dM^-(t) + \sum_{j=1}^{K} \left(J^{L_j^-}\left(\boldsymbol{a}(t)\right) - \boldsymbol{a}(t)\right)_i dL_j^-(t) \tag{2}$$

$$+ \sum_{j=1}^{K} \left(J^{C_j^-}\left(\boldsymbol{a}(t)\right) - \boldsymbol{a}(t)\right)_i dC_j^-(t) \tag{3}$$

where $x_+ := \max\{0, x\}$ for $x \in \mathbb{R}$, $\boldsymbol{x}_i$ denotes the i-th entry of a vector $\boldsymbol{x}$, and J is a shift operator corresponding to the renumbering of the ask side of the book after an event that affects the bid side of the book, and vice versa (it will be explained and illustrated in depth in the coming text).

Explanation of the above terms of the expression:

1. *For the first term of the RHS of (1):* If there are no sell limit orders in the order book at levels less than i at time t (i.e. $A_{i-1}(t) = 0$), then this means that, given that the number of sell limit orders at level i in the order book are more than 0 (i.e. $a_i(t) \neq 0$), the best sell limit order at time t, $P^A(t)$, is precisely at level i, and hence, if a buy market order $M^+(t)$ is submitted at time t, then $a_i(t)$ will get reduced by an amount q shares as a result of the transaction. Otherwise, the buy market order $M^+(t)$ will have no effect on $a_i(t)$. This is illustrated in Figure 2.

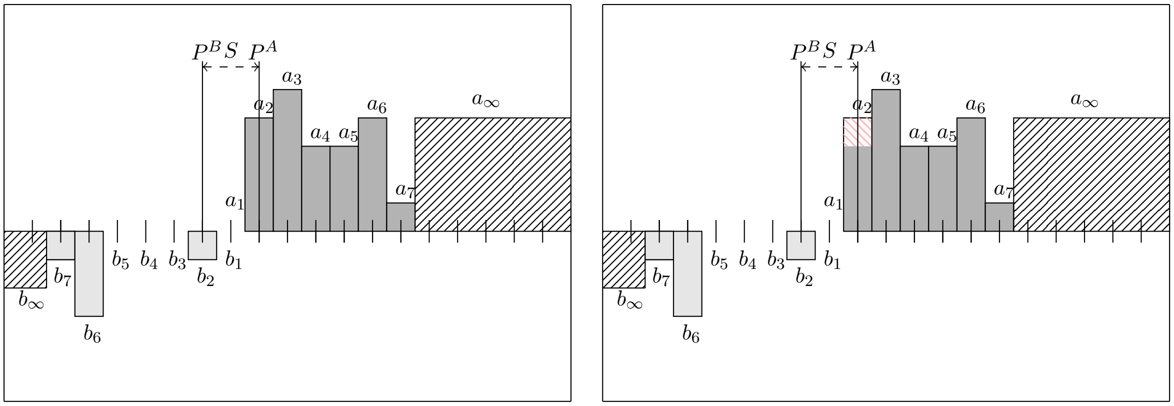

Figure 2: Incoming buy market order dM^+ (depicted by the red lines) in an order book with $K = 7$ and $q = 1$. We initially have $\boldsymbol{a} = (0, 4, 5, 3, 3, 4, 1)$. Hence, $A_1 = 0$ and $a_2 = 4 \neq 0$ is the P^A. Once the buy market order is submitted, $a_2 = 4$ becomes $a_2 = 4 - q = 4 - 1 = 3$. None of the other levels a_i are affected by the buy market order (because by definition, the buy market order interacts only with the best ask).

2. *For the second term of the RHS of (1):* The submission of a sell limit order at level i at time t, $L_i^+(t)$, would obviously increase the number of shares available at level i of the ask side of the book, $a_i(t)$, by an amount q shares. For an example, see Figure 3.

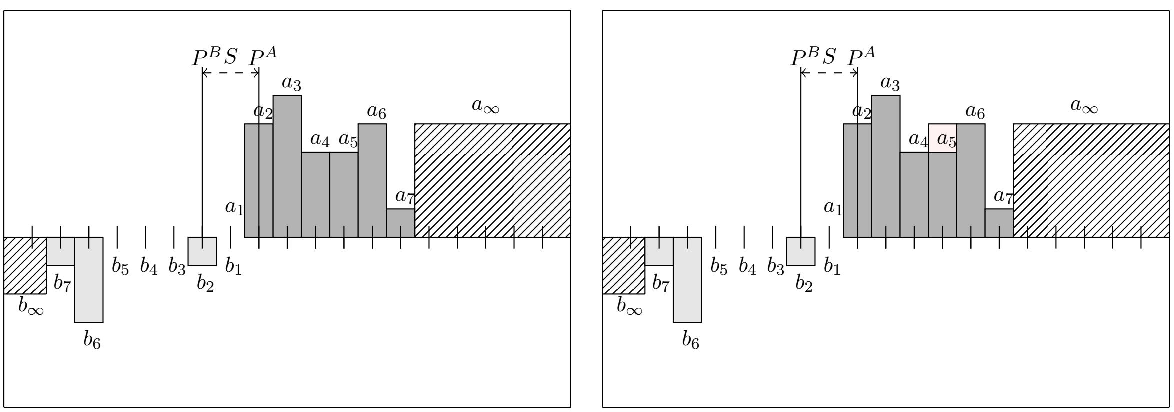

Figure 3: Incoming ask limit order at level $i = 5$, dL_5^+ (depicted by the pink box), in an order book with $K = 7$ and $q = 1$. Initially we had an ask side $\boldsymbol{a} = (0, 4, 5, 3, 3, 4, 1)$. After the ask limit order, the ask side became $\boldsymbol{a} = (0, 4, 5, 3, 4, 4, 1)$ i.e. from $a_5 = 3$, it turned to $a_5 = 3 + q = 3 + 1 = 4$.

3. *For the third term of the RHS of (1):* Assuming that a cancellation of an ask limit order at level i is feasible at time t in the order book i.e. there exist at least 1 ask limit order at level i in the book ($a_i(t) \neq 0$), a cancellation of a sell limit order at level i, $C_i^+(t)$, would obviously reduce the number of shares available at level i of the ask side of the book, $a_i(t)$,

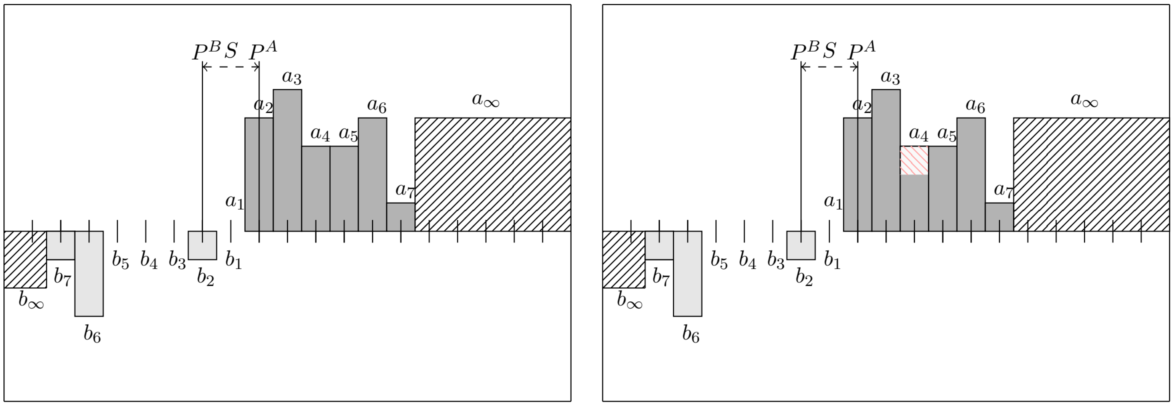

Figure 4: Incoming cancellation of an ask limit order at level $i = 4$ (depicted by the red lines), dC_4^+, in an order book with $K = 7$ and $q = 1$. In this case, $a_4 = 3$ initially. After the cancellation, $a_4 = 3 - q = 3 - 1 = 2$.

by an amount q shares. For an example, see Figure 4.

4. *For the first term of the RHS of (2):* In the case of a submission of a sell market order, $M^-(t)$, the corresponding shift operator J^{M^-} is:

$$J^{M^-}(\boldsymbol{a}(t)) = (\underbrace{0, ..., 0}_{k \text{ times}}, a_1(t), a_2(t), ..., a_{K-k}(t))$$

where $k := \inf \{p \in \{1, ..., K\} : \sum_{j=1}^{p} |b_j(t)| > q\} - \inf\{p \in \{1, ..., K\} : |b_p(t)| > 0\}$.
And so as a result of $dM^-(t)$, the level numbering of the ask side of the book has shifted by k, and hence, the value of $a_i(t)$ has changed. The quantity $(J^{M^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t))_i$ gives this changed value less the initial quantity $a_i(t)$ (before the shifting of the ask side of the book had taken place). See Figure 5 for an illustration of these dynamics.

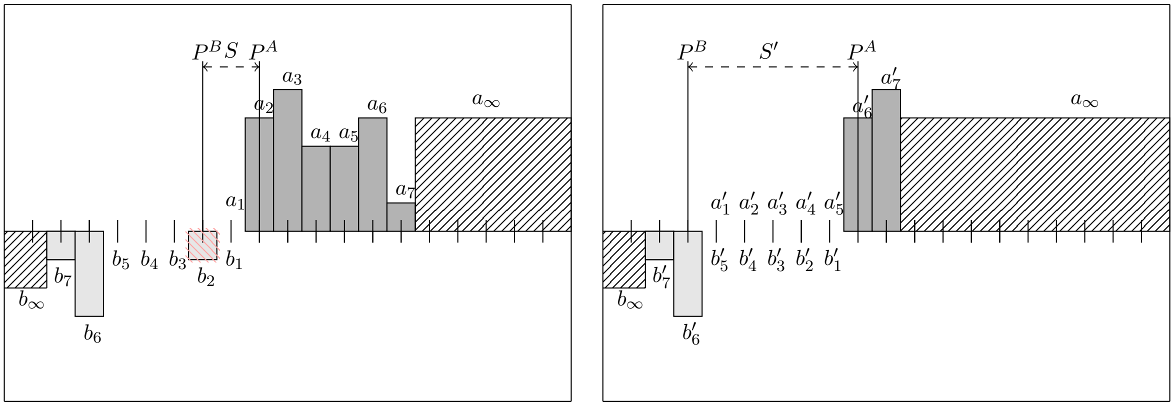

Figure 5: LOB dynamics under the submission of a sell market order: in this example, $K = 7$, $q = 1$, $a_\infty = 4$, $b_\infty = -2$. The shape of the order book is such that $\boldsymbol{a} = (0, 4, 5, 3, 3, 4, 1)$ and $\boldsymbol{b} = (0, -1, 0, 0, 0, -3, -1)$. The spread in ticks is given by $i_S = 1$. Suppose that a sell market order is submitted. Then, the best buy limit order, which is at level 2 of the bid side, gets 'annihilated' (depicted on the diagram by the red lines). This causes a shift on the ask side: $\boldsymbol{a}$ now becomes $\boldsymbol{a'} = (0, 0, 0, 0, 0, 4, 5) = J^{M^-}(\boldsymbol{a})$. Observe that $\boldsymbol{b} = \boldsymbol{b'}$ and the spread in number of ticks is now $i_{S'} = 6$. Note that, had b_2 been greater than $q = 1$, the sell market order would not have caused a shift, but merely would have reduced b_2 by q.

5. *For the second term of the RHS of (2):* In the event of a submission of a buy limit order within the spread i.e. a better buy limit order from the current best buy limit order at time t, k levels away from the previous best bid price, the ask side of the book will shift in the following way:

$$J^{L_i^-}(\boldsymbol{a}(t)) = (a_{1+k}(t), a_{2+k}(t), ..., a_K(t), \underbrace{a_\infty, ..., a_\infty}_{k \text{ times}})$$

where $k = i_S - i$ (i.e. the index of the best bid price less the index of the new better bid). Notice that if $dL_i^-(t)$ occurs where i is such that $b_i(t)$ doesn't lie in the spread, then this has absolutely no effect on the ask side of the book (i.e. no shift on the ask side) and thus, $J^{L_i^-}(\boldsymbol{a}(t)) = \boldsymbol{a}(t)$ for such an i. The quantity $\sum_{j=1}^{K}(J^{L_j^-}(\boldsymbol{a}(t)) - \boldsymbol{a}(t))_i$, therefore, gives the total effect of all possible submissions of limit buy orders at all possible levels at time t ($dL_j^-(t)$ for all $j \in \{1, ..., K\}$) on the i-th level of the ask side of the order book, $a_i(t)$. See Figure 6 for an illustration of these dynamics.

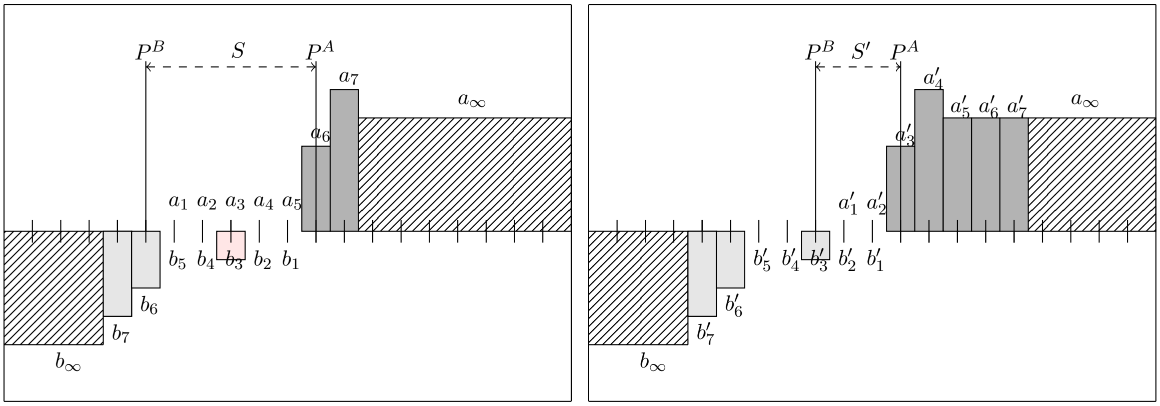

Figure 6: LOB dynamics under the submission of a buy limit order within the spread S: in this example, $K = 7$, $q = 1$, $a_\infty = 4$, $b_\infty = -4$. The shape of the order book is such that $\boldsymbol{a} = (0,0,0,0,0,3,5)$ and $\boldsymbol{b} = (0,0,0,0,0,-2,-3)$. The spread in ticks is given by $i_S = 6$. Suppose that a buy limit order is submitted at level $i = 3$ of the bid side (depicted in pink on the top diagram) $\implies$ $k = i_S - i = 6 - 3 = 3$. This causes a shift on the ask side: $\boldsymbol{a}$ now becomes $\boldsymbol{a'} = (0,0,3,5,4,4,4) = J^{L_3^-}(\boldsymbol{a})$. Observe how $\boldsymbol{b} = \boldsymbol{b'}$ and the spread in number of ticks is now $i_{S'} = 3$.

6. *For the last term of the SDE:* A cancellation of a bid limit order at the best bid will have the exact same affect on the ask side of the book as would a market sell order (i.e. the same effect as explained on the 4th point above). A cancellation of a bid limit order at any other level other than on the best bid will not have an effect on the ask side of the book. We can express this mathematically as

$$J^{C_{i_S}^-}(\boldsymbol{a}(t)) = J^{M^-}(\boldsymbol{a}(t)) \text{ and } J^{C_j^-}(\boldsymbol{a}(t)) = \boldsymbol{a}(t) \text{ for all } j \in \{1, ..., K\} \setminus \{i_S\}.$$

See Figure 7 for an illustration of these dynamics.

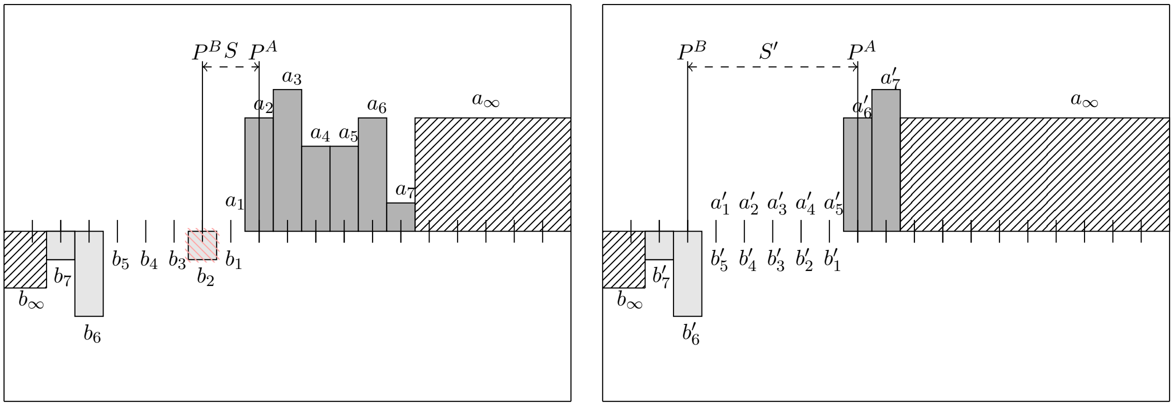

Figure 7: Resulting order book after a cancellation of a bid limit order at level $i = 2$ (depicted by the red lines), dC_2^-, in an order book with $K = 7$ and $q = 1$.

Next, we also express, in an analogous way as for the dynamics of the ask side of the book, the dynamics of the bid side, $\boldsymbol{b}(t)$, of our order book at time t by the following SDE:

$$db_i(t) = \mathbb{1}_{\{b_i(t)\neq 0\}}(q - B_{i-1}(t))_+ dM^-(t) - q dL_i^-(t) + \mathbb{1}_{\{b_i(t)\neq 0\}} q dC_i^-(t) \tag{4}$$

$$+ \Big(J^{M^+}\big(\boldsymbol{b}(t)\big) - \boldsymbol{b}(t)\Big)_i dM^+(t) + \sum_{j=1}^{K} \Big(J^{L_j^+}\big(\boldsymbol{b}(t)\big) - \boldsymbol{b}(t)\Big)_i dL_j^+(t) \tag{5}$$

$$+ \sum_{j=1}^{K} \Big(J^{C_j^+}\big(\boldsymbol{b}(t)\big) - \boldsymbol{b}(t)\Big)_i dC_j^+(t) \tag{6}$$

Comment 3.1.3. Some of the signs of the above expression are opposite to those in the SDE of $a_i(t)$ because we have to account for the fact that, by convention, the $b_i(t)$'s are non-positive.

3.2 Mathematical Modeling of the Evolution of the Order Book

We model all the counting processes mentioned in the previous subsection $(M^{\pm}(t), L_i^{\pm}(t), C_i^{\pm}(t), t \geq 0)$, which represent all possible events that affect our order book, as *independent Poisson processes*:

- We model $(M^+(t))_{t\geq 0}$ as a Poisson process with intensity λ^{M^+}

$$\implies \mathsf{P}(M^+(t) = k) = (\lambda^{M^+} t)^k \frac{e^{-\lambda^{M^+} t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall t \geq 0. \tag{7}$$

- We model $(M^-(t))_{t\geq 0}$ as a Poisson process with intensity λ^{M^-}

$$\implies \mathsf{P}(M^-(t)=k) = (\lambda^{M^-}t)^k \frac{e^{-\lambda^{M^-}t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall t \geq 0. \tag{8}$$

- We model $(L_i^+(t))_{t\geq 0}$ for $i \in \{1,...,K\}$ as a Poisson process with intensity $\lambda_i^{L^+}$

$$\implies \mathsf{P}(L_i^+(t)=k) = (\lambda_i^{L^+}t)^k \frac{e^{-\lambda_i^{L^+}t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall i \in \{1,...,K\}, \forall t \geq 0. \tag{9}$$

- We model $(L_i^-(t))_{t\geq 0}$ for $i \in \{1,...,K\}$ as a Poisson process with intensity $\lambda_i^{L^-}$

$$\implies \mathsf{P}(L_i^-(t)=k) = (\lambda_i^{L^-}t)^k \frac{e^{-\lambda_i^{L^-}t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall i \in \{1,...,K\}, \forall t \geq 0. \tag{10}$$

- We model $(C_i^+(t))_{t\geq 0}$ for $i \in \{1,...,K\}$ as a Poisson process with intensity $a_i(t)\lambda_i^{C^+}$

$$\implies \mathsf{P}(C_i^+(t)=k) = (a_i(t)\lambda_i^{C^+}t)^k \frac{e^{-a_i(t)\lambda_i^{C^+}t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall i \in \{1,...,K\}, \forall t \geq 0. \tag{11}$$

- We model $(C_i^-(t))_{t\geq 0}$ for $i \in \{1,...,K\}$ as a Poisson process with intensity $|b_i(t)|\lambda_i^{C^-}$

$$\implies \mathsf{P}(C_i^-(t)=k) = (|b_i(t)|\lambda_i^{C^-}t)^k \frac{e^{-|b_i(t)|\lambda_i^{C^-}t}}{k!} \qquad \forall k \in \mathbb{N}_0, \forall i \in \{1,...,K\}, \forall t \geq 0. \tag{12}$$

Notice the dependency that we've imposed on the intensities of $(C_i^+(t))_{t\geq 0}$ and $(C_i^-(t))_{t\geq 0}$ on the state of the order book at level i, $a_i(t)$ and $b_i(t)$ respectively. This is to ensure that if at level i there are currently no limit orders, then the probability of a cancellation order at level i is 0. We will see what difference it makes to our analysis when we relax this dependency later on.

From [12], we see that our assumption that order flows follow independent Poisson processes is not very compatible with empirical observations of order books. However, the simplicity of this assumption allows us to establish interesting formulae, and more importantly the stability and ergodicity of the LOB model – both, testable on market data. Although we will not deal with this in this dissertation, there have been developments in LOB modeling whereby the order flows are modeled as Hawkes processes instead of Poisson (cf. [6]). The modeling of order flows as Hawkes processes are very much supported by some empirical properties of the flow of market and limit orders at the microscopic level. In particular, the *time clustering* and *mutual excitation* properties that Hawkes processes exhibit reflect the fact that order arrivals alternate between busy and quiet periods (cf. [21]).

In [11], the idea of modeling the arrivals of the different orders in the order book as independent Poisson processes (like we do here) is not used. Instead, a game-theoretic approach is ensued where the intensities of the arrival of orders are state dependent and the strategic behaviours of market participants are incorporated, assuming that their decisions are based on a price reference (this is usually taken to be the mid-price P) and the current shape of the order book.

Finally, let's now model the order book $(\boldsymbol{a}(t); \boldsymbol{b}(t))$ as a time-homogeneous Markov process. Consequently, we define the stochastic process $(\boldsymbol{X}(t) : t \geq 0)$ where $\boldsymbol{X}(t) := (\boldsymbol{a}(t); \boldsymbol{b}(t)) = (a_1(t), ..., a_K(t); b_1(t), ..., b_K(t))$ The boundary conditions we had imposed on our order book in the previous subsection and the modeling of the counting processes that represent all the different possible events that affect the order book as Poisson processes (which, is well-known, satisfy the Markov property), ensure the Markovianity of $(\boldsymbol{X}(t) : t \geq 0)$.

Let us also introduce *the embedded Markov chain* $(\boldsymbol{Z}_n)_{n \in \mathbb{Z}_+}$ associated with $(\boldsymbol{X}(t) : t \geq 0)$, explained below.

When we are working with $(\boldsymbol{Z}_n)_{n \in \mathbb{Z}_+}$, we are in "event-time", meaning that, if we consider the sequence $(T_n)_{n \in \mathbb{Z}_+}$ of jump times of the continuous-time Markov process $(\boldsymbol{X}(t) : t \geq 0)$, which is defined (as in the book [19]) recursively as:

$$T_0 = 0, \quad T_{n+1} = \inf \{t > T_n : \boldsymbol{X}(t) \neq \boldsymbol{X}(T_n)\} \text{ for } n \in \mathbb{N}_0,$$

then $(\boldsymbol{Z}_n)_{n \in \mathbb{Z}_+}$ is precisely

$$\boldsymbol{Z}_0 = \boldsymbol{X}(0), \quad \boldsymbol{Z}_n = \boldsymbol{X}(T_n) \text{ for } n \in \mathbb{N}$$

The difference between working in event-time and continuous-time is that, in event-time, the probability of each possible event in the order book is "normalized". This means that for $\boldsymbol{x} = (\boldsymbol{a}; \boldsymbol{b}) = (a_1, ..., a_K; b_1, ..., b_K)$ and $n \in \mathbb{N}_0$,

$$p^{M^+} := \mathsf{P}(\text{"Buy market order"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{x})} \tag{13}$$

$$p^{M^-} := \mathsf{P}(\text{"Sell market order"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{\lambda^{M^-}}{\Lambda(\boldsymbol{x})} \tag{14}$$

$$p^{L_i^+} := \mathsf{P}(\text{"Sell limit order at level i"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{\lambda_i^{L^+}}{\Lambda(\boldsymbol{x})}, \text{ for } i \in \{1, ..., K\} \tag{15}$$

$$p^{L_i^-} := \mathsf{P}(\text{"Buy limit order at level i"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{\lambda_i^{L^-}}{\Lambda(\boldsymbol{x})}, \text{ for } i \in \{1, ..., K\} \tag{16}$$

$$p^{C_i^+} \coloneqq \mathsf{P}(\text{"Cancellation of a sell limit order at level i"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{a_i \lambda_i^{C^+}}{\Lambda(\boldsymbol{x})}, \text{ for } i \in \{1, ..., K\} \tag{17}$$

$$p^{C_i^-} \coloneqq \mathsf{P}(\text{"Cancellation of a buy limit order at level i"} | \boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \frac{|b_i| \lambda_i^{C^-}}{\Lambda(\boldsymbol{x})}, \text{ for } i \in \{1, ..., K\} \tag{18}$$

where

$$\Lambda(\boldsymbol{x}) \coloneqq \lambda^{M^+} + \lambda^{M^-} + \sum_{i=1}^{K} \lambda_i^{L^+} + \sum_{i=1}^{K} \lambda_i^{L^-} + \sum_{i=1}^{K} a_i \lambda_i^{C^+} + \sum_{i=1}^{K} |b_i| \lambda_i^{C^-} \tag{19}$$

$$= \lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \Lambda^{C^+} + \Lambda^{C^-} \tag{20}$$

where $\Lambda^{L^+} \coloneqq \sum_{i=1}^{K} \lambda_i^{L^+}$, $\Lambda^{L^-} \coloneqq \sum_{i=1}^{K} \lambda_i^{L^-}$, $\Lambda^{C^+} \coloneqq \sum_{i=1}^{K} a_i \lambda_i^{C^+}$, $\Lambda^{C^-} \coloneqq \sum_{i=1}^{K} |b_i| \lambda_i^{C^-}$.

4 Infinitesimal Generator of LOB

In this section our goal is to derive the infinitesimal generator of our Markov process, $(\boldsymbol{X}(t) : t \geq 0)$, as well as the transition operator of the embedded Markov chain associated to it, $(\boldsymbol{Z}_n)_{n \in \mathbb{N}_0}$. We will then use these in the next section to prove the ergodicity of our model.

4.1 Functional Analysis Preliminaries

In this subsection we will present important definitions and results, taken from [2] and [10], that will be used in the next subsections.

Let V and U be vector spaces over $\mathbb{R}$.

Definition 4.1.1. A function $A : V \to U$ is called a *linear operator* if

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y) \qquad \text{for all } x, y \in V \text{ and } \alpha, \beta \in \mathbb{R}. \tag{21}$$

Definition 4.1.2. A linear operator $A : V \to U$ is *bounded* if there exists a $M > 0$ such that

$$\|Ax\|_U \leq M \|x\|_V \qquad \text{for all } x \in V, \tag{22}$$

where $\| \cdot \|_U$ and $\| \cdot \|_V$ are norms on U and V respectively.

Definition 4.1.3. The *operator norm* of a bounded linear operator $A : V \to U$ is

$$\|A\|_{\text{op}} = \sup_{\|x\|_V=1} \|A(x)\|_U \tag{23}$$
$$= \sup_{x\neq 0} \frac{\|A(x)\|_U}{\|x\|_V}. \tag{24}$$

Comment 4.1.1. We denote the space of all bounded linear operators by:

$$B(V,U) = \{A : V \to \text{ is a bounded linear operator}\}.$$

Theorem 4.1.1. *If V is a normed space with norm $\|\cdot\|_V$ and U is a Banach space (with norm $\|\cdot\|_U$), then $(B(V,U), \|\cdot\|_{op})$ is a Banach space.*

Proof. See [16]. □

Definition 4.1.4. For a set S, a family of mappings $(U^{t,r} : r \leq t)$ from S to S, parameterized by the pairs of numbers $r \leq t$ from a given finite or infinite interval is called a *propagator* in S if:

1. $U^{t,t}$ is the identity operator in S for all t.
2. The following *propagator equation* holds:

$$\forall r \leq s \leq t, \;\; U^{t,s}U^{s,r} = U^{t,r}. \tag{25}$$

Similarly, for a set S, the family of mappings $(U^{t,r} : t \leq r)$ from S to S, is called a *backward propagator* in S if:

1. $U^{t,t}$ is the identity operator in S for all t.
2. The following *propagator equation* holds:

$$\forall t \leq s \leq r, \;\; U^{t,s}U^{s,r} = U^{t,r}. \tag{26}$$

Definition 4.1.5. A family of mappings $(T^t : t \geq 0)$ from S to S, parameterized by non-negative numbers t is said to form a *semigroup* if:

1. T^0 is the identity mapping in S.
2. $T^tT^s = T^{t+s} \qquad \text{for all } \; t, s \geq 0.$

Comment 4.1.2. Note that, if the mappings $(U^{t,r} : t \leq r)$ forming a backward propagator depend only on the differences $r - t$, then the family $(T^t := U^{0,t} : t \geq 0)$ forms a semigroup.

Definition 4.1.6. A semigroup $(T^t : t \geq 0)$ of bounded linear operators on a Banach space V is called *strongly continuous* if

$$\|T^t(f) - f\| \longrightarrow 0 \text{ as t} \rightarrow 0 \qquad \text{for all } f \in V, \tag{27}$$

where $\|\cdot\|$ is the norm of the Banach space V.

Comment 4.1.3. We illustrate the above concepts with the following example:

Let A be a bounded linear operator on a Banach space V (with norm $\|\cdot\|$). Then $(T_t : t \geq 0)$ where $T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ defines a strongly continuous semi-group.

Indeed, let $\alpha, \beta \in \mathbb{R}$ and $f, g \in V$, then

$$\begin{aligned} T_t(\alpha f + \beta g) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(\alpha f + \beta g) \\ &= \alpha \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(f) + \beta \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(g) \quad \text{(easy to see that } A \text{ is linear } \implies A^n \text{ is linear)} \\ &= \alpha T_t(f) + \beta T_t(g). \end{aligned}$$

$\therefore T_t$ is a linear operator for all $t \geq 0$.

It's clear that T_0 is simply the identity operator. Also,

$$\begin{aligned} e^{(t+s)A} &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!} A^n \\ &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k t^k}{k!} \frac{A^{n-k} s^{n-k}}{(n-k)!} \\ &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{n=k}^{\infty} \frac{A^{n-k} s^{n-k}}{(n-k)!} \\ &= e^{tA} e^{sA}. \end{aligned}$$

And so the propagator equation $T_t T_s = T_{t+s}$ for all $t, s \geq 0$ from Definition 4.1.5 is also satisfied.

$\therefore (T_t : t \geq 0)$ is in fact a semigroup.

We also have that A is bounded. This means that there exists an $M > 0$ such that $\|A(f)\| \leq M\|f\|$ for all $f \in V$.

$$\begin{aligned}
\therefore \|T_t(f)\| = \|\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n(f)\| &\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n(f)\| && \text{(triangle inequality of } \|\cdot\|) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \|A(A^{n-1}(f))\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} M\|A^{n-1}(f)\| && \text{(by boundedness of A and the fact that } A^{n-1}(f) \in V) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} M\|A(A^{n-2}(f))\| \\
&\leq \sum_{n=0}^{\infty} \frac{t^n}{n!} M^2\|A^{n-2}(f)\| && \text{(by boundedness of A and the fact that } A^{n-2}(f) \in V) \\
&\leq \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} M^n\right) \|f\| && \text{(iteratively)} \\
&= e^{tM}\|f\| \\
&= K\|f\| && \text{(where } K := e^{tM} > 0)
\end{aligned}$$

$\therefore T_t$ is a bounded linear operator for all $t \geq 0$.

So, in fact, $(T_t : t \geq 0)$ is a semigroup of bounded linear operators on the Banach space V. All that remains to show is that $(T_t : t \geq 0)$ is strongly continuous.
i.e. we would like to show that $\lim_{t \to 0} \|T_t(f) - f\| = 0$ for all $f \in V$.

Notice how

$$\begin{aligned}
\|T_t(f) - f\| &= \|(T_t - I)(f)\| && \text{(where } I \text{ is the identity operator)} \\
&\leq \|T_t - I\|_{\text{op}} \|f\| && (\|T_t - I\|_{\text{op}} < \infty \text{ by boundedness of } T_t)
\end{aligned}$$

And so it suffices to show that $\|T_t - I\|_{\text{op}} \longrightarrow 0$ as $t \to 0$.

$$\begin{aligned}
\|T_t - I\|_{\text{op}} &= \|\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} - I\|_{\text{op}} \\
&= \|(I - I) + \sum_{n=1}^{\infty} \frac{t^n A^n}{n!}\|_{\text{op}} && \text{(since } A^0 = I \text{ is the identity operator)}
\end{aligned}$$

$$= \| \sum_{n=0}^{\infty} \frac{t^{n+1} A^{n+1}}{(n+1)!} \|_{\text{op}} = \| tA \sum_{n=0}^{\infty} \frac{t^n A^n}{(n+1)!} \|_{\text{op}}$$

$$\leq t \|A\|_{\text{op}} \sum_{n=0}^{\infty} \frac{t^n \|A^n\|_{\text{op}}}{(n+1)!} \qquad \text{(by triangle inequality of } \|\cdot\|_{\text{op}})$$

$$\leq t \|A\|_{\text{op}} \sum_{n=0}^{\infty} \frac{t^n \|A\|_{\text{op}}^n}{n!} \qquad \left(\text{as } \frac{1}{(n+1)!} \leq \frac{1}{n!} \ \forall n \in \mathbb{N}_0 \text{ and } \|A^n\|_{\text{op}} \leq \|A\|_{\text{op}}^n\right)$$

$$= t\|A\|_{\text{op}} e^{t\|A\|_{\text{op}}} < \infty \qquad (\text{A is bounded} \implies \|A\|_{\text{op}} < \infty)$$

$$\Big\downarrow \text{ as } t \to 0$$

$$0$$

$\implies \lim_{t \to 0} \|T_t(f) - f\| = 0$ for all $f \in V$.

$\therefore$ $(T_t : t \geq 0)$ is a strongly continuous semigroup by Definition 4.1.6. □

We are now able give the most important definition in this section.

Definition 4.1.7. Let $(T_t : t \geq 0)$ be a strongly continuous semigroup of linear operators on a Banach space V. The *infinitesimal generator* (or simply, *generator*) of $(T_t : t \geq 0)$ is defined as the operator:

$$A(f) = \lim_{t \to 0+} \frac{T_t(f) - f}{t}, \qquad f \in V \tag{28}$$

which is defined on the linear subspace $D_A \subset V$ where this limit exists.

Comment 4.1.4. Notice how

$$A(f) = \lim_{t \to 0+} \frac{T_t(f) - f}{t} = \frac{d}{dt} T_t(f)\Big|_{t=0}. \tag{29}$$

4.2 Application to Markov processes

We follow [14] in this subsection. We first recall some fundamental definitions:

Definition 4.2.1. Analogous to the discrete-time definition of Markov processes (actually, we *usually* refer to Markov processes in discrete-time as Markov *chains* instead), a *Markov process* in $\mathbb{R}$ is a family of processes $((X_{t \geq s}^{s,x}) : s \in \mathbb{R}_+, x \in \mathbb{R})$ depending on $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$ as parameters (the process $(X_{t \geq s}^{s,x})$ is a process with initial value x at time s) such that there exists a family of transition probability kernels $p_{s,t}(x, A)$ from $\mathbb{R}$ to $\mathbb{R}$, $0 \leq s \leq t$ where $A \in \mathscr{B}(\mathbb{R})$ (where $\mathscr{B}(\mathbb{R})$ denotes the Borel sigma algebra of $\mathbb{R}$), which we'll call *transition probabilities*, such that

$$\mathbb{E}(f(X_t^{s,x}) | \mathcal{F}_u)) = \mathbb{E}(f(X_t^{s,x}) | X_u^{s,x})) \tag{30}$$

$$= \int_{\mathbb{R}} f(y) p_{u,t}(X_u^{s,x}, dy) \text{ almost surely} \tag{31}$$

for all $f \in B(\mathbb{R})$ and $0 \leq s \leq u \leq t$.

Definition 4.2.2. The operator $\Phi^{s,t}(f(x)) = \int_{\mathbb{R}} f(y) p_{s,t}(x, dy)$ from (31) is called the *transition operator* of the Markov process $((X_{t \geq s}^{s,x}) : s \in \mathbb{R}_+, x \in \mathbb{R})$.

Now let's start building towards the definition of a Markov semi-group.

Definition 4.2.3. A linear operator L on a functional space (e.g. $B(\mathbb{R})$) is called *positive* if

$$f \geq 0 \implies L(f) \geq 0.$$

Definition 4.2.4. An operator A in a Banach space (e.g. $B(\mathbb{R})$) is called a *linear contraction* if $\|A\|_{\text{op}} \leq 1$.

Definition 4.2.5. A backward propagator (resp. a semigroup) of positive linear contractions in $B(\mathbb{R})$ is said to be a *sub-Markov backward propagator* (resp. a *sub-Markov semigroup*).

Moreover, if all these contractions are *conservative* i.e. they map any constant function to itself, then the former is called a *Markov backward propagator* (resp. a *Markov semigroup*).

We can now prove the main theorem of this subsection which will allow us to derive the infinitesimal generator of the order book in the next subsection.

Theorem 4.2.1. *The family of the transition operators* $(\Phi^{s,t} : s \leq t)$ *of a Markov process in* $\mathbb{R}$ *forms a Markov propagator in* $B(\mathbb{R})$*.*

Moreover, if this Markov process is time-homogeneous, the family $\Big(\Phi_t\big(f(x)\big) : t \geq 0\Big)$*, where* $\Phi_t\big(f(x)\big) = \mathbb{E}_x\big(f(X_t)\big)$*, forms a Markov semi-group.*

Proof. It's clear that $(\Phi^{s,t} : s \leq t)$ is a family of mappings from $B(\mathbb{R})$ to $B(\mathbb{R})$ such that

$$\begin{aligned} \Phi^{t,t}(f(x)) &= \int_{\mathbb{R}} f(y) p_{t,t}(x, dy) \\ &= \int_{\mathbb{R}} f(y) \delta_x(dy) \qquad \text{(if the process at time } t \text{ is at state } x\text{, then at time } t \text{ it has to remain at } x) \\ &= f(x). \end{aligned}$$

i.e. $\Phi^{t,t} = I$ for all $t \geq 0$ is the identity operator.

And for fixed $0 \le r \le s \le t$,

$$\begin{aligned}
\Phi^{r,t}(f(x)) &= \int_{\mathbb{R}} f(y)p_{r,t}(x,dy) = \int_{\mathbb{R}} f(y)p_{r,t}(X_r = x, dy) \\
&= \mathbb{E}(f(X_t)|X_r = x) \\
&= \mathbb{E}(\mathbb{E}(f(X_t)|\mathcal{F}_s)|X_r = x) && \text{(by tower law of conditional expectation since } r \le s\text{)} \\
&= \mathbb{E}(\mathbb{E}(f(X_t)|X_s)|X_r = x) && \text{(by Markov Property)} \\
&= \mathbb{E}(\int_{\mathbb{S}} f(y)p_{s,t}(X_s, dy)|X_r = x) \\
&= \mathbb{E}(\Phi^{s,t}(f(X_s))|X_r = x) \\
&= \int_{\mathbb{R}} \Phi^{s,t}(f(y))p_{r,s}(X_r = x, dy) \\
&= \Phi^{r,s}(\Phi^{s,t}(f(x))) = (\Phi^{r,s} \circ \Phi^{s,t})(f(x)).
\end{aligned}$$

And so, $(\Phi^{s,t} : s \le t)$ also satisfies the propagator equation from Definition 4.1.4.

$\therefore$ $(\Phi^{s,t} : s \le t)$ is in fact a propagator in $B(\mathbb{R})$.

Now, in order to show it's a Markov propagator, we need to also show that $(\Phi^{s,t} : s \le t)$ is positive, and that for all s, t, $\Phi^{s,t}$ is a linear contraction and is conservative.

As $p_{s,t}(x, \cdot)$ for a fixed $x \in \mathbb{R}$ is a probability measure as a function of $A \in \mathscr{B}(\mathbb{R})$ (by the definition of a probability kernel), it's obvious that for $f \ge 0$ in $B(\mathbb{R})$,

$$\Phi^{s,t}(f(x)) = \int_{\mathbb{R}} f(y)p_{s,t}(x,dy) \ge 0$$

i.e. $(\Phi^{s,t} : s \le t)$ is positive.

We would like to show that $\|\Phi^{s,t}\|_{\text{op}} = \sup_{f\in B(\mathbb{R})} \frac{\|\Phi^{s,t}(f)\|}{\|f\|} \le 1$. As the space $(B(\mathbb{R}), \|\cdot\|_\infty)$ is Banach, we can take $\|\cdot\| \equiv \|\cdot\|_\infty$ here.

Notice that for all $f \in B(\mathbb{R})$,

$$\begin{aligned}
\|\Phi^{s,t}(f)\|_\infty &= \sup_{x\in\mathbb{R}} |\int_{\mathbb{R}} f(y)p_{s,t}(x,dy)| \\
&\le \sup_{x\in\mathbb{R}} |\int_{\mathbb{R}} \|f\|_\infty p_{s,t}(x,dy)| \\
&= \|f\|_\infty \cdot \sup_{x\in\mathbb{R}} |\int_{\mathbb{R}} p_{s,t}(x,dy)| \\
&= \|f\|_\infty \cdot \sup_{x\in\mathbb{R}} |p_{s,t}(x,\mathbb{R})|
\end{aligned}$$

$$= \|f\|_\infty \qquad (p_{s,t}(x,\cdot) \text{ is a probability measure on } (\mathbb{R},\mathscr{B}(\mathbb{R})) \implies p_{s,t}(x,\mathbb{R})=1)$$

Hence, $\frac{\|\Phi^{s,t}(f)\|_\infty}{\|f\|_\infty} \leq 1$ for all $f \in B(\mathbb{R})$
$\therefore \|\Phi^{s,t}(f)\|_{\text{op}} \leq 1$ as required. And so, $\Phi^{s,t}$ is a linear contraction.

And finally, to show that $\Phi^{s,t}$ is conservative, take any constant function, say $a \in \mathbb{R}$, then

$$\Phi^{s,t}(a) = \int_{\mathbb{R}} a p_{s,t}(x,dy) = a \int_{\mathbb{R}} p_{s,t}(x,dy) = a \cdot p_{s,t}(x,\mathbb{R}) = a \cdot 1 = a.$$

So, by Definition 4.2.5 ($\Phi^{s,t} : s \leq t$) is a Markov backward propagator, as required.

Now, if this Markov process is time-homogeneous i.e. $\Phi^{s,t}$ and $p_{s,t}(x,A)$ depend on the difference $t-s$ only, then by Comment 4.1.2 the family $(\Phi_t : t \geq 0)$ given by $\Phi_t := \Phi^{0,t}$ forms a Markov semigroup, where

$$\begin{aligned}
\Phi^{0,t}(f(x)) &= \int_{\mathbb{R}} f(y) p_{0,t}(x,dy) \\
&= \int_{\mathbb{R}} f(y) p_{0,t}(X_0 = x, dy) \\
&= \mathbb{E}(f(X_t)|X_0 = x) \\
&= \mathbb{E}_x(f(X_t)).
\end{aligned}$$

□

4.3 Derivation of the Infinitesimal Generator of LOB

We are now in the position to derive the generator of our Markovian model of LOB, $(\boldsymbol{X}(t) : t \geq 0)$. We provide the proof to the result only stated in [1]:

Proposition 4.3.1. *The infinitesimal generator associated to the dynamics of the LOB model* $(\boldsymbol{X}(t) : t \geq 0)$ *is the operator* $\mathcal{L}$ *defined by:*

$$\mathcal{L}f(\boldsymbol{a};\boldsymbol{b}) = \lambda^{M^+}(f([a_i - (q - A_{i-1})_+]_+^{i=1,\ldots,K}; J^{M^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \tag{32}$$

$$+ \sum_{i=1}^{K} \lambda_i^{L^+}(f(a_i + q; J^{L_i^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \tag{33}$$

$$+ \sum_{i=1}^{K} \lambda_i^{C^+} a_i (f(a_i - q; J^{C_i^+}(\boldsymbol{b})) - f(\boldsymbol{a};\boldsymbol{b})) \tag{34}$$

$$+ \lambda^{M^-}(f(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\ldots,K}) - f(\boldsymbol{a};\boldsymbol{b})) \tag{35}$$

$$+\sum_{i=1}^{K} \lambda_i^{L^-} (f(J^{L_i^-}(\boldsymbol{a}); b_i - q) - f(\boldsymbol{a}; \boldsymbol{b})) \tag{36}$$

$$+\sum_{i=1}^{K} \lambda_i^{C^-} |b_i| (f(J^{C_i^-}(\boldsymbol{a}); b_i + q) - f(\boldsymbol{a}; \boldsymbol{b})) \tag{37}$$

for sufficiently regular[1] *functions* $f : \mathbb{Z}^{2K} \to \mathbb{R}$*, where we have used the following shorthand notations:*

$$f(a_i + c; \boldsymbol{b}) \equiv f(a_1, ..., a_{i-1}, a_i + c, a_{i+1} ..., a_K; \boldsymbol{b}), \quad \textit{etc.}$$

$$f((a_i + c)^{i=1,...,K}; \boldsymbol{b}) \equiv f(a_1 + c, a_2 + c, ..., a_K + c; \boldsymbol{b}), \quad \textit{etc.}$$

$$x_+ = \max\{x, 0\}, \qquad x_- = \min\{x, 0\}$$

Proof. Recall that $(\boldsymbol{X}(t) : t \geq 0)$, where $\boldsymbol{X}(t) \coloneqq (\boldsymbol{a}(t); \boldsymbol{b}(t)) = (a_1(t), ..., a_K(t); b_1(t), ..., b_K(t))$, is the $2K$-dimensional Markovian model of the LOB.

Let $\boldsymbol{x} = (\boldsymbol{a}; \boldsymbol{b}) = (a_1, ..., a_K; b_1, ..., b_K)$ where $a_i \in q\mathbb{Z}_+$ and $b_i \in q\mathbb{Z}_-$ (to be consistent with our construction of the order book where we have that each order contains exactly q shares).

By assumption, our Markov process is time-homogeneous. Hence, Theorem 4.2.1 tells us that the family $(\Phi_t(f(\boldsymbol{x})) : t \geq 0)$, where $\Phi_t(f(\boldsymbol{x})) = \mathbb{E}(f(\boldsymbol{X}(t))|\boldsymbol{X}(0) = \boldsymbol{x})$, forms a Markov semigroup. This, together with the fact that our function f is sufficiently regular, allows us to compute the infinitesimal generator of the process (recall Definition 4.1.7):

$$\mathcal{L}f(\boldsymbol{x}) = \mathcal{L}f(\boldsymbol{a}; \boldsymbol{b}) = \lim_{t \to 0+} \frac{\mathbb{E}(f(\boldsymbol{X}(t))|\boldsymbol{X}(0) = \boldsymbol{x}) - f(\boldsymbol{x})}{t} = \lim_{t \to 0+} \frac{\mathbb{E}(f(\boldsymbol{X}(t)) - f(\boldsymbol{x})|\boldsymbol{X}(0) = \boldsymbol{x})}{t} \tag{38}$$

To ease our notations further, we'll introduce some further shorthand notations:

$$\mathbb{E}(f(\boldsymbol{X}(t)) - f(\boldsymbol{x})|\boldsymbol{X}(0) = \boldsymbol{x}) \equiv \mathbb{E}_{\boldsymbol{x}}(f(\boldsymbol{X}(t) - f(\boldsymbol{x})))$$

$$\Delta f(t) \equiv f(\boldsymbol{X}(t)) - f(\boldsymbol{x}).$$

And from now on, for simplicity and for the sake of this proof, we'll denote $f \coloneqq f(\boldsymbol{x}) = f(\boldsymbol{a}; \boldsymbol{b}) = f(a_1, ..., a_K; b_1, ..., b_K)$.

We'll first compute $\mathbb{E}_{\boldsymbol{x}}(f(\boldsymbol{X}(t) - f(\boldsymbol{x})))$. We will do this by conditioning on every type of event that can occur in the order book (given in Section 3) and multiplying by the probability of such an event occurring (all these probabilities were given in Section 3), and then finally applying the law of total expectation.

[1] By sufficiently regular we mean that they are in the linear subspace $D_{\mathcal{L}}$ from Definition 4.1.7

So, we have

$$\mathbb{E}_{\boldsymbol{x}}(f(X(t) - f(\boldsymbol{x}))) = \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dM^+(t))\mathsf{P}(\text{"market buy order at time t"}) \tag{39}$$

$$+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dL_i^+(t))\mathsf{P}(\text{"limit sell order at level i at time t"}) \tag{40}$$

$$+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dC_i^+(t))\mathsf{P}(\text{"cancellation of a sell limit order at level i at time t"}) \tag{41}$$

$$+ \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dM^-(t))\mathsf{P}(\text{"market sell order at time t"}) \tag{42}$$

$$+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dL_i^-(t))\mathsf{P}(\text{"limit buy order at level i at time t"}) \tag{43}$$

$$+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|dC_i^-(t))\mathsf{P}(\text{"cancellation of a buy limit order at level i at time t"}) \tag{44}$$

Now, to compute the RHS of (39), we consider what happens to the order book when a market buy order is submitted. From the dynamics of the order book described in Section 3 by the SDEs (1)-(3) and (4)-(6), we get, from the first term of the RHS of (1), that the random variable $\boldsymbol{a}(t)$ from $\boldsymbol{X}(t) := (\boldsymbol{a}(t); \boldsymbol{b}(t))$ is affected in the following way:

From $\boldsymbol{a}(t)$, it becomes:

$$\Big([a_1(t) - q]_+, [a_2(t) - (q - A_1(t))_+]_+, [a_3(t) - (q - A_2(t))_+]_+, ..., [a_K(t) - (q - A_{K-1}(t))_+]_+\Big),$$

which, by the shorthand notation we've introduced, is equal to $[a_i(t) - (q - A_{i-1}(t))_+]_+^{i=1,...,K}$ (where we set $A_0(t) \equiv 0$).

And, by using the J (shift operator) notation introduced in Section 3, we get that the random variable $\boldsymbol{b}(t)$ from $\boldsymbol{X}(t) := (\boldsymbol{a}(t); \boldsymbol{b}(t))$ is affected in the following way:

From $\boldsymbol{b}(t)$, it becomes $J^{M^+}(\boldsymbol{b}(t))$.

As for the P("market buy order at time t") term, this is just (7) with $k = 1$, i.e.

$$\mathsf{P}(\text{"market buy order at time t"}) = (\lambda^{M^+}t)^1 \frac{e^{-\lambda^{M^+}t}}{1!} = \lambda^{M^+}te^{-\lambda^{M^+}t}.$$

And so, we have the RHS of (39) being equal to

$$(f([a_i(t) - (q - A_{i-1}(t))_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b}(t))) - f) \cdot \lambda^{M^+} t e^{-\lambda^{M^+} t}. \tag{45}$$

And, by the exact same reasoning, now making use of the first term from the RHS of (4), and (8) with $k = 1$ (and with $B_0(t) \equiv 0$), we can compute the term from (42):

$$(f(J^{M^-}(\boldsymbol{a}(t); [b_i(t) + (q - B_{i-1}(t))_+]_-^{i=1,\dots,K}) - f) \cdot \lambda^{M^-} t e^{-\lambda^{M^-} t}. \tag{46}$$

By the same reasoning, we also have:

The summands from (40) being equal to

$$(f(a_i(t) + q; J^{L_i^+}(\boldsymbol{b}(t))) - f) \cdot \lambda_i^{L^+} t e^{-\lambda_i^{L^+} t}, \qquad i \in 1, ..., K \tag{47}$$

(by making use of the second term from the RHS of (1), and (9) with $k = 1$).

The summands from (41) being equal to

$$(f(a_i(t) - q; J^{C_i^+}(\boldsymbol{b}(t))) - f) \cdot a_i(t) \lambda_i^{C^+} t e^{-a_i(t) \lambda_i^{C^+} t}, \qquad i \in \{1, ..., K\} \tag{48}$$

(by making use of the third term from the RHS of (1), and (11) with $k = 1$).

The summands from (43) being equal to

$$(f(J^{L_i^-}(\boldsymbol{a}(t)); b_i - q) - f) \cdot \lambda_i^{L^-} t e^{-\lambda_i^{L^-} t}, \qquad i \in \{1, ..., K\} \tag{49}$$

(by making use of the second term from the RHS of (4), and (10) with $k = 1$).

The summands from (44) being equal to

$$(f(J^{C_i^-}(\boldsymbol{a}(t)); b_i + q) - f) \cdot |b_i(t)| \lambda_i^{C^-} t e^{-|b_i(t)| \lambda_i^{C^-} t}, \qquad i \in \{1, ..., K\} \tag{50}$$

(by making use of the third term from the RHS of (4), and (12) with $k = 1$).

So, by summing up all the terms from (45)-(50) for every $i \in \{1, ..., K\}$, we get $\mathbb{E}_{\boldsymbol{x}}(f(\boldsymbol{X}(t) - f(\boldsymbol{x})))$ from the LHS of (39), i.e.,

$$\mathbb{E}_{\boldsymbol{x}}(f(\boldsymbol{X}(t) - f(\boldsymbol{x}))) = (f([a_i(t) - (q - A_{i-1}(t))_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b}(t))) - f) \cdot \lambda^{M^+} t e^{-\lambda^{M^+} t}$$
$$+ \sum_{i=1}^{K} (f(a_i(t) + q; J^{L_i^+}(\boldsymbol{b}(t))) - f) \cdot \lambda_i^{L^+} t e^{-\lambda_i^{L^+} t}$$

$$+\sum_{i=1}^{K}(f(a_i(t)-q;J^{C_i^+}(\boldsymbol{b}(t)))-f)\cdot a_i(t)\lambda_i^{C^+}te^{-a_i(t)\lambda_i^{C^+}t}$$
$$+(f(J^{M^-}(\boldsymbol{a}(t);[b_i(t)+(q-B_{i-1}(t))_+]_-^{i=1,\dots,K})-f)\cdot\lambda^{M^-}te^{-\lambda^{M^-}t}$$
$$+\sum_{i=1}^{K}(f(J^{L_i^-}(\boldsymbol{a}(t));b_i-q)-f)\cdot\lambda_i^{L^-}te^{-\lambda_i^{L^-}t}$$
$$+\sum_{i=1}^{K}(f(J^{C_i^-}(\boldsymbol{a}(t));b_i+q)-f)\cdot|b_i(t)|\lambda_i^{C^-}te^{-|b_i(t)|\lambda_i^{C^-}t}.$$

And now, to obtain our infinitesimal generator (38), all that remains is to divide the above expression by t and then take its limit as $t\to 0+$. As a result of this, all the t's from above disappear and all the exponential terms will tend to 1. Moreover, as $t\to 0+$,

$$\boldsymbol{a}(t)=(a_1(t),...,a_K(t))\longrightarrow\boldsymbol{a}(0)=\boldsymbol{a}=(a_1,...,a_K)$$

and

$$\boldsymbol{b}(t)=(b_1(t),...,b_K(t))\longrightarrow\boldsymbol{b}(0)=\boldsymbol{b}=(b_1,...,b_K),$$

where we have $\boldsymbol{a}(0)=\boldsymbol{a}$ and $\boldsymbol{b}(0)=\boldsymbol{b}$, because the above expectation is conditioned on $\boldsymbol{X}(0)=(\boldsymbol{a}(0);\boldsymbol{b}(0))=\boldsymbol{x}=(\boldsymbol{a};\boldsymbol{b})$.

It is then easy to see that the final expression we get is precisely the expression (32)-(37) as required. □

We can also derive the transition operator of the embedded Markov chain associated with $(\boldsymbol{X}(t):t\geq 0)$ that we had introduced in Section 3, $(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$.

Proposition 4.3.2. *The transition operator of $(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$ is the operator $\mathcal{D}$ defined by:*

$$\mathcal{D}f(\boldsymbol{a};\boldsymbol{b})=\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\cdot\mathcal{L}f(\boldsymbol{a};\boldsymbol{b}),\tag{51}$$

for sufficiently regular functions $f:\mathbb{Z}^{2K}\to\mathbb{R}$, where, recall the notations from (19) and (20), $\Lambda(\boldsymbol{a};\boldsymbol{b}):=\lambda^{M^+}+\lambda^{M^-}+\Lambda^{L^+}+\Lambda^{L^-}+\Lambda^{C^+}+\Lambda^{C^-}$.

Proof. Let $\boldsymbol{a}=(a_1,...,a_K)\in q\mathbb{Z}_+^K$ and $\boldsymbol{b}=(b_1,...,b_K)\in q\mathbb{Z}_-^K$ and let $\boldsymbol{x}=(\boldsymbol{a};\boldsymbol{b})$.

The transition operator $\mathcal{D}$ is computed by:

$$\mathbb{E}(f(\boldsymbol{Z}_n)|\boldsymbol{Z}_{n-1}=\boldsymbol{x})-f(\boldsymbol{x})=\mathbb{E}(f(\boldsymbol{Z}_n)-f(\boldsymbol{x})|\boldsymbol{Z}_{n-1}=\boldsymbol{x}).$$

We again introduce the shorthand notation $\Delta f(t) \equiv f(\boldsymbol{Z}_n) - f(\boldsymbol{x})$. By the same reasoning as in the previous proof, we have

$$\begin{aligned}
\mathbb{E}(f(\boldsymbol{Z}_n) - f(\boldsymbol{x})|\boldsymbol{Z}_{n-1} = \boldsymbol{x}) = \mathbb{E}_{\boldsymbol{x}}&(\Delta f(t)|\text{"Buy market order at time n"}) \cdot p^{M^+} \\
&+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|\text{"Sell limit order at level i at time n"}) \cdot p^{L_i^+} \\
&+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|\text{"Cancellation of a sell limit order at level i at time n"}) \cdot p^{C_i^+} \\
&+ \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|\text{"Sell market order at time n"}) \cdot p^{M^-} \\
&+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|\text{"Buy limit order at level i at time n"}) \cdot p^{L_i^-} \\
&+ \sum_{i=1}^{K} \mathbb{E}_{\boldsymbol{x}}(\Delta f(t)|\text{"Cancellation of a buy limit order at level i at time n"}) \cdot p^{C_i^-},
\end{aligned}$$

where we use the notations and expressions from (13)-(20). This, together with the fact that the expectations on the LHS of the above equation have already been found in the proof of Proposition 4.3.1 (they are, respectively, given just on the LHS of each of the products from (45)-(50)), yields the following equation:

$$\begin{aligned}
\mathbb{E}(f(\boldsymbol{Z}_n) - f(\boldsymbol{x})|\boldsymbol{Z}_{n-1} = \boldsymbol{x}) = (f(&[a_i(t) - (q - A_{i-1}(t))_+]_+^{i=1,\ldots,K}; J^{M^+}(\boldsymbol{b}(t))) - f) \cdot \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{x})} \\
&+ \sum_{i=1}^{K} (f(a_i(t) + q; J^{L_i^+}(\boldsymbol{b}(t))) - f) \cdot \frac{\lambda_i^{L^+}}{\Lambda(\boldsymbol{x})} \\
&+ \sum_{i=1}^{K} (f(a_i(t) - q; J^{C_i^+}(\boldsymbol{b}(t))) - f) \cdot \frac{a_i \lambda_i^{C^+}}{\Lambda(\boldsymbol{x})} \\
&+ (f(J^{M^-}(\boldsymbol{a}(t); [b_i(t) + (q - B_{i-1}(t))_+]_-^{i=1,\ldots,K}) - f) \cdot \frac{\lambda^{M^-}}{\Lambda(\boldsymbol{x})} \\
&+ \sum_{i=1}^{K} (f(J^{L_i^-}(\boldsymbol{a}(t)); b_i - q) - f) \cdot \frac{\lambda_i^{L^-}}{\Lambda(\boldsymbol{x})} \\
&+ \sum_{i=1}^{K} (f(J^{C_i^-}(\boldsymbol{a}(t)); b_i + q) - f) \cdot \frac{|b_i| \lambda_i^{C^-}}{\Lambda(\boldsymbol{x})}.
\end{aligned}$$

Notice that the RHS of the above equation is precisely $\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})} \cdot \mathcal{L}f(\boldsymbol{a};\boldsymbol{b})$, as required. □

5 Stability and Ergodicity of LOB

This section will contain the most important results of the dissertation that all the previous sections have been building up towards. That is, the proofs for the ergodicity of our Markovian models of LOB.

5.1 Importance and Some Definitions

Firstly, what is "*stability*"? We shall follow [17] and [18] in this and the next subsection to build up towards the notion of ergodicity of a Markov process.

One way to explain "stability" is by making use of the notions of *communication* and *recurrence* of a stochastic process.

Let $(\Phi_n)_{n\geq 0}$ be a Markov process on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with state space $(S, \mathscr{S})$. Let $\tau_A := \inf\{n \geq 0 : \Phi_n \in A\}$, $A \in \mathscr{S}$, be the first hitting time of the process to the set A.

In one sense, the least restrictive form of stability we might require is *irreducibility*. That is, regardless of our starting point, we are always able to reach the same collection of sets (from $\mathscr{S}$). To this end, we introduce the notion of ϕ-irreducibility here in the uncountable/measure-theoretic setting:

Definition 5.1.1. *ϕ-irreducibility* for $(\Phi_n)_{n\geq 0}$ means that there exists a measure ϕ on $(S, \mathscr{S})$ with the property that for every starting point $x \in S$ of the process,

$$\phi(A) > 0 \implies \mathsf{P}_x(\tau_A < \infty) > 0.$$

The intuitive meaning of Definition 5.1.1 is that 'all reasonably sized sets, as "measured" by ϕ, can be reached from every possible starting point of the process $x \in S$'.

Now, a stronger notion of stability would be for the Markov process to actually be **guaranteed** to eventually reach like states from unlike starting points. To this end we introduce the notion of *recurrence* in the uncountable/measure-theoretic setting:

Definition 5.1.2. A Markov process $(\Phi_n)_{n\geq 0}$ is said to be *ϕ-recurrent* if there exists a measure ϕ on $(S, \mathscr{S})$ such that for all starting points $x \in S$,

$$\phi(A) > 0 \implies \mathsf{P}_x(\tau_A < \infty) = 1.$$

Comment 5.1.1. A further strengthening of the above notion is to require for every starting point $x \in S$,

$$\phi(A) > 0 \implies \mathbb{E}_x(\tau_A) < \infty.$$

This condition is stronger than the one in Definition 5.1.2 as it alludes a faster recurrence of the chain to a particular state.

We would like to establish a more "long term" version of stability in terms of the convergence of the distributions of the process as time progresses. This is where the notion of *ergodicity* and limiting behaviour of a process come in the discussion. Ergodicity presents a much stronger notion of stability than all the concepts we've explored thus far in this subsection: there is an 'invariant regime' described by a measure π on $(S, \mathscr{S})$ such that if the process starts in this regime (that is, Φ_0 has distribution π) then it shall remain in this regime, and in the case where the process starts in some other 'regime' (i.e. Φ_0 doesn't have distribution π), then it converges in a strong probabilistic sense, to π. In the next subsection we will make these notions more mathematically precise.

5.2 Ergodic Markov processes

In this subsection, we will build up to an equivalent definition of ergodicity which will ultimately allow us in the next subsection to prove the ergodicity of the Markov process modelling the LOB, $(\boldsymbol{X}(t) : t \geq 0)$, as well as that of the embedded Markov chain (in event-time), $(\boldsymbol{Z}_t)_{t \in \mathbb{N}_0}$.

Our Markov process in this project has a countable state space. The results in this subsection, although presented in a measure-theoretic setting, of course are applicable in the countable state space setting as well.

Let $(\Phi_n)_{n \in \mathbb{N}}$ be our Markov chain on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with state space $(S, \mathscr{S})$. We now set $\Omega \coloneqq S^\infty$, $\mathcal{F}$ to be the product σ-algebra on Ω and $\mathscr{S}$ to be the set of all subsets of S. Let's also introduce the *one-step transition probability*

$$\mathsf{P}(x, A) = \mathsf{P}(\Phi_n \in A | \Phi_{n-1} = x) \text{ for } x \in S \text{ and } A \in \mathscr{S}.$$

For a given initial probability measure μ on $\mathscr{S}$, we construct for our Markov chain $(\Phi_n)_{n \in \mathbb{N}}$ to have an initial distribution P_μ such that for $x_0 \in S$, $\mathsf{P}_\mu(\Phi_0 = x_0) = \mu(x_0)$ and for $A \in \mathcal{F}$,

$$\mathsf{P}_\mu(\boldsymbol{\Phi} \in A | \Phi_0 = x_0) = \mathsf{P}_\mu((\Phi_0, \Phi_1, \Phi_2, ...) \in A | \Phi_0 = x_0) = \mathsf{P}_{x_0}(\boldsymbol{\Phi} \in A),$$

where P_{x_0} is the probability distribution on $\mathcal{F}$ which is obtained when the initial distribution is the point mass δ_{x_0} at x_0 (because under the condition that the process starts at x_0, we must not have our Φ_0 to be anything other than x_0).

The n-th step transition probability is given by

$$\mathsf{P}_\mu(\Phi_n = y|\Phi_0 = x) = \mathsf{P}^n(x, y)$$

for $x, y \in S$.

We say that the state $x \in S$ *leads* to state $y \in S$ ($x \to y$ for short) if there exists an $n(x, y) \in \mathbb{N}$ such that $\mathsf{P}^{n(x,y)}(x, y) > 0$.

For $A \in \mathscr{S}$, the random variable

$$\eta_A \coloneqq \sum_{i=1}^{\infty} \mathbb{1}_{\{\Phi_n \in A\}}$$

counts the number of visits of the chain to a set A.

Definition 5.2.1. We say a set $A \in \mathscr{S}$ is *uniformly transient* if there exists an $M < \infty$ such that $\mathbb{E}_x(\eta_A) \leq M$ for all $x \in A$.

A set $A \in \mathscr{S}$ is called *recurrent* if $\mathbb{E}_x(\eta_A) = \infty$ for all $x \in A$.

State $\alpha \in S$ is called *recurrent* if $\mathbb{E}_\alpha(\eta_\alpha) = \infty$ and *transient* if $\mathbb{E}_\alpha(\eta_\alpha) < \infty$.

Definition 5.2.2. A σ-finite measure π on $\mathscr{S}$ with the property

$$\pi(A) = \int_S \mathsf{P}(x, A)\pi(dx), \quad \text{for all } A \in \mathscr{S}$$

is called *invariant*.

Markov processes with the property that for all $k \geq 1$ the marginal distribution of $\{\Phi_n, ..., \Phi_{n+k}\}$ doesn't change as n varies, are called *stationary processes*.

Comment 5.2.1. It's clear that a Markov chain/process will not, in general, be stationary, since, in a particular realization we may have that $\Phi_0 = x$ with probability 1 for some fixed $x \in S$. However, it is possible that with an appropriate choice of initial distribution for Φ_0 we may produce a stationary process $(\Phi_n)_{n\in\mathbb{N}}$.

Indeed, let π be an initial invariant probability measure i.e. $\pi(A) = \int_S \pi(dw)\mathsf{P}(w, A)$.

Then, by iterating

$$\begin{aligned}
\pi(A) &= \int_S (\int_S \pi(dx)\mathsf{P}(x,dw))\mathsf{P}(w,A) \\
&= \int_S \pi(dx) \int_S \mathsf{P}(x,dw)\mathsf{P}(w,A) \\
&= \int_S \pi(dx)\mathsf{P}^2(x,A) && \text{(by Chapman-Kolmogorov equation)} \\
&= ... \\
&= \int_S \pi(dx)\mathsf{P}^n(x,A) \\
&= \mathsf{P}_\pi(\Phi_n \in A) \qquad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathscr{S}.
\end{aligned}$$

And so, from the Markov property, it's clear that $(\Phi_n)_{n\in\mathbb{N}}$ is stationary if and only if the distribution of Φ_n doesn't vary with time n.

Invariant probability measures are not only important because they define stationary processes (as we saw in Comment 5.2.1 above), but they will also turn out to be the measures which define the long term/ergodic behaviour of the chain.

In the preceding text we developed ideas of stability largely in terms of recurrence structures. We were concerned with how likely it would be for the chain to return to some state or set; and whether this would happen in a finite expected/mean time or not.

Ergodicity deals with a more important concept of stability: the concept of the chain "settling down", converging, to a stable/stationary regime.

We are thus interested in the question of, given the existence of an invariant measure π on $\mathscr{S}$ (without going into a detailed discussion into its existence and even uniqueness, which is a very involved discussion, to which we direct the reader to [17]), when do the n-step transition probabilities converge, in a suitable way, to π?

We consider the *signed measure* (a generalization to the concept of a measure by allowing it to have negative values) $P^n(x,\cdot) - \pi(\cdot)$ on $\mathscr{S}$.

We shall consider a global type of convergence of the measure P^n to π. That is, in terms of the total variation norm:

Definition 5.2.3. Let μ be a signed measure on $\mathscr{S}$, then the total variation norm $\|\mu\|$ is given by

$$\|\mu\| := \sup_{A\in\mathscr{S}} \mu(A) - \inf_{A\in\mathscr{S}} \mu(A).$$

The key limit of interest for this dissertation will be of the form:

$$\lim_{n\to\infty} \|P^n(x,\cdot) - \pi(\cdot)\| = \lim_{n\to\infty} [\sup_{A\in\mathscr{S}} (P^n(x,A) - \pi(A)) - \inf_{A\in\mathscr{S}} (P^n(x,A) - \pi(A))] \tag{52}$$

$$= 2 \lim_{n\to\infty} \sup_{A\in\mathscr{S}} |P^n(x,A) - \pi(A)| \tag{53}$$

$$= 0. \tag{54}$$

We'll adopt the term *ergodic* for chains/processes where the above limit (52) holds as the time sequence n goes to infinity (rather than as n goes to infinity through some subsequence).

The period of a state $x \in S$ is given by

$$\gcd\{n \in \mathbb{N} : P^n(x,x) > 0\}$$

and we say that state x is *aperiodic* if the above is equal to 1 and *periodic* if it's greater than 1. Moreover, it's well-known that periodicity is a class property.

And so, a periodic Markov chain, which is irreducible (and so, as periodicity is a class property, all states are periodic), is one in which for all its states, it can only return to them at times that are multiples of some integer greater than 1.

Unfortunately, in the case when states are periodic for an irreducible Markov chain, the limit from (52) can hold at best as we go through a periodic sequence nd, where d is the period of the chain, as $n \to \infty$. Thus, we define ergodic chains to be aperiodic.

To make the concept of ergodic Markov chains/processes more mathematically precise, we give the following definition:

Definition 5.2.4. A Markov process/chain is *ergodic* if it's aperiodic and irreducible and if there exists an invariant probability measure π such that

$$\lim_{t\to\infty} \|\mathsf{P}^t(x,\cdot) - \pi(\cdot)\| = 0 \qquad \text{for all } x \in S$$

where P^t is the Markov transition probability function.

Definition 5.2.5. Let $f : S \to \mathbb{R}$ be a measurable function. We define the *f-norm* as

$$\|\nu\|_f := \sup_{g:|g|\le f} |\nu(g)| = \sup_{g:|g|\le f} |\int_S g d\nu|$$

where ν is any sign measure on $\mathscr{S}$.

Definition 5.2.6. Let P_1 and P_2 be Markov transition functions, and for a positive function $1 \leq V < \infty$, define the <u>*V-norm distance*</u> between P_1 and P_2 as

$$\|\!|P_1 - P_2|\!\|_V := \sup_{x\in S} \frac{\|P_1(x,\cdot) - P_2(x,\cdot)\|_V}{V(x)} = \sup_{x\in S} \frac{\sup_{f:|f|\leq V} |P_1(x,f) - P_2(x,f)|}{V(x)}.$$

Definition 5.2.7. Define the <u>*outer product*</u> of the function 1 and the measure π by

$$[1 \otimes \pi](x, A) = \pi(A) \quad \text{for } x \in S \text{ and } A \in \mathscr{S}.$$

Definition 5.2.8. An ergodic Markov process/chain $(\Phi_n)_{n\geq 0}$ is called <u>*V-uniformly ergodic*</u> if

$$\|\!|P^n - 1 \otimes \pi|\!\|_V = \sup_{x\in S} \frac{\|P^n(x,\cdot) - [1 \otimes \pi](x,\cdot)\|_V}{V(x)} \longrightarrow 0 \quad \text{as } n \to 0.$$

Definition 5.2.9. A set $C \in \mathscr{S}$ is called a <u>*small (or petite) set*</u> if there exists an $m > 0$ and a non-trivial measure ν_m on $\mathscr{S}$ such that for all $x \in C$ and $B \in \mathscr{S}$,

$$\mathsf{P}^m(x, B) \geq \nu_m(B).$$

We are now able to state vital results that will be used to prove the ergodicity of the order book in the next subsection.

Theorem 5.2.1. *Suppose that $(\Phi_n)_{n\geq 0}$ is a ψ-irreducible and aperiodic Markov process. Then the following are equivalent for any measurable function $V \geq 1$:*

i $(\Phi_n)_{n\geq 0}$ *is V-uniformly ergodic.*

ii There exists a petite set C and constants $\beta > 0, b < \infty$, such that the following drift condition holds

$$\mathcal{L}V(x) \leq -\beta V(x) + b\mathbb{1}_C(x), \quad x \in S \tag{55}$$

where $\mathcal{L}$ is the infinitesimal generator of $(\Phi_n)_{n\geq 0}$ (assuming it's well-defined on the function V i.e. V is in the domain of $\mathcal{L}$)

Proof. See [17]. □

We now turn to the book [18]. Theorem 6.1 on Page 536 substantially builds upon the results of [22] and [8], which allows us to conclude the statement of Theorem 7.1 on Page 537 of [18] (which is an extension of Theorem 3 (ii) of [23]).

These results are fruitful in the dissertation's endeavours as they provide an equivalent definition of V-uniform ergodicity. They are summarized below.

Proposition 5.2.1. *The following are equivalent:*

i A Markov process $(\Phi_n)_{n\geq 0}$ is V-uniformly ergodic.

ii There exists a function $V > 1$, an invariant distribution π and constants $0 < r < 1$ and $R < \infty$ such that

$$\|P^n(x,\cdot) - \pi(\cdot)\| \leq Rr^n V(x), \quad x \in S, n > 0 \tag{56}$$

where $\|\cdot\|$ is the total variation norm.

iii $V > 1$ is a coercive function that satisfies the following geometric drift condition

$$\mathcal{L}V(x) \leq -cV(x) + d \tag{57}$$

for some $c > 0$ and $d < \infty$, where $\mathcal{L}$ is the infinitesimal generator of $(\Phi_n)_{n\geq 0}$ (as shown in the proof of Theorem 8.1 in [18]).

Comment 5.2.2. We call the function V above the *"Lyapunov test function"*.

Comment 5.2.3. By comparing Definition 5.2.4 with inequality (56), V-uniform ergodicity means that the convergence of $\|P^n(x,\cdot) - \pi(\cdot)\|$ to 0 as $n \to \infty$ is exponentially fast. Thus, V-uniform ergodicity is a much stronger statement than just ergodicity.

Comment 5.2.4. The above Proposition basically tells us that V-uniform ergodicity can be characterized in terms of the infinitesimal generator of the Markov process (or the transition operator of the Markov chain, in the discrete time setting). Therefore, proving that a Markov process is V-uniformly ergodic is equivalent to showing the existence of a coercive function $V > 1$ such that the geometric drift condition (57), or, by Theorem 5.2.1, the drift condition (55) holds. As we have derived the infinitesimal generator of $(\boldsymbol{X}(t) : t \geq 0)$, and the transition operator of $(\boldsymbol{Z}_n)_{n\in\mathbb{N}}$ (recall Proposition 4.3.1 and Proposition 4.3.2), we are in a position to prove their ergodicities with the help of Proposition 5.2.1 and Theorem 5.2.1.

5.3 Ergodicity and Stability of the Order Book

An original approach, as well as a **significant** amount of detail has been given in the following proofs, compared to [1].

Theorem 5.3.1. *Under the assumption that* $\underline{\lambda_C} := \min_{1\leq i\leq K}\{\lambda_i^{C^\pm}\} < \infty$, $(\boldsymbol{X}(t))_{t\geq 0} = (\boldsymbol{a}(t);\boldsymbol{b}(t))_{t\geq 0}$ *is an ergodic Markov process. To be more precise, it is* V*-uniformly ergodic.*

Proof. Let $\boldsymbol{x} := (\boldsymbol{a};\boldsymbol{b}) = (a_1,...,a_K;b_1,...,b_K)$ where $a_i \in q\mathbb{Z}_+$ and $b_i \in q\mathbb{Z}_-$.

Define $V(\boldsymbol{a};\boldsymbol{b}) := \sum_{i=1}^K a_i + \sum_{i=1}^K |b_i| + q$. As $q > 1$, it follows that $V > 1$. Moreover, V is clearly a coercive function.

We would like to find an upper bound for $\mathcal{L}V(\boldsymbol{x})$, where $\mathcal{L}$ was found in Proposition 4.3.1. To be more explicit, we want to upper bound the following expression:

$$\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) = \lambda^{M^+}(V([a_i - (q-A_{i-1})_+]_+^{i=1,...,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \tag{58}$$

$$+ \sum_{i=1}^K \lambda_i^{L^+}(V(a_i+q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \tag{59}$$

$$+ \sum_{i=1}^K \lambda_i^{C^+} a_i(V(a_i-q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) \tag{60}$$

$$+ \lambda^{M^-}(V(J^{M^-}(\boldsymbol{a}); [b_i + (q-B_{i-1})_+]_-^{i=1,...,K}) - V(\boldsymbol{a};\boldsymbol{b})) \tag{61}$$

$$+ \sum_{i=1}^K \lambda_i^{L^-}(V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{a};\boldsymbol{b})) \tag{62}$$

$$+ \sum_{i=1}^K \lambda_i^{C^-}|b_i|(V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{a};\boldsymbol{b})). \tag{63}$$

First, let's find an upper bound for the expression on the RHS of (58), which is equal to:

$$\lambda^{M^+}\left[\sum_{i=1}^K [a_i - (q-A_{i-1})_+]_+ + \sum_{i=1}^K |J^{M^+}(b_i)| + q - (\sum_{i=1}^K a_i + \sum_{i=1}^K |b_i| + q)\right]. \tag{64}$$

What's the largest possible value the above expression can take whilst adhering to the dynamics of the order book (given by the SDEs from Section 3)? Well, we can see that this is obtained when the buy market order dM^+ does not cause a shift in the bid side of the order book, i.e., when $J^{M^+}(b_i) = b_i$ for every $i = 1,...,K$. But this happens precisely when $a_{i_S} > q$, where the buy market order wouldn't be able to "annihilate" the best ask level a_{i_S} (i.e. turn it to 0); but merely reduce it by q. And, as no "annihilation" occurs on the ask side of the order book, there is no shift on the bid side.

To further convince ourselves that the case where $a_{i_S} > q$ indeed gives us the largest possible value for the expression from (64) after a buy market order, just consider the scenario where we have $a_{i_S} = q$ instead. Then the submission of a buy market order would cause a_{i_S} to turn to 0 (recall that the buy market order interacts only with the best ask price), which in turn would shift the bid side of the order book by $A^{-1}(q) - i_S$, and so the first $A^{-1}(q) - i_S$ levels of $\boldsymbol{b}$ would turn to 0 i.e. $J^{M^+}(b_i) = 0$ for every $i = 1, 2, ..., A^{-1}(q) - i_S$. This would obviously make the expression from (64) smaller than if we had $a_{i_S} > q$ (and hence $J^{M^+}(b_i) = b_i$ for every $i = 1, ..., K$).

The above reasoning is illustrated diagrammatically, on a simplistic order book in the Appendix (see Figures 1 and 2).

Therefore, the expression from (64) is bounded above by:

$$
\begin{aligned}
(64) \leq \lambda^{M^+} & \left[\sum_{i=1}^{i_S - 1} [a_i - (q - A_{i-1})_+]_+ + (a_{i_S} - q) + \sum_{i=i_S+1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q - \left(\sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q\right) \right] \\
& \qquad \text{(as explained above, } a_{i_S} > q \text{ implies } J^{M^+}(b_i) = b_i \text{ for every } i = 1, ..., K) \\
= \lambda^{M^+} & \left[a_{i_S} - q + \sum_{i=i_S+1}^{K} [a_i - (q - A_{i-1})_+]_+ + \sum_{i=1}^{K} |b_i| + q - \left(a_{i_S} + \sum_{i=i_S+1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q\right) \right] \\
& \qquad \text{(by definition of } i_S, \ a_i = 0 \text{ for every } i = 1, ..., i_S - 1) \\
= \lambda^{M^+} & \left[\sum_{i=i_S+1}^{K} [a_i - (q - A_{i-1})_+]_+ + \sum_{i=1}^{K} |b_i| - \left(\sum_{i=i_S+1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q\right) \right] \\
= \lambda^{M^+} & \left[\sum_{i=i_S+1}^{K} a_i + \sum_{i=1}^{K} |b_i| - \left(\sum_{i=i_S+1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q\right) \right] \\
& \text{(by def. of } i_S, \ A_{i-1} \geq q \ \ \forall i = i_S + 1, ..., K \implies [a_i - (q - A_{i-1})_+]_+ = a_i \text{ for all } i = i_S + 1, ..., K) \\
= -\lambda^{M^+} & q. \qquad (65)
\end{aligned}
$$

By the same reasoning and symmetry of the order book, an upper bound for the expression from (61) is the following:

$$
(61) \leq -\lambda^{M^-} q \qquad (66)
$$

Now, let's find an upper bound for the expression in (59), which is equal to:

$$
\sum_{i=1}^{K} \lambda_i^{L^+} \left[(\cancel{a_i} + q) + \cancel{\sum_{\substack{j=1 \\ j: j \neq i}}^{K}} a_j + \sum_{j=1}^{K} |J^{L_i^+}(b_j)| + \cancel{q} - \left(\cancel{\sum_{i=1}^{K}} a_i + \sum_{i=1}^{K} |b_i| + \cancel{q}\right) \right]. \qquad (67)
$$

For $i \in \{1, ..., i_S - 1\}$, a sell limit order at level i, dL_i^+, will cause a shift in the bid side of the order book (because i is in the spread). To be precise,

$$
J^{L_i^+}(b_j) = b_{j+(i_S - i)} \qquad \text{for all } j \in \{1, ..., K - (i_S - i)\}
$$

and

$$J^{L_i^+}(b_j) = b_\infty \qquad \text{for all } j \in \{K - (i_S - i) + 1, ..., K\}.$$

For $i \in \{i_S, ..., K\}$, a sell limit order at level i, dL_i^+, will not cause a shift in the bid side of the book (because i is not in the spread) i.e. we'll have $J^{L_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$.

The above two points are illustrated diagrammatically for simplistic order books in the Appendix (see Figures 3 and 4).

Combining the above facts, we get:

$$\begin{aligned}
\sum_{i=1}^{i_S-1} \lambda_i^{L^+}(V(a_i+q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) &= \sum_{i=1}^{i_S-1} \lambda_i^{L^+}\left[q + \sum_{j=1}^{K-(i-i_S)} |b_{j+(i_S-i)}| + \sum_{j=K-(i_S-i)+1}^{K} |b_\infty| - \sum_{j=1}^{K} |b_j|\right] \\
&= \sum_{i=1}^{i_S-1} \lambda_i^{L^+}\left[q + \sum_{j=1}^{K-(i-i_S)} |b_{j+(i_S-i)}| + (i_S - i)|b_\infty| - \sum_{j=1}^{K} |b_j|\right] \\
&= \sum_{i=1}^{i_S-1} \lambda_i^{L^+}\left[q + \sum_{j=1}^{K-(i-i_S)} |b_{j+(i_S-i)}| + (i_S - i)|b_\infty| - \sum_{j=1}^{i_S-i} |b_j| \right. \\
&\qquad \left. - \sum_{j=1}^{K-(i-i_S)} |b_{j+(i_S-i)}| - \sum_{j=K-(i_S-i)+1}^{K} |b_j|\right] \\
&= \sum_{i=1}^{i_S-1} \lambda_i^{L^+}\left[q + (i_S - i)|b_\infty| - \sum_{j=1}^{i_S-i} |b_j| - \sum_{j=K-(i_S-i)+1}^{K} |b_j|\right] \\
&\leq \sum_{i=1}^{i_S-1} \lambda_i^{L^+} q + \sum_{i=1}^{i_S-1} \lambda_i^{L^+}(i_S - i)|b_\infty| \qquad (68)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=i_S}^{K} \lambda_i^{L^+}(V(a_i+q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a};\boldsymbol{b})) &= \sum_{i=i_S}^{K} \lambda_i^{L^+}\left[q + \sum_{j=1}^{K} |b_j| - \sum_{j=1}^{K} |b_j|\right] \\
&= \sum_{i=i_S}^{K} \lambda_i^{L^+} q. \qquad (69)
\end{aligned}$$

So by combining the derived inequality from (68), and equation from (69), and applying them to the expression from (67), we get the following inequality for the expression from (59):

$$
\begin{aligned}
(59) = \sum_{i=1}^{K} \lambda_i^{L^+} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{a}; \boldsymbol{b})) &\leq \sum_{i=1}^{i_S - 1} \lambda_i^{L^+} q + \sum_{i=1}^{i_S - 1} \lambda_i^{L^+} (i_S - i)|b_\infty| + \sum_{i=i_S}^{K} \lambda_i^{L^+} q \\
&= \sum_{i=1}^{K} \lambda_i^{L^+} q + \sum_{i=1}^{i_S - 1} \lambda_i^{L^+} (i_S - i)|b_\infty| \\
&= \sum_{i=1}^{K} \lambda_i^{L^+} q + \sum_{i=1}^{K} \lambda_i^{L^+} (i_S - i)_+ |b_\infty|. \qquad (70)
\end{aligned}
$$

By the same reasoning and symmetry of the order book, an upper bound for the expression from (62) is the following:

$$
(62) \leq \sum_{i=1}^{K} \lambda_i^{L^-} q + \sum_{i=1}^{K} \lambda_i^{L^-} (i_S - i)_+ a_\infty. \qquad (71)
$$

Now, we want to find an upper bound for the expression in (60), which is equal to:

$$
\sum_{i=1}^{K} \lambda_i^{C^+} a_i \Bigg[(\cancel{a_i} - q) + \cancel{\sum_{\substack{j=1 \\ j: j \neq i}}^{K} a_j} + \sum_{j=1}^{K} J^{C_i^+}(b_j) + \cancel{q} - (\cancel{\sum_{j=1}^{K} a_i} + \sum_{i=1}^{K} |b_i| + \cancel{q}) \Bigg]. \qquad (72)
$$

Let $i \in \{1, ..., K\}$ and consider

$$
\lambda_i^{C^+} a_i \Bigg[-q + \sum_{j=1}^{K} J^{C_i^+}(b_j) - \sum_{i=1}^{K} |b_i| \Bigg]. \qquad (73)
$$

When $i = i_S$, the largest possible value that the expression from (73) can take follows the exact same reasoning as the way we obtained our upper bound for the expression from (64) earlier. That is, the largest possible value that the above expression can take occurs when the sell cancellation limit order at level i, dC_i^+, doesn't cause a shift in the bid side of the order book. This occurs precisely when $a_i > q$, where we have that $J^{C_i^+}(b_j) = b_j$ for all $j \in \{1, ..., K\}$. For a diagrammatic illustration see Figure 5 in the Appendix.

Therefore, when $i = i_S$, we have the expression from (73) bounded above by:

$$
\begin{aligned}
(73) &\leq \lambda_i^{C^+} a_i \Bigg[-q + \sum_{j=1}^{K} |b_j| - \sum_{i=1}^{K} |b_i| \Bigg] \\
&= -\lambda_i^{C^+} a_i q. \qquad (74)
\end{aligned}
$$

Now, as for the case when $i \in \{1, ..., i_S - 1\}$, by the definition of i_S, $a_i = 0$. This means that no sell cancellation order can be submitted at level i (since no order even exists at that level!). This leaves the entire order book completely unchanged. It follows, that for such an i, the expression from (73) is equal to 0:

$$\text{The expression from (73) equals 0.} \tag{75}$$

When $i \in \{i_S + 1, ..., K\}$, no shift is caused in the bid side of the order book as a result of dC_i^+ i.e. we have $J^{C_i^+}(b_j) = b_j$ for every $j \in \{1, ..., K\}$. This is because the cancellation does not interfere with any sell limit order at level i_S (for an illustration, see Figure 6 in the Appendix). Thus, under the assumption that $a_i \geq q$ (when a cancellation dC_i^+ is actually possible!), we get the expression from (73) being equal to:

$$\begin{aligned}(73) &= \lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} |b_j| - \sum_{i=1}^{K} |b_i| \right] \\ &= -\lambda_i^{C^+} a_i q.\end{aligned} \tag{76}$$

Otherwise, if $a_i = 0$, then a cancellation order, dC_i^+, is not even possible and hence, the expression from (73) is equal to 0:

$$\text{The expression from (73) equals 0.} \tag{77}$$

So by combining the equations and inequalities from (74)-(77) (which cover all the possibilities $i \in \{1, ..., K\}$ for the expression from (73)), and applying them to the expression from (72), we get:

$$\begin{aligned}(60) &= \sum_{i=1}^{K} \lambda_i^{C^+} a_i \left[-q + \sum_{j=1}^{K} J^{C_i^+}(b_j) - \sum_{i=1}^{K} |b_i| \right] \\ &\leq -\sum_{i=1}^{K} \lambda_i^{C^+} a_i q.\end{aligned} \tag{78}$$

By the same reasoning and symmetry of the order book, an upper bound for the expression from (63) is the following:

$$(63) \leq -\sum_{i=1}^{K} \lambda_i^{C^-} |b_i| q. \tag{79}$$

Finally, we have obtained upper bounds for all the terms (58)-(63). These upper bounds were explicitly given in (65), (66), (70), (71), (78), (79). Now, by combining these upper bounds, we get an upper bound for $\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b})$:

$$\mathcal{L}V(\boldsymbol{a}; \boldsymbol{b}) \leq -(\lambda^{M^+} + \lambda^{M^-})q + \sum_{i=1}^{K} (\lambda_i^{L^+} + \lambda_i^{L^-})q - \sum_{i=1}^{K} (\lambda_i^{C^+} a_i + \lambda_i^{C^+} |b_i|)q$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&= -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \sum_{i=1}^{K}(\lambda_i^{C^+} a_i + \lambda_i^{C^+}|b_i|)q \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&\qquad\qquad \text{(Recall the notations from Section 4.3)} \\
&\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \sum_{i=1}^{K}(\underline{\lambda_C} a_i + \underline{\lambda_C}|b_i|)q \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&\qquad\qquad \text{(by our assumption that } \underline{\lambda_C} < \infty) \\
&= -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda_C} q(\sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i| + q - q) \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&= -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda_C} q(V(\boldsymbol{a};\boldsymbol{b}) - q) \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-}(i_s - i)_+ a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+}(i_s - i)_+ |b_\infty| \\
&\leq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q - \underline{\lambda_C} q(V(\boldsymbol{a};\boldsymbol{b}) - q) \\
&\quad + \sum_{i=1}^{K} \lambda_i^{L^-} K a_\infty + \sum_{i=1}^{K} \lambda_i^{L^+} K |b_\infty| \\
&\text{(boundary conditions from Section 3.1} \implies i_S \leq K+1 \implies (i_S - i)_+ \leq K \;\; \forall i \in \{1, ..., K\}) \\
&= -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q + \underline{\lambda_C} q^2 - \underline{\lambda_C} q V(\boldsymbol{a};\boldsymbol{b}) \\
&\quad + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+}|b_\infty|).
\end{aligned}
$$

Finally, let's set

$$c \coloneqq \underline{\lambda_C} q > 0$$

and

$$d \coloneqq -(\lambda^{M^+} + \lambda^{M^-})q + (\Lambda^{L^-} + \Lambda^{L^+})q + \underline{\lambda_C} q^2 + K(\Lambda^{L^-} a_\infty + \Lambda^{L^+}|b_\infty|) < \infty.$$

Therefore, we have $\mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) \leq -cV(\boldsymbol{a};\boldsymbol{b}) + d$ with $c > 0$ and $d < \infty$. And so, the geometric drift condition (57) is satisfied. Thus, by Proposition 5.2.1, $(\boldsymbol{X}(t) : t \geq 0)$ is V-uniformly ergodic and hence converges to its stationary state exponentially fast. □

Let's now show that $(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$ is also ergodic.

Theorem 5.3.2. *$(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$ is an ergodic Markov chain. To be more precise, it's V-uniformly ergodic.*

Proof. Let $z > 1$ be fixed.

Define the function

$$V(\boldsymbol{a};\boldsymbol{b}) \coloneqq z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i|}.$$

Clearly $V \geq 1$ and V is coercive.

We would like to find an upper bound for $\mathcal{D}V(\boldsymbol{x})$, where $\mathcal{D}$ was found in Proposition 4.3.2. To be more precise, we want to upper bound the following expression:

$$\begin{aligned}
\mathcal{D}V(\boldsymbol{a};\boldsymbol{b}) &= \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})} \cdot \mathcal{L}V(\boldsymbol{a};\boldsymbol{b}) && (80)\\
&= (V([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b})) - V(\boldsymbol{x})) \cdot \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{x})} && (81)\\
&+ \sum_{i=1}^{K} (V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{x})) \cdot \frac{\lambda_i^{L^+}}{\Lambda(\boldsymbol{x})} && (82)\\
&+ \sum_{i=1}^{K} (V(a_i - q; J^{C_i^+}(\boldsymbol{b})) - V(\boldsymbol{x})) \cdot \frac{a_i \lambda_i^{C^+}}{\Lambda(\boldsymbol{x})} && (83)\\
&+ (V(J^{M^-}(\boldsymbol{a}; [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}) - V(\boldsymbol{x})) \cdot \frac{\lambda^{M^-}}{\Lambda(\boldsymbol{x})} && (84)\\
&+ \sum_{i=1}^{K} (V(J^{L_i^-}(\boldsymbol{a}); b_i - q) - V(\boldsymbol{x})) \cdot \frac{\lambda_i^{L^-}}{\Lambda(\boldsymbol{x})} && (85)\\
&+ \sum_{i=1}^{K} (V(J^{C_i^-}(\boldsymbol{a}); b_i + q) - V(\boldsymbol{x})) \cdot \frac{|b_i| \lambda_i^{C^-}}{\Lambda(\boldsymbol{x})}. && (86)
\end{aligned}$$

First, let's find an upper bound for the term from (81), which is equal to:

$$\frac{\lambda^{M^+}}{\Lambda(\boldsymbol{a};\boldsymbol{b})} \left[z^{\sum_{i=1}^{K} [a_i - (q - A_{i-1})_+]_+ + \sum_{i=1}^{K} |J^{M^+}(b_i)|} - z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i|} \right]. \quad (87)$$

For the same reasons as in the proof of Theorem 5.3.1, the largest possible value of the above expression is when we have $a_{i_S} > q$.

And as we've seen, when we have $a_{i_S} > q$, we get that $J^{M^+}(b_i) = b_i$ for every $i \in \{1, ..., K\}$, $[a_i - (q - A_{i-1})_+]_+ = a_i$ for every $i \in \{i_S + 1, ..., K\}$ and also, $a_i = 0$ for every $i \in \{1, ..., i_S - 1\}$. This implies that the expression from (87) is upper-bounded by:

$$(87) \leq \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\left[z^{\sum_{\substack{i=1\\ i\neq i_S}}^{K} a_i + (a_{i_S} - q) + \sum_{i=1}^{K}|b_i|} - z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i|}\right]$$
$$= \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\left[z^{\sum_{i=1}^{K} a_i - q + \sum_{i=1}^{K}|b_i|} - z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i|}\right]. \tag{88}$$

By symmetry of the order book, we also have that the expression from (84) is bounded above by:

$$(84) \leq \frac{\lambda^{M^-}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\left[z^{\sum_{i=1}^{K} a_i - q + \sum_{i=1}^{K}|b_i|} - z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i|}\right]. \tag{89}$$

Now let's find an upper bound for the expression from (82), which is equal to

$$\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^+}\left[z^{\sum_{\substack{j=1\\ j\neq i}}^{K} a_j + (a_i + q) + \sum_{j=1}^{K}|J^{L_i^+}(b_j)|} - z^{\sum_{j=1}^{K} a_j + \sum_{j=1}^{K}|b_j|}\right]. \tag{90}$$

As explained in the proof of Theorem 5.3.1, for $i \in \{1, ..., i_S - 1\}$,

$$J^{L_i^+}(b_j) = b_{j+(i_S - i)} \quad \forall j \in \{1, ..., K - (i_S - i)\}$$

and

$$J^{L_i^+}(b_j) = b_\infty \quad \forall j \in \{K - (i_S - i) + 1, ..., K\}.$$

And for $i \in \{i_S, ..., K\}$, $J^{L_i^+}(b_j) = b_j$ for every $j \in \{1, ..., K\}$.

And so by combining the above facts, we get:

$$\sum_{i=1}^{i_S - 1}(V(a_i + q; J^{L_i^+}(\boldsymbol{b})) - V(\boldsymbol{x}))\cdot\frac{\lambda_i^{L^+}}{\Lambda(\boldsymbol{x})} = \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{i_S - 1}\lambda_i^{L^+}\left[z^{\sum_{j=1}^{K} a_j + q + \sum_{j=1}^{K-(i_S - i)}|b_{j+(i_S - i)}| + \sum_{j=K-(i_S - i)+1}^{K}|b_\infty|} - z^{\sum_{j=1}^{K} a_j + \sum_{j=1}^{K}|b_j|}\right]$$
$$\leq \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{i_S - 1}\lambda_i^{L^+}\left[z^{\sum_{j=1}^{K} a_j + q + \sum_{j=1}^{K}|b_j| + (i_S - i)|b_\infty|} - z^{\sum_{j=1}^{K} a_j + \sum_{j=1}^{K}|b_j|}\right] \tag{91}$$

and

$$\sum_{i=i_S}^{K}(V(a_i+q;J^{L_i^+}(\boldsymbol{b}))-V(\boldsymbol{x}))\cdot\frac{\lambda_i^{L^+}}{\Lambda(\boldsymbol{x})}=\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{i_S-1}\lambda_i^{L^+}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]. \tag{92}$$

So, by combining the inequality from (91), and the equation from (92), and applying them to (90), we get the following upper bound for the expression from (82):

$$\begin{aligned}(82)\le{}&\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{i_S-1}\lambda_i^{L^+}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|+(i_S-i)|b_\infty|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]\\&+\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{i_S-1}\lambda_i^{L^+}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]\\={}&\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^+}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|+(i_S-i)_+|b_\infty|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]\\\le{}&\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^+}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|+K|b_\infty|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]\qquad\Big((i_S-i)_+\le K\Big)\end{aligned} \tag{93}$$

By symmetry of the order book, we also get an upper bound for the expression from (85):

$$(85)\le\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^-}\left[z^{\sum_{j=1}^K a_j+q+\sum_{j=1}^K|b_j|+Ka_\infty}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]. \tag{94}$$

Now, the expression from (83) is equal to:

$$\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{C^+}a_i\left[z^{\sum_{j=1}^K a_j-q+\sum_{j=1}^K|b_j|}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right], \tag{95}$$

since by definition of a cancellation order dC_i^+, we subtract q from a_i.

And similarly, the expression from (86) is equal to:

$$\frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{C^-}|b_i|\left[z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|-q}-z^{\sum_{j=1}^K a_j+\sum_{j=1}^K|b_j|}\right]. \tag{96}$$

And so by combining (88), (89), (93), (94), (95), (96), we get an upper bound for (80)-(86). By making use of the definition of our function V, the upper bound is precisely this:

$$\begin{aligned}
\mathcal{D}V(\boldsymbol{a};\boldsymbol{b}) \leq\ & \frac{\lambda^{M^+}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\Big[z^{-q}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big] \\
&+ \frac{\lambda^{M^-}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\Big[z^{-q}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big] \\
&+ \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^+}\Big[z^{q+K|b_\infty|}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big] \\
&+ \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{L^-}\Big[z^{q+Ka_\infty}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big] \\
&+ \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{C^+}\Big[z^{-q}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big] \\
&+ \frac{1}{\Lambda(\boldsymbol{a};\boldsymbol{b})}\sum_{i=1}^{K}\lambda_i^{C^-}\Big[z^{-q}V(\boldsymbol{a};\boldsymbol{b}) - V(\boldsymbol{a};\boldsymbol{b})\Big].
\end{aligned}$$

Now, by setting $d_\infty := \max\{a_\infty, |b_\infty|\}$ and factoring out $V(\boldsymbol{a};\boldsymbol{b})$ from the above inequality, we get:

$$\begin{aligned}
\frac{\mathcal{D}V(\boldsymbol{a};\boldsymbol{b})}{V(\boldsymbol{a};\boldsymbol{b})} \leq\ & \frac{\lambda^{M^+} + \lambda^{M^-}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}(z^{-q} - 1) \\
&+ \frac{\Lambda^{L^+} + \Lambda^{L^-}}{\Lambda(\boldsymbol{a};\boldsymbol{b})}(z^{q+Kd_\infty} - 1) \\
&+ \frac{\sum_{j=1}^{K}\lambda_j^{C^+}a_j + \sum_{j=1}^{K}\lambda_j^{C^-}|b_j|}{\Lambda(\boldsymbol{a};\boldsymbol{b})}(z^{-q} - 1).
\end{aligned}$$

Let $\underline{\lambda_C} := \min_{1\leq i\leq K}\lambda_i^{C^\pm}$ and $\overline{\lambda_C} := \max_{1\leq i\leq K}\lambda_i^{C^\pm}$ and $\phi(\boldsymbol{a};\boldsymbol{b}) := \sum_{i=1}^{K} a_i + \sum_{i=1}^{K}|b_i|$.

Notice how

$$\begin{aligned}
\Lambda(\boldsymbol{x}) &:= \lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \sum_{i=1}^{K} a_i\lambda_i^{C^+} + \sum_{i=1}^{K}|b_i|\lambda_i^{C^-} \\
&\geq \lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \underline{\lambda_C}\cdot\phi(\boldsymbol{a};\boldsymbol{b}).
\end{aligned}$$

Therefore,

$$\frac{\mathcal{D}V(\boldsymbol{a};\boldsymbol{b})}{V(\boldsymbol{a};\boldsymbol{b})} \leq \frac{\lambda^{M^+} + \lambda^{M^-}}{\lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \underline{\lambda_C}\cdot\phi(\boldsymbol{a};\boldsymbol{b})}(z^{-q} - 1) \tag{97}$$

$$+ \frac{\Lambda^{L^+} + \Lambda^{L^-}}{\lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \underline{\lambda_C}\cdot\phi(\boldsymbol{a};\boldsymbol{b})}(z^{q+Kd_\infty} - 1) \tag{98}$$

$$+\frac{\overline{\lambda_C}\cdot\phi(\boldsymbol{a};\boldsymbol{b})}{\lambda^{M^+}+\lambda^{M^-}+\Lambda^{L^+}+\Lambda^{L^-}+\underline{\lambda_C}\cdot\phi(\boldsymbol{a};\boldsymbol{b})}(z^{-q}-1). \tag{99}$$

Denote the RHS of the above inequality as $B(\boldsymbol{a};\boldsymbol{b})$.

$$\begin{aligned}\lim_{\phi(\boldsymbol{a};\boldsymbol{b})\to\infty} B(\boldsymbol{a};\boldsymbol{b}) &= \lim_{\phi(\boldsymbol{a};\boldsymbol{b})\to\infty}\left[\frac{\overline{\lambda_C}}{\frac{(\lambda^{M^+}+\lambda^{M^-}+\Lambda^{L^+}+\Lambda^{L^-})}{\phi(\boldsymbol{a};\boldsymbol{b})}+\underline{\lambda_C}}(z^{-q}-1)\right] && \text{((97) and (98) tend to 0 as } \phi(\boldsymbol{a};\boldsymbol{b})\to\infty) \\ &= \frac{\overline{\lambda_C}}{\underline{\lambda_C}}\cdot(z^{-q}-1)<0 && \text{(since } z>1 \text{ and } q\geq 1)\end{aligned}$$

So by the classic definition of convergence of sequences, for all $\epsilon>0$ there exists an $N>0$ such that for all $(\boldsymbol{a};\boldsymbol{b})$ with $\phi(\boldsymbol{a};\boldsymbol{b})>N$ in our state space, we have

$$|B(\boldsymbol{a};\boldsymbol{b})-\frac{\overline{\lambda_C}}{\underline{\lambda_C}}\cdot(z^{-q}-1)|\leq\epsilon.$$

So let's set $\epsilon:=-\frac{1}{2}\cdot\frac{\overline{\lambda_C}}{\underline{\lambda_C}}\cdot(z^{-q}-1)>0$.

Then, there exists an $A>0$ such that for every $(\boldsymbol{a};\boldsymbol{b})$ with $\phi(\boldsymbol{a};\boldsymbol{b})>A$ in our state space, we have

$$|B(\boldsymbol{a};\boldsymbol{b})-\frac{\overline{\lambda_C}}{\underline{\lambda_C}}\cdot(z^{-q}-1)|\leq-\frac{1}{2}\cdot\frac{\overline{\lambda_C}}{\underline{\lambda_C}}\cdot(z^{-q}-1)$$

$$\implies B(\boldsymbol{a};\boldsymbol{b})\leq\frac{\overline{\lambda_C}}{2\underline{\lambda_C}}\cdot(z^{-q}-1)=:-\beta<0. \tag{100}$$

Define

$$C:=\{(\boldsymbol{a};\boldsymbol{b}):\phi(\boldsymbol{a};\boldsymbol{b})\leq A\},$$

which is clearly of finite cardinality because $(\boldsymbol{a};\boldsymbol{b})$ lives in our state space i.e. $a_i\in q\mathbb{Z}_+$ and $b_i\in -q\mathbb{Z}_+$ for every $i\in\{1,...,K\}$. Moreover, by the Heine-Borel Theorem, it's a compact set (C is closed and bounded in $\mathbb{R}^{2K}$), and hence is a petite set (recall Definition 5.2.9), since we are in a countable state space setting (such criteria for identifying compact sets as petite sets are covered in detail in Chapter 6 of [17]).

Finally, if $(\boldsymbol{a};\boldsymbol{b})$ in our state space is such that $\phi(\boldsymbol{a};\boldsymbol{b})>A$, then from (97)-(99) and (100) we get the following inequality:

$$\mathcal{D}V(\boldsymbol{a};\boldsymbol{b})\leq B(\boldsymbol{a};\boldsymbol{b})\cdot V(\boldsymbol{a};\boldsymbol{b})\leq-\beta V(\boldsymbol{a};\boldsymbol{b}). \tag{101}$$

And if $(\boldsymbol{a};\boldsymbol{b})$ in our state space is such that $\phi(\boldsymbol{a};\boldsymbol{b})\leq A$ i.e. $(\boldsymbol{a};\boldsymbol{b})\in C$, then we have the following inequality:

$$\mathcal{D}V(\boldsymbol{a};\boldsymbol{b}) \leq b, \tag{102}$$

where

$$b := \max_{(\boldsymbol{u};\boldsymbol{v})\in C} \mathcal{D}V(\boldsymbol{u};\boldsymbol{v}).$$

And b is finite because C is a finite set.

By combining (101) and (102), we get that the drift condition (55) is satisfied:

$$\mathcal{D}V(\boldsymbol{x}) \leq -\beta V(\boldsymbol{x}) + b\mathbb{1}_C(\boldsymbol{x}).$$

And so by Theorem 5.2.1, $(\boldsymbol{Z}_n)_{n\in\mathbb{N}_0}$ is V-uniformly ergodic. □

Recall the dependency of our Poisson processes $(C_i^+(t))_{t\geq 0}$ and $(C_i^-(t))_{t\geq 0}$ on the state of the order book (their intensities are $a_i(t)\lambda_i^{C^+}$ and $|b_i(t)|\lambda_i^{C^-}$ respectively). We can actually define a Markov process modeling the LOB on which we can prove its V-uniform ergodicity whilst having the previously stated assumptions on our cancellation processes relaxed (under some conditions).

To this end, to highlight the fact that we are now considering the order book with the assumption of cancellation rates depending on the state of the order book relaxed, we denote the order book as well as the embedded Markov chain associated to it by $(\boldsymbol{X'}(t) : t \geq 0)$ and $(\boldsymbol{Z'_n})_{n\in\mathbb{N}}$ respectively. This means that we shall now model $(C_i^+(t))_{t\geq 0}$ and $(C_i^-(t))_{t\geq 0}$ as Poisson processes with intensities $\lambda_i^{C^+}$ and $\lambda_i^{C^-}$ respectively, for $i \in \{1, ..., K\}$.

Theorem 5.3.3. *Set* $\Lambda^{C^\pm} := \sum_{i=1}^K \lambda_i^{C^\pm}$ *and* $\Lambda^{L^\pm} := \sum_{i=1}^K \lambda_i^{L^\pm}$.

Under the condition $\lambda^{M^+}+\lambda^{M^-}+\Lambda^{C^+}+\Lambda^{C^-} > (\Lambda^{L^+}+\Lambda^{L^-})(1+Kd_\infty)$ *where* $d_\infty := \max\{a_\infty, |b_\infty|\}$, $(\boldsymbol{Z'_n})_{n\in\mathbb{N}}$ *is* V*-uniformly ergodic.*

Proof. We get a similar inequality as in the proof of Theorem 5.3.2 by the same arguments and with the same 'Lyapunov test function' $V(\boldsymbol{a};\boldsymbol{b}) = z^{\sum_{i=1}^K a_i + \sum_{i=1}^K |b_i|}$, with the only difference that we now don't have the dependence of the cancellation rates on $\boldsymbol{x'} := (a_1, ..., a_K; b_1, ..., b_K)$.

This means that now, $\Lambda(\boldsymbol{a};\boldsymbol{b}) =: \Lambda = \lambda^{M^+} + \lambda^{M^-} + \Lambda^{L^+} + \Lambda^{L^-} + \Lambda^{C^+} + \Lambda^{C^-}$ (no longer dependent on $(\boldsymbol{a};\boldsymbol{b})$), and the terms $\sum_{j=1}^K \lambda_j^{C^+} a_j$ and $\sum_{j=1}^K \lambda_j^{C^-} |b_j|$ in the inequality from Page 45 right after "Now, by setting $d_\infty := \max\{a_\infty, |b_\infty|\}$ and factoring out...", are now Λ^{C^+} and Λ^{C^-} respectively. Hence,

$$\frac{\mathcal{D}V(\boldsymbol{x'})}{V(\boldsymbol{x'})} \leq \frac{\lambda^{M^+} + \lambda^{M^-}}{\Lambda}(z^{-q} - 1) \tag{103}$$

$$+\frac{\Lambda^{L^+}+\Lambda^{L^-}}{\Lambda}(z^{q+Kd_\infty}-1) \tag{104}$$

$$+\frac{\Lambda^{C^+}+\Lambda^{C^-}}{\Lambda}(z^{-q}-1) \tag{105}$$

where $z = 1+\epsilon$ for some $\epsilon > 0$.

By considering the Taylor expansion around 1 of x^{-q}, and taking ϵ sufficiently small, we get

$$\begin{aligned} z^{-q} = (1+\epsilon)^{-q} &= 1 + (-q)\epsilon + \frac{(-q)(-q-1)}{2}\epsilon^2 + ... \\ &= 1 - q\epsilon + ... \end{aligned}$$

and

$$\begin{aligned} z^{q+Kd_\infty} = (1+\epsilon)^{q+Kd_\infty} &= 1 + (q+Kd_\infty)\epsilon + \frac{(q+Kd_\infty)(q+Kd_\infty-1)}{2}\epsilon^2 + ... \\ &= 1 + (q+Kd_\infty)\epsilon + ... \end{aligned}$$

And so the inequality (103)-(105) becomes

$$\begin{aligned} \frac{\Lambda \mathcal{D} V(\boldsymbol{x'})}{V(\boldsymbol{x'})} &\leq (\lambda^{M^+}+\lambda^{M^-})(-q\epsilon) \\ &+ (\Lambda^{L^+}+\Lambda^{L^-})(q+Kd_\infty)\epsilon \\ &+ (\Lambda^{C^+}+\Lambda^{C^-})(-q\epsilon) + g(\epsilon) \\ &\leq (\lambda^{M^+}+\lambda^{M^-})(-q\epsilon) \\ &+ (\Lambda^{L^+}+\Lambda^{L^-})(q+qKd_\infty)\epsilon \\ &+ (\Lambda^{C^+}+\Lambda^{C^-})(-q\epsilon) + g(\epsilon), \end{aligned}$$

where g is a function of ϵ that tends to 0 as $\epsilon \to 0$.

And so for negligible $\epsilon > 0$, the sign of

$$(\lambda^{M^+}+\lambda^{M^-})(-q\epsilon) + (\Lambda^{L^+}+\Lambda^{L^-})(q+qKd_\infty)\epsilon + (\Lambda^{C^+}+\Lambda^{C^-})(-q\epsilon) + g(\epsilon)$$

is determined by the quantity

$$-(\lambda^{M^+}+\lambda^{M^-}) + (\Lambda^{L^+}+\Lambda^{L^-})(1+Kd_\infty) - (\Lambda^{C^+}+\Lambda^{C^-})$$

which doesn't depend on $\boldsymbol{x'}$.

And so if the condition of the Theorem holds, then the above quantity is negative and, since $\Lambda > 0$, we get that

$$\mathcal{D}V(\boldsymbol{x'}) \leq -\beta V(\boldsymbol{x'}) + d$$

where

$$\beta := -\frac{\Big((\lambda^{M^+} + \lambda^{M^-})(-q\epsilon) + (\Lambda^{L^+} + \Lambda^{L^-})(q + qKd_\infty)\epsilon + (\Lambda^{C^+} + \Lambda^{C^-})(-q\epsilon) + g(\epsilon)\Big)}{\Lambda} > 0$$

and

$$d = 0 < \infty.$$

And so by Proposition 5.2.1 we are done. □

Corollary 5.3.1. *Under the same condition as the previous Theorem,* $(\boldsymbol{X'}(t) : t \geq 0)$ *is* V*-uniformly ergodic.*

Proof. By Proposition 4.3.2 and the fact that under the $(\boldsymbol{X'}(t) : t \geq 0)$ process $\Lambda(\boldsymbol{a}; \boldsymbol{b}) =: \Lambda > 0$ is no longer dependent on $\boldsymbol{x'} := (\boldsymbol{a}; \boldsymbol{b})$, $\mathcal{L}V(\boldsymbol{x'}) = \Lambda\mathcal{D}V(\boldsymbol{x'})$ where V is as in the previous Theorem.

And so by the previous Theorem,

$$\mathcal{L}V(\boldsymbol{x'}) \leq -\gamma V(\boldsymbol{x'}) + d,$$

where $\gamma := \Lambda\beta > 0$ where β is from the last Theorem and $d = 0 < \infty$.

And so by Proposition 5.2.1 we are done. □

Interestingly, if we append an 'indicator' process that indicates the last event that occurred in the order book before (or at) time t, to our $(\boldsymbol{X}(t) : t \geq 0)$, we get that this new process is V-uniformly ergodic.

To this end, let's define this 'indicator' process $(\epsilon(t) : t \geq 0)$ where $\epsilon(t)$ indicates the last event that occurred in the order book before (or at) time $t \geq 0$. All the possible events that could have occurred are: a buy market order (i.e. M^+), a sell market order (i.e. M^-), a limit sell order at level $i \in \{1, ..., K\}$ (i.e. L_i^+), a limit buy order at level $i \in \{1, ..., K\}$ (i.e. L_i^-), a cancellation of a limit sell order at level $i \in \{1, ..., K\}$ (i.e. C_i^+), or a cancellation of a limit buy order at level $i \in \{1, ..., K\}$ (i.e. C_i^-)... A total of $2 + 2K + 2K = 4K + 2$ events.

Let's enumerate the above events in the following way: $1 \equiv M^+$, $2 \equiv M^-$, $2 + i \equiv L_i^+$ for $i \in \{1, ..., K\}$, $2 + K + i \equiv L_i^-$ for $i \in \{1, ..., K\}$, $2 + 2K + i \equiv C_i^+$ for $i \in \{1, ..., K\}$, $2 + 3K + i \equiv C_i^-$ for $i \in \{1, ..., K\}$.

Based on the above enumeration of the events, we define for $t \geq 0$, $\epsilon(t) \in \{1, 2, ..., 4K + 2\}$. So for example, if the last event that had occured in the order book before time t was a cancellation of a limit buy order at level 4, then $\epsilon(t) = 2 + 3K + 4 = 3K + 6$.

Let's now define this appended process as $(\boldsymbol{Y}(t) : t \geq 0)$ where $\boldsymbol{Y}(t) := (\boldsymbol{X}(t), \epsilon(t)) = (\boldsymbol{a}(t); \boldsymbol{b}(t), \epsilon(t))$.

Corollary 5.3.2. *The infinitesimal generator of the Markov process $(\boldsymbol{Y}(t) : t \geq 0)$ is the operator $\mathcal{P}$ defined by:*

$$\mathcal{P}f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) = \lambda^{M^+}(f([a_i - (q - A_{i-1})_+]_+^{i=1,\dots,K}; J^{M^+}(\boldsymbol{b}), 1) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{106}$$

$$+ \sum_{i=1}^{K} \lambda_i^{L^+}(f(a_i + q; J^{L_i^+}(\boldsymbol{b}), 2 + i) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{107}$$

$$+ \sum_{i=1}^{K} \lambda_i^{C^+} a_i(f(a_i - q; J^{C_i^+}(\boldsymbol{b}), 2 + 2K + i) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{108}$$

$$+ \lambda^{M^-}(f(J^{M^-}(\boldsymbol{a}); [b_i + (q - B_{i-1})_+]_-^{i=1,\dots,K}, 2) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{109}$$

$$+ \sum_{i=1}^{K} \lambda_i^{L^-}(f(J^{L_i^-}(\boldsymbol{a}); b_i - q, 2 + K + i) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{110}$$

$$+ \sum_{i=1}^{K} \lambda_i^{C^-}|b_i|(f(J^{C_i^-}(\boldsymbol{a}); b_i + q, 2 + 3K + i) - f(\boldsymbol{a}; \boldsymbol{b}, \epsilon) \tag{111}$$

for $f : \mathbb{Z}^{2K+1} \to \mathbb{R}$ sufficiently regular.

Proof. $(\boldsymbol{Y}(t) : t \geq 0)$ is just $(\boldsymbol{X}(t) : t \geq 0)$ with the 'indicator' process $(\epsilon(t) : t \geq 0)$ appended. Therefore, to derive $\mathcal{P}$ we essentially use the exact same arguments as in the proof of Proposition 4.3.1 with very minor modifications. To be precise, the only difference between this and the proof of Proposition 4.3.1 is that here we have to make use of the definition of $\epsilon(t)$ and the value it takes when we're conditioning on a particular event of the order book. □

Corollary 5.3.3. *The process $(\boldsymbol{Y}(t) : t \geq 0)$ is V-uniformly ergodic.*

Proof. The proof is the same as that of Theorem 5.3.1 but with a minor modification to the Lyapunov test function. To be precise, due to the expression of the infinitesimal generator of $(\boldsymbol{Y}(t) : t \geq 0)$, $\mathcal{P}$, here we need $V(\boldsymbol{x}) := V(\boldsymbol{a}; \boldsymbol{b}, \epsilon) = \sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i| + \epsilon + q$ in order to satisfy the drift condition $\mathcal{P}V(\boldsymbol{x}) \leq -cV(\boldsymbol{x}) + d$ where c and d are as in the proof of Theorem 5.3.1. And so by Proposition 5.2.1 we are done. □

In much the same way as we had done with the embedded Markov chain of $(\boldsymbol{X}(t) : t \geq 0)$, $(\boldsymbol{Z}_n)_{n \in \mathbb{N}}$, in proving its ergodicity in Theorem 5.3.2, we can do in proving the ergodicity of the embedded Markov chain of $(\boldsymbol{Y}(t) : t \geq 0)$, with the only difference that here we would need to account for the appended process i.e. our choice of Lyapunov test function would now be slightly modified to $V(\boldsymbol{a}; \boldsymbol{b}, \epsilon) = z^{\sum_{i=1}^{K} a_i + \sum_{i=1}^{K} |b_i| + \epsilon}$.

6 Conclusion

The aim of the dissertation was to investigate the long term behaviour of the Markovian model of the order book. We proved that the process, as well as some variants of it, are ergodic, meaning that they converge, in a strong probabilistic sense (recall the total variation norm), to their unique invariant distributions. The property of ergodicity is desirable, as it's a long term notion of stability, and means that probabilities of the different events in the order book become increasingly stable over time. We actually proved something stronger than this though, we proved that this stabilization happens exponentially fast (our process is V-uniformly ergodic). What is more, the eminent Ergodic theorem is, as a result of our Markovian models being ergodic, applicable.

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