Spring, 2015

# ME 612 – Continuum Mechanics

## Lecture 3
## Introduction to vectors and tensors

PURDUE ENGINEERING

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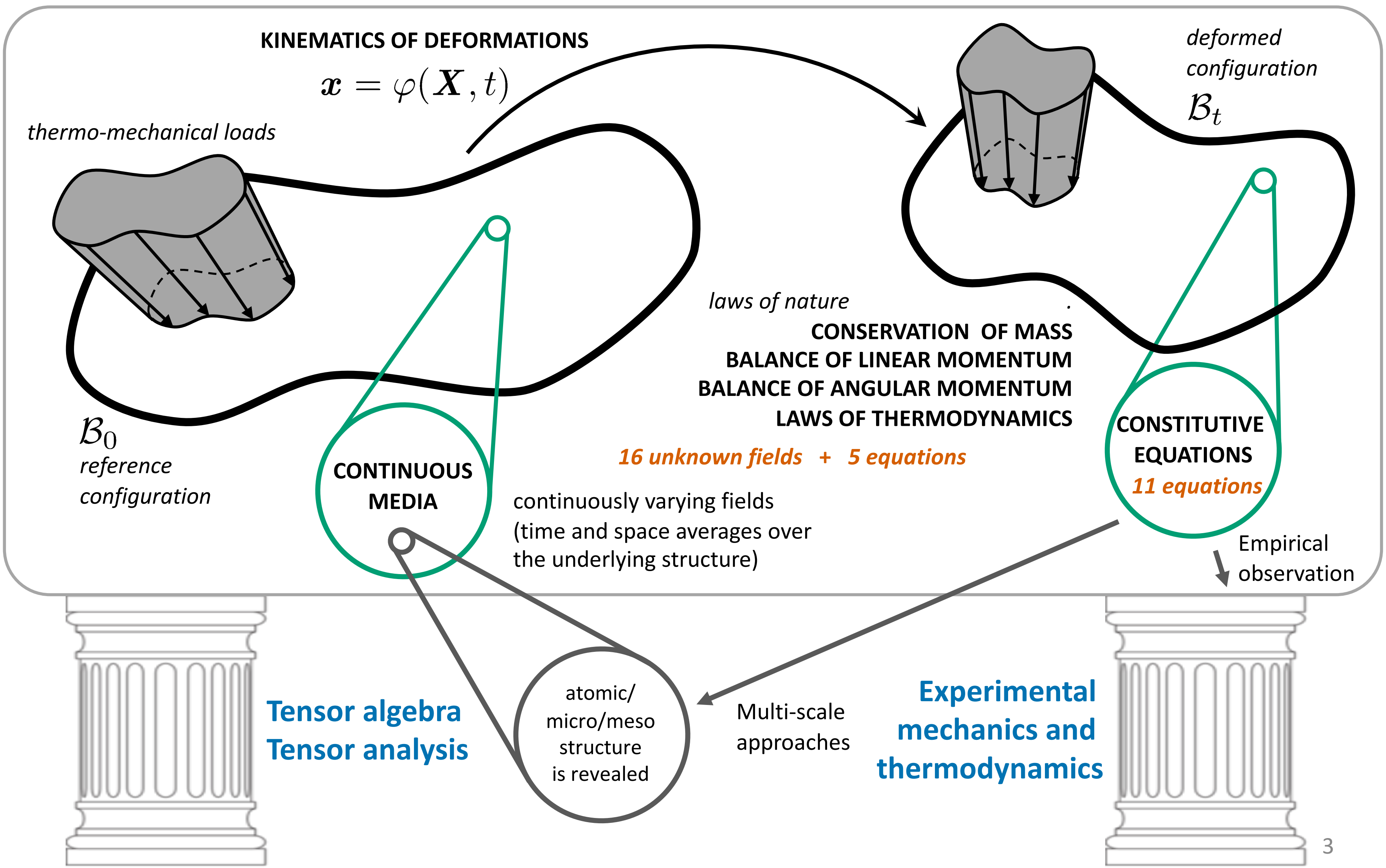
# Announcements

Midterm Exam

- Thursday, April 2rd, 3-5 p.m., ME 2054

# Lecture 3 – Introduction to tensors and vectors



# Lecture 3 – Introduction to tensors and vectors

## Review (transformation rules)

DIY

**Tensors**: real-valued multi-linear functions of vectors

1-order tensor (vector) | 2-order tensor

$$a_i \equiv \boldsymbol{a}[\boldsymbol{e}_i] \qquad A_{ij} \equiv \boldsymbol{A}[\boldsymbol{e}_i, \boldsymbol{e}_j]$$

coordinate invariant

$$a'_i = \mathrm{Q}_{\alpha i}\ a_\alpha \qquad A'_{ij} = \mathrm{Q}_{\alpha i} A_{\alpha\beta} \mathrm{Q}_{\beta j}$$

$$[\boldsymbol{a}]' = \mathbf{Q}^T[\boldsymbol{a}] \qquad [\boldsymbol{A}]' = \mathbf{Q}^T[\boldsymbol{A}]\mathbf{Q}$$

transformation rules

$$\mathbf{Q}^T = \mathbf{Q}^{-1}$$
$$\det(\mathbf{Q}) = 1$$

Properties of the transformation matrix
(proper orthogonal = orthogonal + positive determinant)

# Tensor algebra

## Tensor operations

- Addition: $\boldsymbol{C}[\boldsymbol{a},\boldsymbol{b}] = \boldsymbol{A}[\boldsymbol{a},\boldsymbol{b}] + \boldsymbol{B}[\boldsymbol{a},\boldsymbol{b}]$

  DIY — Prove

  $$C_{ij} = A_{ij} + B_{ij} \Longleftrightarrow \boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$$

- Magnification: $\boldsymbol{B}[\boldsymbol{a},\boldsymbol{b}] = \lambda\boldsymbol{A}[\boldsymbol{a},\boldsymbol{b}]$ $\quad B_{ij} = \lambda A_{ij} \Longleftrightarrow \boldsymbol{B} = \lambda\boldsymbol{A}$

- Transpose: $\boldsymbol{B}[\boldsymbol{a},\boldsymbol{b}] = \boldsymbol{A}[\boldsymbol{b},\boldsymbol{a}]$ $\quad B_{ij} = A_{ji} \Longleftrightarrow \boldsymbol{B} = \boldsymbol{A}^T$

- Tensor product: $\boldsymbol{D}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}] = \boldsymbol{A}[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{v}[\boldsymbol{c}]$

  $$D_{ijk} = A_{ij}v_k \Longleftrightarrow \boldsymbol{D} = \boldsymbol{A} \otimes \boldsymbol{v}$$

Dyad: *a second-order tensor formed by the tensor product of two vectors*

$$A_{ij} = a_i b_j \Longleftrightarrow \boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$$

$$\boldsymbol{a} \otimes \boldsymbol{b} \neq \boldsymbol{b} \otimes \boldsymbol{a}$$

DIY — Dyad in matrix notation

# Tensor algebra

## Tensor operations

- Contraction:

$$\mathrm{Cont}_{ij}\,\boldsymbol{T} = \boldsymbol{T}[\boldsymbol{a}_1, ..., \boldsymbol{a}_{i-1}, \boldsymbol{e}_k, \boldsymbol{a}_{i+1}, ..., \boldsymbol{a}_{j-1}, \boldsymbol{e}_k, \boldsymbol{a}_{j+1} ..., \boldsymbol{a}_n] \quad (\text{sum on } k)$$

$$\boldsymbol{v}[\boldsymbol{a}] = \mathrm{Cont}_{23}\boldsymbol{D} = \boldsymbol{D}[\boldsymbol{a}, \boldsymbol{e}_k, \boldsymbol{e}_k] \iff v_i = D_{ikk}$$

- Contracted multiplication:

  + $\boldsymbol{v}[\boldsymbol{a}] = \mathrm{Cont}_{23}(\boldsymbol{A}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{u}[\boldsymbol{c}]) = \boldsymbol{A}[\boldsymbol{a}, \boldsymbol{e}_j]\boldsymbol{u}[\boldsymbol{e}_j]$

    $v_i = A_{ik}u_k \iff \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$

    DIY — Matrix notation

  + $\boldsymbol{C}[\boldsymbol{a}, \boldsymbol{b}] = \mathrm{Cont}_{23}(\boldsymbol{A}[\boldsymbol{a}, \boldsymbol{c}]\boldsymbol{B}[\boldsymbol{d}, \boldsymbol{b}]) = \boldsymbol{A}[\boldsymbol{a}, \boldsymbol{e}_k]\boldsymbol{B}[\boldsymbol{e}_k, \boldsymbol{b}]$

    $C_{ij} = A_{ik}B_{kj} \iff \boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$

  + $C_{ij} = A_{ik}B_{jk} \iff \boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}^T$ $\quad C_{ij} = A_{ki}B_{jk} \iff \boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{B}^T$

    $C_{ij} = A_{ki}B_{kj} \iff \boldsymbol{C} = \boldsymbol{A}^T\boldsymbol{B}$

# Tensor algebra

## Tensor operations

- Contraction:

$$\mathrm{Cont}_{ij}\,\boldsymbol{T} = \boldsymbol{T}[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i-1}, \boldsymbol{e}_k, \boldsymbol{a}_{i+1}, \ldots, \boldsymbol{a}_{j-1}, \boldsymbol{e}_k, \boldsymbol{a}_{j+1} \ldots, \boldsymbol{a}_n]$$

(sum on $k$)

$$\boldsymbol{v}[\boldsymbol{a}] = \mathrm{Cont}_{23}\boldsymbol{D} = \boldsymbol{D}[\boldsymbol{a}, \boldsymbol{e}_k, \boldsymbol{e}_k] \iff v_i = D_{ikk}$$

- Scalar contraction:

(2-order) $\mathrm{tr}\,\boldsymbol{A} = \mathrm{Cont}_{12}\boldsymbol{A} = \mathrm{tr}[\boldsymbol{A}] = A_{ii}$ $\det\boldsymbol{A} \equiv \det[\boldsymbol{A}]$

(1-order) $\boldsymbol{a}\cdot\boldsymbol{b} = \mathrm{Cont}_{12}(\boldsymbol{a}\otimes\boldsymbol{b}) = a_i b_i$

DIY $\det(\boldsymbol{a}\otimes\boldsymbol{b})$

(double contraction of two 2-order tensors)

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij} \qquad \boldsymbol{A}\cdot\cdot\boldsymbol{B} = A_{ij}B_{ji}$$

- _____ (double contraction of a 4-order tensor and a 2-order tensor)

$$[\boldsymbol{E} : \boldsymbol{B}]_{ij} = E_{ijkl}B_{kl} \qquad [\boldsymbol{E}\cdot\cdot\boldsymbol{B}]_{ij} = E_{ijkl}B_{lk}$$

# Tensor algebra

## Tensor operations

DIY

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{v} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{v})$$

# Tensor algebra

## Tensor basis

- Tensor product of vectors can be used to define basis for tensors.
-

(**2-order**) – can be written as a linear combination of dyads (i.e., a dyadic)

A dyadic of two (linearly independent) dyads for $n_\mathrm{d} = 2$

$$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{d}$$

A dyadic of three (linearly independent) dyads for $n_\mathrm{d} = 3$

$$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{d} + \boldsymbol{e} \otimes \boldsymbol{f}$$

…. but also (i.e., the decomposition is not unique)

$$\boldsymbol{A} = A_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \qquad A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j$$

where $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ form linearly independent basis.

(**4-order**)

$$\boldsymbol{E} = E_{ijkl}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l)$$

# Properties of (second order) tensors

## Orthogonal tensors

$$\boldsymbol{Q} \text{ is } \underline{\text{orthogonal}} \text{ if } (\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} \quad \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{n_\mathrm{d}}$$

The tensor preservers the magnitude of, and the angle between, the vectors on which it operates.

Necessary and sufficient condition: $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$ $\quad (\det \boldsymbol{Q} = 1)$

proper orthogonal

Proper orthogonal tensors represent rotations, .e.g., 

$$\boldsymbol{e}_1' = \boldsymbol{Q}\boldsymbol{e}_1$$
$$\boldsymbol{e}_2' = \boldsymbol{Q}\boldsymbol{e}_2$$
$$\boldsymbol{e}_3' = \boldsymbol{Q}\boldsymbol{e}_3$$

<u>NOTE</u>: Recall the matrix that links two bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}_i'\}$ … $\boldsymbol{e}_j' = \mathrm{Q}_{ij}\ \boldsymbol{e}_i$

Though $[\boldsymbol{Q}] = \mathbf{Q}$ , they represent very different ideas!

# Properties of (second order) tensors

## Symmetrical and antisymmetrical tensors

$\boldsymbol{S}$ is a symmetric if $S_{ij} = S_{ji} \iff \boldsymbol{S} = \boldsymbol{S}^T$

$\boldsymbol{A}$ is an antisymmetric if $A_{ij} = -A_{ji} \iff \boldsymbol{A} = -\boldsymbol{A}^T$

Any tensor $T_{ij}$ can be decomposed into a symmetric part $T_{(ij)} \equiv \frac{1}{2}(T_{ij} + T_{ji})$ and an antisymmetric $T_{[ij]} \equiv \frac{1}{2}(T_{ij} - T_{ji})$ so that

$$T_{ij} = T_{(ij)} + T_{[ij]}$$

DIY

Axial vector: Given $\boldsymbol{A}$, $\exists\, \boldsymbol{\omega}$ s.t. $\boldsymbol{A}\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{a} \quad \forall \boldsymbol{a} \in \mathbb{R}^3$

*show that* $\omega_k = -\frac{1}{2}\epsilon_{ijk}A_{ij}$

Contraction of symmetric and antisymmetric tensors:

*show that* $\boldsymbol{S} : \boldsymbol{A} = S_{ij}A_{ij} = 0$

# Properties of (second order) tensors

## Principal values and directions

$$G_{ij}\Lambda_j = \lambda\Lambda_i \iff \boldsymbol{G}\boldsymbol{\Lambda} = \lambda\boldsymbol{\Lambda} \iff (G_{ij} - \lambda\delta_{ij})\Lambda_j = 0 \iff (\boldsymbol{G} - \lambda\boldsymbol{I})\boldsymbol{\Lambda} = \boldsymbol{0}$$

Nontrivial solutions [i.e., $\lambda^{\boldsymbol{G}}$ (eigenvalue) and $\boldsymbol{\Lambda}^{\boldsymbol{G}}$ (eigenvector)] require

$$\det(\boldsymbol{G} - \lambda\boldsymbol{I}) = 0$$

## Characteristic equation and principal invariants

The characteristic equation of $\boldsymbol{G}$ (for $n_\mathrm{d} = 3$) is

$$-\lambda^3 + I_1(\boldsymbol{G})\lambda^2 - I_2(\boldsymbol{G})\lambda + I_3(\boldsymbol{G}) = 0$$

where the principal invariants of $\boldsymbol{G}$ are

$$I_1(\boldsymbol{G}) = \mathrm{tr}(\boldsymbol{G})$$
$$I_2(\boldsymbol{G}) = \tfrac{1}{2}[(\mathrm{tr}\,\boldsymbol{G})^2 - \mathrm{tr}\,\boldsymbol{G}^2]$$
$$I_3(\boldsymbol{G}) = \det\boldsymbol{G}$$

- Symmetric tensor: DIY

$\lambda_\alpha^{\boldsymbol{S}} \in \mathbb{R}$ $\quad$ $\sum_{\alpha=1}^{3} \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} = \boldsymbol{I}$

$\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \cdot \boldsymbol{\Lambda}_\beta^{\boldsymbol{S}} = \delta_{\alpha\beta}$ $\quad$ $\boldsymbol{S} = \sum_{\alpha=1}^{3} \lambda_\alpha^{\boldsymbol{S}}\, \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}}$

# Properties of (second order) tensors

## Principal values and directions

(principal basis, completeness relation, spectral decomposition)

DIY

Symmetric tensor $\boldsymbol{S} \in \mathbb{R}^3$

# Properties of (second order) tensors

## Cayley-Hamilton theorem ($\boldsymbol{T}$ on $\mathbb{R}^3$)

$$\boldsymbol{T}^3 = I_1(\boldsymbol{T})\boldsymbol{T}^2 - I_2(\boldsymbol{T})\boldsymbol{T} + I_3(\boldsymbol{T})\boldsymbol{I}$$

(i.e., $\boldsymbol{T}$ satisfies its own characteristic equation).

$$\boldsymbol{T}^4 = (I_1^2 - I_2)\boldsymbol{T}^2 + (I_3 - I_1 I_2)\boldsymbol{T} + I_1 I_3 \boldsymbol{I}$$

$$\boldsymbol{T}^5 = \ldots$$

NOTE: Recall that $[\boldsymbol{T}^3]_{ij} = T_{im}T_{mn}T_{nj}$

# Properties of (second order) tensors

## Symmetric positive-definite tensors

$\boldsymbol{S}$ is positive definite iff $Q(x) \equiv S_{ij}x_i x_j > 0 \ \forall \boldsymbol{x} \in \mathbb{R}^{n_\mathrm{d}}, \boldsymbol{x} \neq \boldsymbol{0}$

$\boldsymbol{S}$ is positive definite iff $\lambda_\alpha^{\boldsymbol{S}} > 0, \ \forall \alpha$

+ Now we can define the square root:

If $\boldsymbol{S}$ is positive definite, then $\exists \boldsymbol{R}, \text{ s.t. } \boldsymbol{R}^2 = \boldsymbol{S}$

For $\boldsymbol{S}$ on $\mathbb{R}^3$ , $\sqrt{\boldsymbol{S}} = \sum_{\alpha=1}^{3} \sqrt{\lambda_\alpha^{\boldsymbol{S}}} (\boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}} \otimes \boldsymbol{\Lambda}_\alpha^{\boldsymbol{S}})$

…. notice that we choose the 'positive-definite' square root!

Note: the scalar function $Q(x) \equiv S_{ij}x_i x_j$
is called the quadratic form of $\boldsymbol{S}$ .

# Properties of (second order) tensors

## Isotropic tensors (… important for constitutive relations)

- An isotropic tensor is a tensor whose components are unchanged by coordinate transformation.

$$\mathrm{Q}_{\alpha i}\mathrm{Q}_{\beta j}T_{\alpha\beta} = T_{ij} \ , \forall \mathbf{Q} \in \text{ group of rotation matrices}$$

+ Zeroth-order: All zeroth-order tensors are isotropic.
+ First-order: The only isotropic first-order tensor is the zero vector.
+ Second-order: All isotropic second-order tensors are proportional to the identity tensor.
+ Third-order: All isotropic (hemitropic) third-order tensors are proportional to the permutation symbol.
+ Forth-order: All isotropic fourth-order tensors can be written as

$$C_{ijkl} = \alpha\ \delta_{ij}\delta_{kl} + \beta\ \delta_{ik}\delta_{jl} + \gamma\ \delta_{il}\delta_{jk}$$

Any questions?