THE MYSTERIOUS GEOMETRY OF ARTIN GROUPS

JON MCCAMMOND

Abstract. Artin groups are easily defined but most of them are poorly understood. In this survey I try to highlight precisely where the problems begin. The first part reviews the close connection between Coxeter groups and Artin groups as well as the associated topological spaces used to investigate them. The second part describes the location of the border between the Artin groups we understand at a very basic level and those that remain fundamentally mysterious. The third part highlights those collections of Artin groups (and their relatives) that are not currently understood but which we are likely to understand sometime soon.

Artin groups, also known as Artin-Tits groups, are easy to define via presentations but they are often very poorly understood. In this article, when I say that we "understand" a particular group, what I mean is do we know how to solve its word problem. As is well-known, this is equivalent to being able to construct arbitrarily large portions of its Cayley graph and arbitrarily large portions of the universal cover of its presentation 2-complex. It is in this sense that most Artin groups are poorly understood. For most Artin groups we do not know how to solve the word problem.

In this survey I try to highlight exactly where the problems begin. It is divided into three parts and each part corresponds to one of the lectures in the short course I gave at the winter braids conference in Caen in early March 2017. The first part reviews the close connection between Coxeter groups and Artin groups with a particular focus on the associated topological spaces used to investigate them. The second part summarizes exactly which collections of Artin groups we understand at a basic level and which ones remain mysterious. And the third part highlights those collections of Artin groups (and their relatives) that are not currently understood but which we are likely to understand in the near future. In keeping with the informal nature of this survey, I have retained some of the blackboard images used in my talks.

Date: October 9, 2017.

2010 *Mathematics Subject Classification.* 20F36, 20F55.

Key words and phrases. Artin groups.

Part 1. Among friends

In Part 1 we are "among friends" since the material is more or less familiar and it is organized around the contributions of the mathematicians H.S.M. Coxeter, Jacques Tits, Mike Davis and Mario Salvetti. It contains a quick review of some background material about Coxeter groups and Artin groups starting with Coxeter presentations, the Tits representation and Davis complexes and then proceeding on to Salvetti complexes and Artin groups. For the purposes of this article, I assume that the reader is familiar with Coxeter groups at the level of Humphreys introductory text [Hum90]. For Artin groups, there is no standard introductory text, but the interested reader can find a more detailed development of some of these ideas in the recent survey articles by Luis Paris [Par14a, Par14b].

1. Coxeter

This section first recalls the basic facts about Coxeter groups and the presentations and diagrams used to define them and then describes the important special cases where these groups act geometrically and irreducibly on spheres and euclidean spaces.

1.1. **Presentations and Diagrams.** The groups known as Coxeter groups and Artin groups are easy to define using presentations and there are two main conventions for encoding these presentations in diagrams with edge labels.

Definition 1.1 (Coxeter groups and Artin groups)**.** A *braid relation of length* m between generators a and b equates the two strictly alternating positive words of length m. For $m = 2, 3, 4, \ldots$ these relations are $ab = ba$, $aba = bab$, $abab = baba$, and so on. An *Artin presentation* has at most one braid relation for each pair of distinct generators and no other relations. *Coxeter presentations* add relations that make each generator an involution. *Artin groups* are defined by *Artin presentations* and *Coxeter groups* are defined by *Coxeter presentations*.

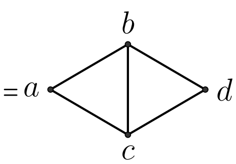
Remark 1.2 (Conventions)*.* There are two conventions for encoding these presentations as an edge-labeled simple graph with vertices indexing generators and decorated edges indicating braid relations. In what I call the *classical notation*, generators that commute correspond to vertices that are not connected by an edge, generators satisfying the classical braid relation $aba = bab$ are connected by an unlabeled edge, braid relations with $m > 3$ require a label and the lack of a relation is indicated by an edge with an infinity symbol. In what I call the

m	Classical	Modern
2	no edge $\bullet\quad\bullet$	no label $\bullet\text{---}\bullet$
3	no label $\bullet\text{---}\bullet$	label $\overset{3}{\bullet\text{---}\bullet}$
>3	label $\overset{m}{\bullet\text{---}\bullet}$	label $\overset{m}{\bullet\text{---}\bullet}$
∞	label $\overset{\infty}{\bullet\text{---}\bullet}$	no edge $\bullet\quad\bullet$

TABLE 1. The two diagram conventions for Coxeter groups and Artin groups.

modern notation, generators that commute correspond to vertices with an unlabeled edge, braid relations with $m > 2$ require a label and the lack of a relation is indicated by the lack of an edge. See Table 1.

Example 1.3 (A small example)**.** Consider the following simple graph.

$\Gamma =$ 

If we view Γ as encoding an Artin presentation using the classical notation, then the corresponding Artin group is the one defined by the following presentation.

$$\text{ART}(\Gamma) = \left\langle a, b, c, d \;\middle|\; \begin{matrix} aba = bab, & ad = da, & bdb = dbd \\ aca = cac, & bcb = cbc, & cdc = dcd \end{matrix} \right\rangle$$

To define the corresponding Coxeter group one would also add the relations $a^2 = b^2 = c^2 = d^2 = 1$.

Remark 1.4 (Structure and use)*.* The two conventions highlight different aspects of the Coxeter groups and Artin groups they define. When a diagram is disconnected in the classical notation, this means that the Coxeter group (and the Artin group) defined by this diagram splits as a *direct product* of the groups defined by its connected components. When a diagram is discconnected in the modern notation, this means that the Coxeter group (and the Artin group) defined by this diagram splits as a *free product* of the groups defined by its connected components. The convention chosen often depends on the type of groups under investigation. The modern notation makes it very easy to define a *right-angled Artin group*, where every relation is a commutation relation, because these diagrams are simple unlabeled graphs without

	Classical	Modern
disconnected	direct product	free product
no labels	small-type	right-angled

TABLE 2. The two conventions highlight different aspects of the structure of the corresponding groups and they make different types of groups easy to define.

loops or multiple edges. In the classical notation these simple unlabeled graphs define the *Artin groups of small type*. See Table 2.

1.2. **Spherical Coxeter groups.** One of the main motivations for introducing a general theory of Coxeter groups is the elegance and importance of the groups generated by reflections acting geometrically and irreducibly on spheres and euclidean spaces. Recall that a group is said to act *geometrically* when its action is properly discontinuous and cocompact by isometries and that an action on euclidean space is *irreducible* if there does not exist a nontrivial orthogonal decomposition of the underlying space so that the group splits as a product of subgroups acting on these subspaces. We begin with the reflection groups acting geometrically and irreducibly on spheres.

Definition 1.5 (Spherical Coxeter groups)**.** The irreducible *spherical Coxeter groups* are those groups generated by reflections that act geometrically and irreducibly on a sphere in some euclidean space fixing its center. They are also known as the *finite Coxeter groups* or *finite reflection groups*. The adjective *spherical* highlights the type of geometry used in its definition. The classification of such groups is classical and essentially due to Coxeter. All of these groups have Coxeter presentations that are encoded in the well-known Dynkin diagrams using the classical notation. The *type* of a Dynkin diagram is its name in the Cartan-Killing classification and it is *crystallographic* or *non-crystalographic* depending on whether or not it extends to a euclidean Coxeter group. The crystallographic types consist of three infinite families (A_n, $B_n = C_n$, and D_n) and five sporadic examples (G_2, F_4, E_6, E_7, and E_8). The non-crystallographic types are H_3, H_4 and $I_2(m)$ for $m \neq 3,4,6$. The subscript is the dimension of the euclidean space containing the sphere on which it acts.

Example 1.6 (Simplices and cubes)**.** The spherical Coxeter groups of types A and B are the best known and represent the symmetry groups of regular simplices and high-dimensional cubes, respectively.

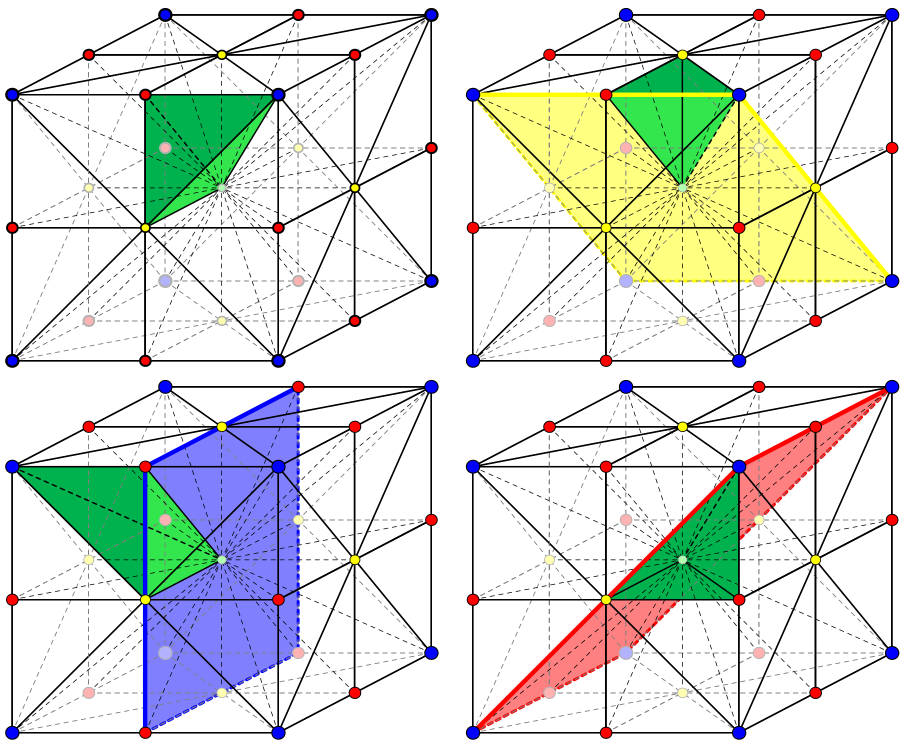


FIGURE 1. The spherical Coxeter group $\textsc{Cox}(B_3)$.

As groups they are the *symmetric groups* and extensions of symmetric groups by elementary 2-groups called *signed symmetric groups*. For example, the group $\textsc{Cox}(A_3) \cong \textsc{Sym}_4$ is the symmetric group of a regular tetrahedron and the group $\textsc{Cox}(B_3) \cong (\mathbb{Z}_2)^3 \rtimes \textsc{Sym}_3$ and is the group of symmetries of the 3-cube shown in Figure 1. More generally, the symmetry group of a regular euclidean polytope is an example of a spherical Coxeter group and every irreducible spherical Coxeter group whose Dynkin diagram has no branch point defines the symmetry group of a regular polytope.

Definition 1.7 (Simplices and tilings)**.** For each irreducible spherical Coxeter group $W = \textsc{Cox}(X_n)$, the fixed hyperplanes of its reflections partition the unit sphere into a tiling by simplicial simplices where every dihedral angle is of the form $\frac{\pi}{m}$ for some integer $m > 1$. Let σ be one such top-dimensional simplex. The tiling can be recovered by reflecting in the sides of σ and the group can be recovered since it acts simply transitively on the images of σ in the tiling. This is illustrated in Figure 1 if one intersects the cell structure shown with a small sphere around the center of the cube. The cube in the upper left

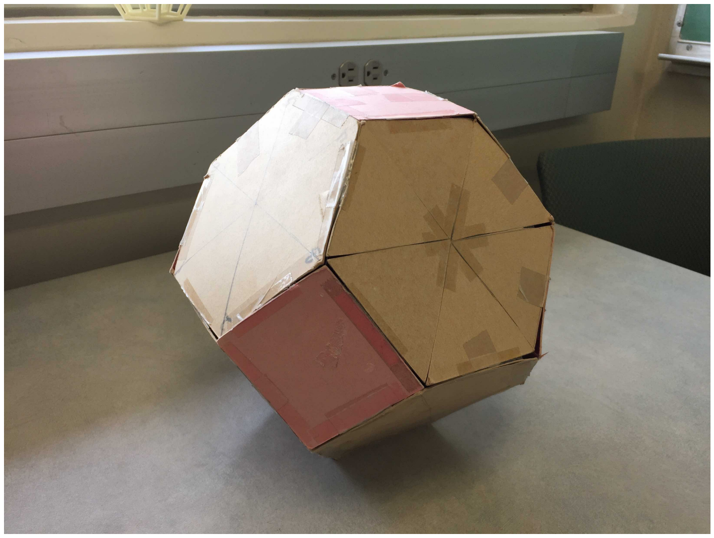


FIGURE 2. The SYM_4 Permutahedron.

shades a tetrahedron which intersects with the small sphere to produce a spherical triangle with dihedral angles $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$. The other three cubes illustrate the image of this tetrahedron under the action of the three reflections in its sides.

Given a spherical Coxeter group W, it is easy to construct a highly regular euclidean polytope called a W-permutahedron.

Definition 1.8 (W-permutahedra)**.** Let $W = \mathrm{COX}(X_n)$ be an irreducible spherical Coxeter group, let σ be one of the top-dimensional simplices in the corresponding tiling of the sphere and let $\mathcal{C}$ be the simplicial euclidean cone generated by non-negative scalar multiples of the points in σ. There is a unique point x in the simplicial cone $\mathcal{C}$ that is distance $\frac{1}{2}$ from each of its facets. The convex hull of the W-orbit of x is euclidean polytope P_W called a (metric) *W-permutahedron*. Because x is distance $\frac{1}{2}$ from each facet, every edge in the 1-skeleton of P_W has unit length. When the spherical Coxeter group W is reducible, one takes a direct metric product of the permutahedra for each of its irreducible components.

The name comes from the special case where W is the symmetric group SYM_n. In this case the convex hull of the $n!$ points formed by all possible permutations of the coordinates of the vector $(1, 2, \ldots, n)$ is a (rescaled) version of the W-permutahedron. The permutahedron associated to the symmetric group on 4 elements is shown in Figure 2.

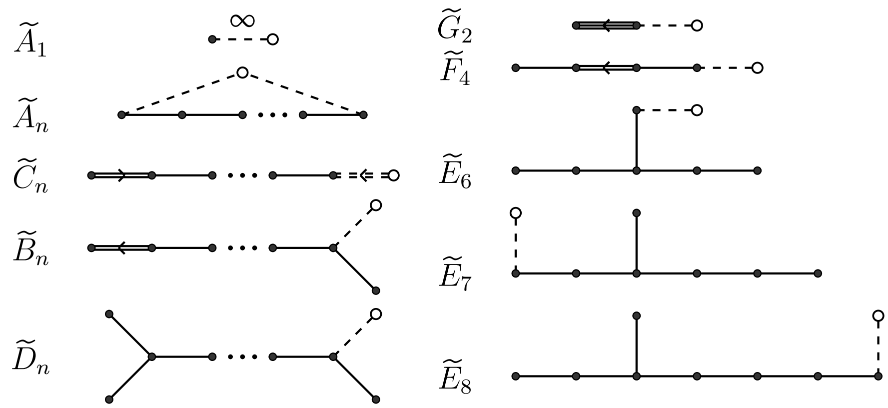


FIGURE 3. Four infinite families and five sporadic examples.

1.3. **Euclidean Coxeter groups.** We now shift our attention to the reflection groups that act geometrically and irreducibly on some euclidean space. Recall that these are the groups whose structure plays a large role in the classification of complex semisimple Lie algebras.

Definition 1.9 (Euclidean Coxeter groups)**.** The irreducible *euclidean Coxeter groups* are the groups generated by reflections that act geometrically and irreducibly on euclidean space. They are also known as *affine Coxeter groups* or *affine reflection groups*. Affine geometry is a type of geoemtry that only preserves parallism and does not preserve distances and angles. Thus *euclidean* is a more approrpriate description of the type of geometry preserved by these groups despite the fact that *affine* is adjective that is standard in the literature. The classification of such groups is also classical and their presentations are encoded in the extended Dynkin diagrams shown in Figure 3. Since the only edge labels that occur in euclidean Coxeter groups are 4 and 6, these are usually replaced by double and triple edges, respectively. There are four infinite families ($\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$) and and five sporadic examples ($\widetilde{G}_2$, $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$). The subscript is the dimension of the euclidean space on which it acts. Removing the white dot and the attached dashed edge or edges from the extended Dynkin diagram $\widetilde{X}_n$ produces the corresponding ordinary Dynkin diagram X_n that defines a closely related spherical Coxeter group.

The extended Dynkin diagrams shown in Figure 3 index many different types of objects. They index, for example, euclidean simplices with restricted dihedral angles.

Definition 1.10 (Euclidean Coxeter simplices)**.** Each extended Dynkin diagram encodes a simplex in euclidean space, unique up to rescaling, with the following properties: the vertices of the diagram are in bijection with the facets of the simplex, i.e. its codimension 1 faces, and vertices s and t in the diagram are connected with 0, 1, 2, or 3 edges iff the corresponding facets intersect with a dihedral angle of $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$, respectively. These conventions are sufficient to describe the simplices associated to each diagram with one exception: the diagram $\widetilde{A}_1$ corresponds to a 1-simplex in $\mathbb{R}^1$ whose facets are its endpoints. These do not intersect and this is indicated by the infinity label on its unique edge. The extended Dynkin diagrams form a complete list of those euclidean simplices where every dihedral angle is of the form $\frac{\pi}{m}$ for some integer $m > 1$. We call these *euclidean Coxeter simplices.*

From these euclidean Coxeter simplices we can construct an associated tiling of euclidean space and recover the corresponding euclidean Coxeter group.

Definition 1.11 (Euclidean tilings)**.** Let $\widetilde{X}_n$ be an extended Dynkin diagram and let σ be the corresponding euclidean n-simplex described above. The group generated by the collection of $n+1$ reflections which fix some facet of σ is the corresponding euclidean Coxeter group $W = \textsc{Cox}(\widetilde{X}_n)$ and the images of σ under the action of W group tile euclidean n-space. As an illustration, consider the extended Dynkin diagram of type $\widetilde{G}_2$. It represents a euclidean triangle with dihedral angles $\frac{\pi}{3}$, $\frac{\pi}{6}$ and $\frac{\pi}{2}$ and the euclidean Coxeter group $\textsc{Cox}(\widetilde{G}_2)$ generated by the reflections in its sides is associated with the tiling of $\mathbb{R}^2$ by congruent 30-60-90 triangles shown in Figure 4. In this recycled figure, the shaded portions can be safely ignored: the focus here is on the underlying tiling.

2. Tits

The general theory of Coxeter groups was pioneered by Jacques Tits in the early 1960s with the spherical and euclidean Coxeter groups as key examples that motivate their introduction. In that first unpublished paper from 1961, Tits defines general Coxeter groups using the simple presentations we know today and he proves that every Coxeter group has a faithful linear representation preserving a symmetric bilinear form (Theorem 2.4). As a consequence, every Coxeter group has a solvable word problem. In 2013 this historically significant early article was finally published as part of his collected works. See Volume I, Chapter 43, p.803-818 in [Tit13]. In this section we recall the definition

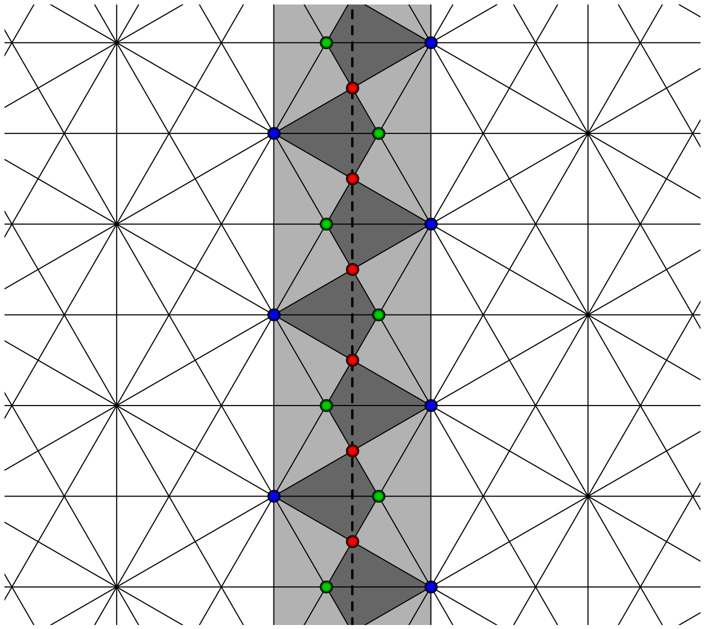


FIGURE 4. The euclidean Coxeter Group $\text{Cox}(\widetilde{G}_2)$.

of Tits' linear representation. For additional details see standard references such as [Hum90], [BB05] or [Dav08]. We begin with the Coxeter matrix.

Definition 2.1 (Coxeter matrix)**.** Let W be a Coxeter group with generators $s_1, s_2, \ldots, s_n$ and let $m_{ij} = m_{ji}$ be the length of the Artin relation involving s_i and s_j. When $i = j$ we define $m_{ii} = 1$ and when there is no relation between s_i and s_j we define $m_{ij} = \infty$. The *Coxeter matrix* M is the n by n matrix whose (i,j)-entry is $\cos(\pi - \frac{\pi}{m_{ij}})$. When $m_{ij} = \infty$, $\frac{\pi}{m_{ij}} = 0$ and the coresponding matrix entry is -1. The Coxeter matrix M can be used to define a *symmetric bilinear form* on $V = \mathbb{R}^n$ by the formula $\langle u, v\rangle = u^{\text{tr}} M v$ for column vectors u and v.

Example 2.2. Let $W = \langle a, b, c \mid ab = ba, bcb = cbc, a^2 = b^2 = c^2 = 1\rangle$ be a Coxeter group with $s_1 = a$, $s_2 = b$ and $s_3 = c$. Then $m_{12} = 2$, $m_{13} = \infty$ and $m_{23} = 3$ and the corresponding Coxeter matrix is

$$M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 1 \end{bmatrix}.$$

This group W is a *hyperbolic triangle group* and we use it as a running example for the remainder of Part 1. For every Coxeter group W Tits defined a linear representation as follows.

Definition 2.3 (Tits representation)**.** Let W be a Coxeter group with generators $s_1, s_2, \ldots, s_n$, let $V = \mathbb{R}^n$ be a vector space with standard basis $\{e_1, e_2, \ldots, e_n\}$ and let $\langle u, v\rangle$ be the symmetric bilinear form on V defined using the Coxeter matrix M. The i-th generator s_i is sent to the linear transformation r_i defined by the equation $r_i(v) = v - 2\langle v, e_i\rangle e_i$. This is a reflection that sends the basis vector e_i to $-e_i$ and it fixes pointwise its orthogonal complement. It is easy to check that each r_i is an involution as required and that all the other relations are satisfied so that this map on generators extends to a group homomorphism from W to $GL(V)$. Moreover, each generator preserves the symmetric bilinear form and thus the image of the entire group W is contained in the corresponding orthogonal group of invertible linear transformations that preserve this form.

Tits proved is that this linear representation is always faithful.

Theorem 2.4 (Faithful representation)**.** *For every Coxeter group W, its Tits representation is faithful.*

Remark 2.5 (Types of Coxeter groups). Since the Coxeter matrix is real and symmetric, it is diagonalizable and all of its eigenvalues are real. The *signature* of a Coxeter matrix is a triple (p, n, z) where p, n and z are the number of positive, negative and zero eigenvalues, respectively, counted with multiplicity. Coxeter groups can be coarsely classified by the signature of the symmetric bilinear forms they preserve. The irreducible spherical and euclidean groups are those that preserve positive definite ($n = z = 0$) and positive semi-definite forms ($n = 0$, $z > 0$), respectively. When the Coxeter matrix has at least one zero eigenvalue, the corresponding symmetric bilinear form is *singular*.

Singular forms cause a slight problem. For now we assume that the form is nonsingular and return to the singular case at the end of the section. For every Coxeter group, one can use its faithful linear representation to can find a nice space on which it acts. This might be a sphere, euclidean space, hyperbolic space, or more generally the interior of its Tits cone.

Definition 2.6 (Tits cone)**.** Let W be a Coxeter group with Coxeter matrix M. When M is nonsingular, the hyperplanes orthogonal to the standard basis vectors e_i bound a closed simplicial cone $\mathcal{C}$. The the union of the images of $\mathcal{C}$ under the action of W is called the *Tits cone.*

Example 2.7 (A hyperbolic example)**.** The Coxeter matrix of the group described in Example 2.2 has two positive eigenvalues and one negative eigenvalue. Thus the linear transformations that preserve the bilinear form preserve the surfaces of the form $x^2 + y^2 - z^2 = k$ (once the matrices are rewritten with respect to a different basis). The level set of this form is a double cone when $k = 0$, a hyperboloid of one sheet when $k = 1$ and a hyperboloid of two sheets when $k = -1$. The original basis vectors e_1, e_2 and e_3 have length 1 and live in the hyperboloid of one sheet. For each e_i we define the half-space of points that have a non-negative inner product with e_i and let $\mathcal{C}$ be the simplicial cone that is their intersection. Two of the three rays of $\mathcal{C}$ intersect the interior of the upper sheet of the hyperboloid of two sheets and the third lies in the light cone. The intersection of $\mathcal{C}$ with the upper sheet of the hyperboloid of two sheets is a hyperbolic triangle with one ideal vertex. This is easier to see in the disk model. The upper sheet of the hyperboloid of two sheets is one of the standard models of the hyperbolic plane and the Coxeter group W acts on this plane generated by reflections. In the hyperbolic plane there is a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and 0 where one of its vertices is an ideal vertex that corresponds to the intersect of $\mathcal{C}$ with the upper sheet. The tiling of the hyperbolic plane by the reflections of this triangle in its sides is sketched in Figure 5. The union of the orbits of the closed hyperbolic triangle under the action of W includes every point in the interior of the disk plus the countable set of points that form the orbit of the ideal vertex of the triangle under the action of W. Back in $\mathbb{R}^3$ the Tits cone contains the positive linear multiples of every point in the upper sheet of the hyperboloid of two sheets plus a countable set of rays in the upper cone of the double cone. Passing to the *interior of the Tits cone* removes the countable set of rays.

When the Coxeter matrix M is singular, an extra step is needed.

Remark 2.8 (Singular forms). When the Coxeter matrix M is singular, the hyperplanes orthogonal to the basis vectors e_i do not bound a simplicial cone. Tits' solution is to replace each matrix in the linear representation with its inverse transpose. In this *contragradient representation* the generators are still reflections that fix a hyperplane and send some vector to its negative, but now these hyperplanes always bound a simplicial cone. The orbit of this cone under the action of W is the official definition of the *Tits cone*. When M is non-singular the two representations are equivalent and this step is optional.

3. Davis

In [Dav83] Mike Davis introduced a very cell complex space for each Coxeter group W. It can be constructed from the cell structure dual

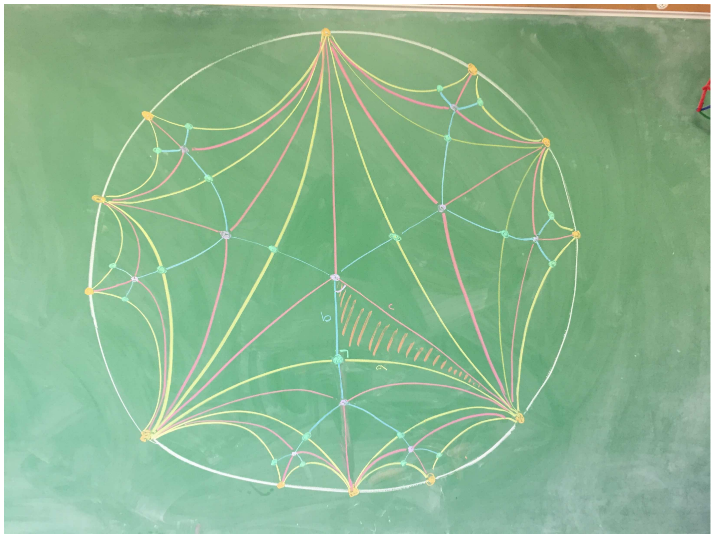


FIGURE 5. A tiling of the hyperbolic plane generated by the reflections in the sides of a triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$ and 0. The original triangle has been shaded.

to the Tits cone or from the Cayley graph by adding various W'-permutahedra. We continue using the $(2,3,\infty)$ Coxeter group from Example 2.2 to illustrate these ideas.

Definition 3.1 (Cayley graph)**.** The unoriented (right) *Cayley graph* of W with respect to its standard generating set S is the 1-skeleton of the cell complex dual to the Tits cone. Concretely, the Cayley graph has a vertex labeled w corresponding to image $w\mathcal{C}$ of the simplicial cone $\mathcal{C}$ for each $w \in W$ and two vertices are connected by an undirected edge if and only if the corresponding simplicial cones share a common codimension 1 face.

Example 3.2 (Cayley graph)**.** A portion of the Cayley graph of the Coxeter group defined in Example 2.2 is shown in Figure 6 superimposed on the triangular tiling of the hyperbolic plane. When the disk model of the hyperbolic plane is replaced by the hyperboloid model and the points in the upper sheet of the hyperboloid replaced by all of their positive scalar multiples, the Cayley graph as shown in Figure 6 can be seen to correspond to the description given in Definition 3.1.

The Davis complex of a Coxeter group W can be constructed by attaching permutahedra to its unoriented Cayley graph.

Definition 3.3 (Davis complex)**.** Let W be a Coxeter group and let Γ be its unoriented Cayley graph. For each subset $S' \subset S$ such that

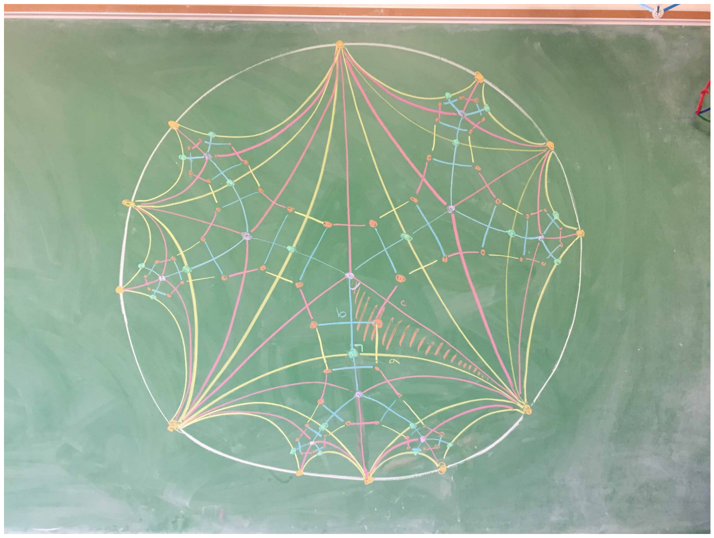


FIGURE 6. A portion of the Cayley graph of our running example superimposed on the corresponding triangular tiling of the hyperbolic plane.

the parabolic subgroup $W' = \langle S' \rangle$ is finite and for each element $w \in W$ we attach a metric W'-permutahedron to the vertices of Γ labeled by the elements in the coset wW' in W. When $w'W'$ and $w''W''$ are two such cosets and $w'W' \subset w''W''$ then we identify the W'-permutahedron attached to the vertices labeled by the elements in $w'W'$ as a face of the W''-permutahedron attached to the vertices labeled by the elements in $w''W''$.

The main idea behind Davis' construction should be clear from our running example.

Example 3.4 (Davis complex)**.** A portion of the Davis complex for our running example is shown in Figure 7. For each subset S' of the standard generating set S generates a finite subgroup W', there are many copies of the 1-skeleton of a W'-permutahedron in the Cayley graph of W whose vertices are labeled by the elements in the coset wW'. In this case, generators a and b generate Klein 4-group and b and c generate a 6-element dihedral group. The corresponding permutahedra are the regular square and the regular hexagon with unit side lengths. A square is attached to every 4-cycle labeled by a and b and a hexagon is attached to every 6-cycle labeled by b and c.

The piecewise euclidean metrics on the various permutahedra added to the Cayley graph to form the Davis complex are compatible where

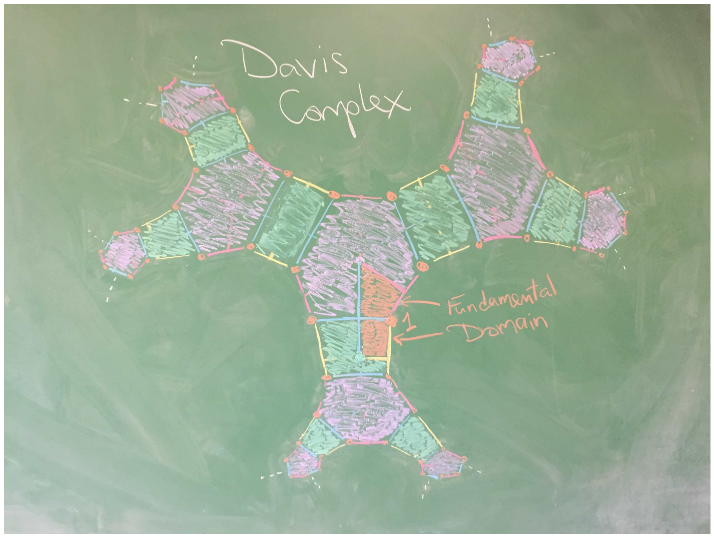


FIGURE 7. A portion of the Davis complex for our running example.

they overlap and the result is a global geodesic metric called the *Moussong metric*. Gabor Moussong was a student of Mike Davis and in his 1988 dissertation he proved that the Davis complex with the Moussong metric is what is known as a complete CAT(0) space.

Theorem 3.5 (CAT(0))**.** *Every Coxeter group W is a* CAT(0) *group. In particular, the Davis complex of W with the Moussong metric is a complete* CAT(0) *space and the action of W on its Davis complex is geometric, i.e. properly discontinuous, cocompact and by isometries.*

A proof of this result can be found in [Dav08]. For more information about CAT(0) spaces and CAT(0) groups see [BH99].

Remark 3.6 (Cell structure and stabilizers). The cell structure of the Davis complex as described here is not quite the same as the one originally used by Davis in [Dav83]. The official version is a subdivision of the one given here and it has a number of technical advantages. In Davis' cell structure the fixed spaces of the generating reflections are subcomplexes, the natural fundamental domain for the action W (which has been shaded in Figures 7 and 8) is a subcomplex and all points in the same cell has the same stabilizer. The cell structure used here, on the other hand, is conceptually easy to describe and it also makes it easy to describe the Salvetti complex as a modified version of the Davis complex as we do in the next section.

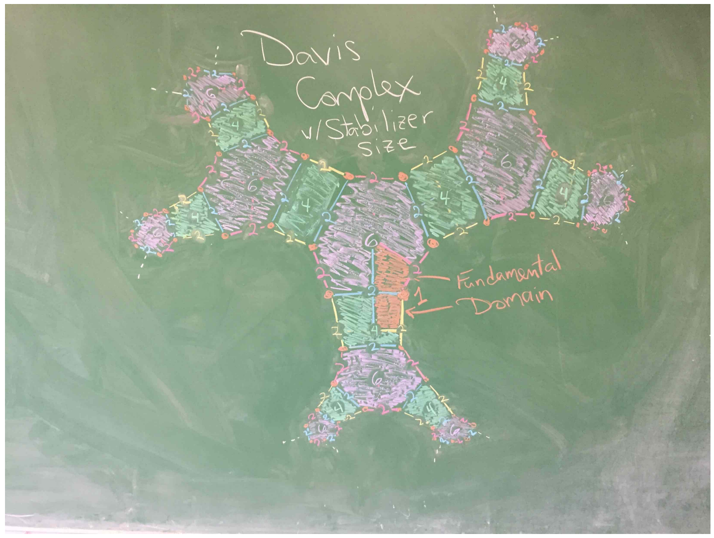


FIGURE 8. A portion of the Davis complex for our running example with stablizer information added.

4. SALVETTI

In this section we very briefly describe the spaces that enable the transition from Coxeter groups to Artin groups. For the details see Luis Paris' survey articles [Par14a, Par14b] and the references therein. We begin with the notion of the braid group of a group action.

Definition 4.1 (Braid group of an action)**.** For any group G acting on a space X a point $x \in X$ is said to be *regular* when its G-stabilizer is trivial, the *space of regular orbits* is the quotient of the subspace of regular points by the free G-action and the *braid group of* G *acting on* X is the fundamental group of the space of regular orbits. Thus, to find the braid group of a group G acting on a space X one first removes the points with non-trivial stabilizers, then quotients by the resulting free G-action and finally takes the fundamental group of the quotient.

The name "braid group" alludes to the fact that when the symmetric group $\textsc{Sym}_n$ acts on $\mathbb{C}^n$ by permuting coordinates, the braid group of this action is Artin's classical braid group $\textsc{Braid}_n$.

Example 4.2 (Artin's braid group)**.** The braid group of $\textsc{Sym}_n$ acting on $\mathbb{C}^n$ by coordinate permutation is Artin's braid group. To see this note that a point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ can be represented by n labeled points in the complex plane. A point in $\mathbb{C}^n$ is not regular if and only if it has coordinates that are equal. Thus the space of regular points are parameterized by n distinct labelled points in $\mathbb{C}$. Quotienting

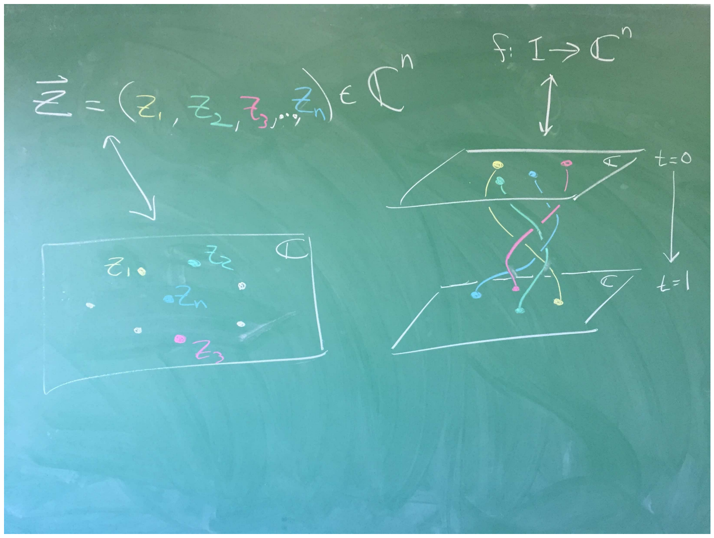


FIGURE 9. The Braid Arrangement

by the free action of SYM_n corresponds to removing the labels. Finally a loop based at a particular point in $\mathbb{C}^n$ corresponds to a motion of n unlabeled points in $\mathbb{C}$ that starts and ends at the same configuration and the points remain distinct throughout. Tracing this out over time reveals a braid. See Figure 9.

For each Coxeter group with n standard generators, the interior of its Tits cone is a subspace of $\mathbb{R}^n$. Artin groups are defined using the corresponding subspace of $\mathbb{C}^n$.

Definition 4.3 (Artin groups)**.** For each labeled diagram Γ the Artin group $\text{ART}(\Gamma)$ can be defined as the braid group of the action of $\text{COX}(\Gamma)$ on the interior of its *complexified Tits cone* which we denote $\mathbb{C}\text{TITS}^\circ(\Gamma)$. The only points with non-trivial stabilizers are those in the fixed hyperplanes of the reflections in $\text{COX}(\Gamma)$ so $\text{COX}(\Gamma)$ acts freely on $\mathbb{C}\text{TITS}^\circ(\Gamma) \setminus \mathcal{H}$ where $\mathcal{H}$ is the union of these hyperplanes.

Rather than work with this space directly, or even define it more precisely, we describe instead a space homotopy equivalent to it which is much easier to visualize and understand. It is defined using oriented permutahedra.

Definition 4.4 (Oriented W-Permutahedra)**.** Let v be a vertex of a polytope P that is a W-permutahedron. There is a unique vertex v' that is directly oppostive v so that the vector from v to v' passes through the center of P. We orient each edge in the 1-skeleton of P so

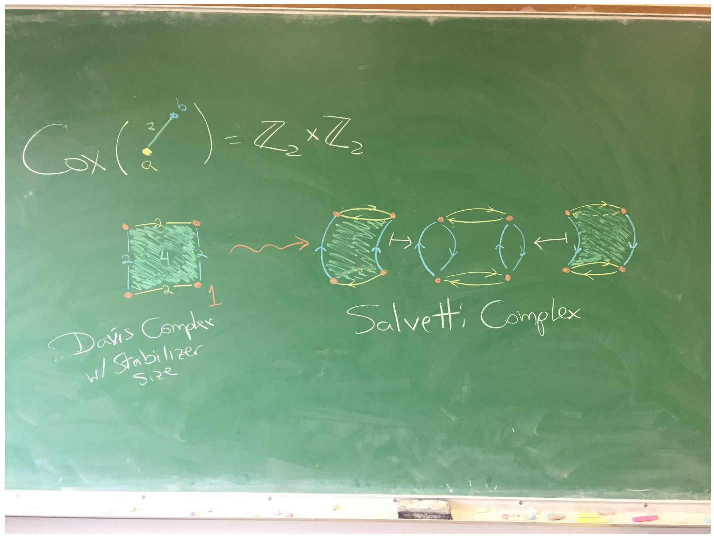


FIGURE 10. An example of the Salvetti complex as an oriented version of the Davis complex.

that its dot product with the vector from v to v' is positive. We are essentially using this vector to define a height function. Note that P has as many orientations as it has vertices.

Definition 4.5 (Oriented Davis Complex)**.** Let Γ be a labeled diagram and let $\textsc{Davis}(\Gamma)$ be the Davis complex of the corresponding Coxeter group $\textsc{Cox}(\Gamma)$. The *oriented Davis complex* keeps the same vertex set but it replaces each W'-permutahedron in the Davis complex with multiple copies, one for each possible orientation. When a permutahedron has a smaller permutahedron as a face, the oriented version of the larger one is attached to the oriented version of the smaller one where the orientations are compatible. The resulting space is also known as the *Salvetti complex* and we denote it be $\textsc{Salv}(\Gamma)$. The Coxeter group $\textsc{Cox}(\Gamma)$ acts freely on the Salvetti complex $\textsc{Salv}(\Gamma)$ and its quotient is a 1-vertex complex that has one oriented W'-permutahedron for each subset S' of the standard generating set S for which $W' = \langle S' \rangle$ is finite. Its fundamental group is the Artin group $\textsc{Art}(\Gamma)$. In some parts of the literature, the name "Salvetti complex" is used to denote this 1-vertex quotient.

Example 4.6 (Oriented Davis Complex)**.** Let $W = \textsc{Cox}(\Gamma)$ be the 2-generator Coxeter group where the generators commute. The group W is the Klein 4 group and its Davis complex is a unit square. The corresponding Salvetti complex has the same 4 vertices, each original

$$\begin{array}{lcll} \textsc{Cox}(\Gamma) & \curvearrowright & \textsc{Tits}^\circ(\Gamma) & \cong \textsc{Davis}(\Gamma) \\ \textsc{Cox}(\Gamma) & \curvearrowright & \mathbb{C}\textsc{Tits}^\circ(\Gamma) \setminus \mathcal{H} & \cong \textsc{Salv}(\Gamma) \\ \textsc{Art}(\Gamma) & = & \pi_1(\textsc{Salv}(\Gamma)/\textsc{Cox}(\Gamma)) & \end{array}$$

FIGURE 11. A summary of the spaces and groups discussed in Part 1.

edge becomes two oriented edges and the original square becoomes 4 oriented squares. The result is a torus with 4 vertices, 8 edges and 4 squares on which the group W acts freely. See Figure 10. In order to make the torus easier to see, I have shown the 4 squares paired up into two annuli being attached to the 1-skeleton. The quotient by W is a torus with 1 vertex, 2 edges and 1 square whose fundamental group is the corresponding Artin group isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

The main theorem is that the Salvetti complex is homotopically equivalent to space used to officially define the Artin groups.

Theorem 4.7 (Salvetti complex)**.** *For each labeled diagram* Γ*, the space* $\mathbb{C}\textsc{Tits}^\circ(\Gamma) \setminus \mathcal{H}$*, interior of the complexified Tits cone with its hyperplanes removed, is equivariantly homotopy equivalent to the Salvetti complex* $\textsc{Salv}(\Gamma)$*.*

Definition 4.8 (Universal cover)**.** The $K(\pi,1)$ *conjecture* asserts that for every diagram Γ the Salvetti complex $\textsc{Salv}(\Gamma)$ is a classifying space for the Artin group $\textsc{Art}(\Gamma)$, which is true if and only if its is contractible. By Theorem 4.7 this is equivalent to studying $\mathbb{C}\textsc{Tits}^\circ(\Gamma) \setminus \mathcal{H}$ and its universal cover.

Remark 4.9 (Summary). A summary of the spaces and groups discussed in Part 1 is shown in Figure 11. For each labeled diagram Γ the Coxeter group $\textsc{Cox}(\Gamma)$ acts on the interior of its Tits cone which is homotopy equivalent to the piecewise euclidean CAT(0) space known as the Davis complex and the corresponding action on the Davis complex is geometric. Next, the Coxeter group $\textsc{Cox}(\Gamma)$ acts freely on the complexified interior of its Tits cone with the fixed hyperplanes removed, a space which is homotopy equivalent to the Salvetti complex $\textsc{Salv}(\Gamma)$. Finally the quotient of the Salvetti complex $\textsc{Salv}(\Gamma)$ by the free $\textsc{Cox}(\Gamma)$ action is a space whose fundamental group is the Artin group $\textsc{Art}(\Gamma)$.

Part 2. On the edge

In Part 2 we are "on the edge" in the sense that the border separating those Artin groups whose structure we basically understand from those about which we know very little has the feeling of the edge of a sharp cliff. On one side there are groups about which we know quite a bit and on the other side there are groups about which we essentially know nothing beyond a few basic facts about small slices of the groups. Continuing the metaphor, the goal of this part is to describe the location of this cliff edge where our basic knowledge of the structure of Artin groups completely disappears. A secondary goal is to explain why I say that *most* Artin groups remain fundamentally mysterious. The sections in this part discuss the big picture, the Artin groups that we understand, the parts of Artin groups that we understand and finally the Artin groups that we do not understand.

5. Big Picture

In this section I focus on the big picture and the long view. Artin groups are closely related to Coxeter groups, are defined by simple presentations and have been studied since the 1970s. Highlights of this early work include the articles by Pierre Deligne [Del72] and by Egbert Brieskorn and Kyoji Saito [BS72] that investigate the Artin groups corresponding to finite Coxeter groups as well as the 1983 dissertation by Harm van der Lek under the supervision of Looijenga [vdL83] that derives presentations for arbitrary Artin groups viewed as the braid groups of Coxeter groups acting on the interior of their complexified Tits cones. We begin with a comment that drives much of our intuition about the class of Artin groups.

Remark 5.1 (Algorithmic properties). Over the years there has been a fair amount of process in understanding some special classes of Artin groups and to date every Artin group that has been understood and every portion of an Artin group that has been understood has turned out to have very good algorithmic properties. The natural conjecture is that all Artin groups are well-behaved.

On the other hand, in the 1990s I heard Ruth Charney give a survey talk about what was known and not known about Artin groups and I remember being amazed at the vast amount that was not known about these groups, particularly since the defining presentations for Artin groups are so nice and so much is known about Coxeter groups. In many ways this part is a lack-of-progress report since the boundary between those Artin groups where we know how to solve the word problem and those where we do not has changed very little over the past two decades. There is also no getting around the fact that the algorithm properties of many Artin groups are very poorly understood. In a relatively recent survey article [GP12] Eddy Godelle and Luis Paris highlight some of the basic conjectures that remain wide open about general Artin groups.

Conjecture 5.2. *There are four basic conjectures about irreducible Artin groups.*

A) *Every Artin group is torsion-free*
B) *Every non-spherical Artin group has trivial center*
C) *Every Artin group has a solvable word problem*
D) *Artin groups satisfy the $K(\pi, 1)$ conjecture*

Remark 5.3 (A small example, revisited). Consider the Artin group defined by the small presentation in Example 1.3. Even for this example there is very little that is known. By the work of Ruth Charney we know that the $K(\pi, 1)$ conjecture is true for this group and as a consequence it is also torsion-free [Cha04], but we do not know how to solve its word problem and it is unclear whether or not it has a trivial center. For graphs that are even slightly more complicated, all four of the basic conjectures are open.

For comparison, consider how much more we know about general Coxeter groups. Every Coxeter group is defined by a simple presentation, has a faithful linear representation, is an automatic group and is a CAT(0) group. In particular, every Coxeter group act geometrically on a simply-connected non-positively curved piecewise euclidean cell complex known as the Davis complex with Moussong's metric. They fit into many of the powerful theories of geometric group theory and are algorithmically very nice. The contrast between what we know about Coxeter groups and what we know about Artin groups is very stark.

6. Known groups

This section describes those classes of Artin groups for which we know how to solve the word problem. After a short structural result that greatly simplifies the types of groups we need to consider, we discuss two broad categories of Artin groups: those where the corresponding Coxeter matrix is positive semi-definite and those where the dimension of the corresponding Salvetti complex is very low.

6.1. **Local properties.** We begin with an early structural result by van der Lek that makes it easy to reduce to the case where every pair of generators satisfies some nontrivial braid relation. Recall the notion of a parabolic subgroup.

Definition 6.1 (Parabolic subgroups)**.** Let Γ be a labeled diagram, let $A = \textsc{Art}(\Gamma)$ be the corresponding Artin group and let the set S indexing the vertices of Γ be the standard generating set of A. For each subset $S' \subset S$ let Γ' be the full subgraph of Γ with vertex set S' and let A' be the subgroup of A generated by S'. The subgroup A' is called a *parabolic subgroup* and since it satisfies the braid relations among the generators in S' that are encoded by the edges of Γ', A' is the homorphic image of the Artin group $\textsc{Art}(\Gamma')$ defined by the portion of the original Artin presentation restricted to S'.

In 1983 Harm van der Lek proved the following result as part of his dissertation [vdL83]. For a modern proof see [Par14a].

Theorem 6.2 (Parabolic subgroups)**.** *Let $A = \textsc{Art}(\Gamma)$ be an Artin group with standard generating set S and diagram Γ. If A' is a parabolic subgroup of A generated by the subset $S' \subset S$ and Γ' is the corresponding subgraph of Γ, then the natural homomorphism from $\textsc{Art}(\Gamma')$ onto A' is an isomorphism. In particular, A' is also an Artin group and its Artin presentation is the obvious one obtained by restricting the Artin presentation of A to the generators in S'.*

Remark 6.3 (Injectivity). Van der Lek proved A' is an Artin group by proving that the map from $\textsc{Art}(\Gamma')$ to A' is injective. We should note that this is somewhat surprising from an algebraic perspective since neither A nor A' necessarily has a decidable word problem. The proof is elementary algebraic topology: if Y is a subspace of X and there is a retraction from X to Y, then $\pi_1(Y)$ injects into $\pi_1(X)$, and this does not require either fundamental group to have a decidable word problem. Van der Lek works with the complexified Tits cones with the fixed hyperplanes removed and finds such a retraction. Recently Ruth Charney and Luis Paris showed that these parabolic subgroups are also convex in their Cayley graphs [CP14] by applying similar ideas to the Salvetti complex.

As an immediate consequence of Theorem 6.2, an Artin group with a pair of standard generators that do not satisfy a nontrivial braid relation (i.e. where some m is equal to ∞) can be decomposed as an amalgamated free product of Artin groups with strictly fewer generators.

Corollary 6.4 (Decompositions)**.** *Let $A = \textsc{Art}(\Gamma)$ be an Artin group with standard generating set S. If there exist generators a and b in S*

that do not satisfy a nontrivial braid relation, then the Artin group A *is an amalgamated free product of the parabolic subgroups generated by* $S \setminus \{a\}$ *and* $S \setminus \{b\}$ *amalgamated along the parabolic subgroup generated by* $S \setminus \{a, b\}$.

Recall that an amalgamated free product has a solvable word problem if and only if the factor groups and the amalgamating subgroup have solvable word problems. Thus it makes sense to make the following definition.

Definition 6.5 (Local properties)**.** An Artin group in which every pair of generators satisfies a non-trivial braid relation is sometimes called 2-*local* in the literature. Let $\mathcal{A}$ be a collection of 2-local Artin groups. We say an Artin group is *locally in* $\mathcal{A}$ if every (irreducible) parabolic subgroup that is 2-local belongs to $\mathcal{A}$. The notation $\textsc{loc}(\mathcal{A})$ refers to the collection of all Artin groups that are locally in $\mathcal{A}$. Note that by Corollary 6.4 the Artin groups in $\textsc{loc}(\mathcal{A})$ can be repeatedly decomposed until we reach a point where the factor groups and amalgamating subgroups are (direct products of groups) in $\mathcal{A}$ and thus the original group can be built up as an iterated amalgamated free products of Artin groups where the groups at the start of the process all belong to $\mathcal{A}$. In particular, if all of the Artin groups in $\mathcal{A}$ have a solvable word problem, then all of the groups in $\textsc{loc}(\mathcal{A})$ have a solvable word problem.

We should note that this notion of a local property is not standard in the literature but as should become clear, it is a useful notation to have around for succinctly describing various classes of Artin groups.

6.2. **Positive semi-definite Coxeter matrix.** We now turn our attention to the Artin groups for which we know how to solve the word problem and they roughly fall into two categories. The first of these is where the Coxeter matrix positive semi-definite. The easiest class to define and the one where we know the most about the groups is the class of right-angled Artin groups.

Definition 6.6 (Right-angled Artin groups)**.** The simplest Artin group is the one with only 1 generator and it is the integers $\mathbb{Z}$. If every irreducible component of an Artin group is $\mathbb{Z}$ then all of its generators commute and it is in the collection $\mathcal{Z}$ of finitely generated free abelian groups. An Artin group is *right-angled* if every relation is a commutation, i.e. every m is either 2 or ∞, and the collection of all right-angled Artin groups is the same as the class $\textsc{loc}(\mathcal{Z})$ of all locally abelian Artin groups or the class $\textsc{loc}(\mathbb{Z})$ of all locally $\mathbb{Z}$ Artin groups.

Every right-angled Artin group is the fundamental group of a non-positively curved cube complex and this makes it easy to solve all four of the basic conjectures for these groups (Conjecture 5.2). Partly because their algebraic and geometric structures are so well understood, they have played a prominent role in the construction of the Bestvina-Brady examples in the late 90s [BB97] and in the recent work of Agol, Wise and their coauthors related to the fine structure of hyperbolic 3-manifolds [Ago08, Ago13, Wis12].

Definition 6.7 (Spherical Artin groups)**.** An Artin group $A = \textsc{Art}(\Gamma)$ is called *spherical* or *finite-type* if the corresponding Coxeter group $W = \textsc{Cox}(\Gamma)$ acts geometrically on a sphere or, equivalently, W is finite.

The systematic study of Artin groups began in 1972 with the pair of adjacent articles in the *Inventiones* by Pierre Deligne [Del72] and by Egbert Brieskorn and Kyoji Saito [BS72] in which they establish all four of the basic conjectures for the class $\mathcal{S}$ of all spherical Artin groups. The Deligne argument is geometric and only covers the crystallographic Artin groups. The Brieskorn-Saito argument is more algebraic and it applies to all spherical Artin groups. Dehornoy and Paris axiomatized these algebraic arguments to define *Garside structures.* The class $\textsc{loc}(\mathcal{S})$ of locally spherical Artin groups is called *FC-type* in the literature.

An Artin group is *euclidean* if the corresponding Coxeter group is euclidean. As noted earlier, these groups are more commonly referred to as *affine* in the literature. A few years ago Robert Sulway and I were able to prove the following result.

Theorem 6.8 (Euclidean Artin groups)**.** *Every irreducible euclidean Artin group is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

Theorem 6.8 answers three of the four basic questions about these Artin groups. The proof uses an alternative infinite presentation called the "dual presentation" and an alternative version of Garside theory suitable for infinite presentations. It does not answer the $K(\pi,1)$ conjecture for these groups because the classifying spaces that we construct are not known to be homotopy equivalent to the one-vertex versions of the Salvetti complexes for these groups. See [MS] for the proof and [McC] for a survey of the results leading up to this result. Let $\mathcal{E}$ denote the class of euclidean Artin groups in the broad sense that also includes the spherical ones. The class $\textsc{loc}(\mathcal{E})$ of locally euclidean Artin groups is a natural extension of the Artin groups of FC-type and all of these groups are now known to have a solvable word problem. We should also note that the $K(\pi, 1)$ conjecture has been shown to hold for some of euclidean Artin groups. ADD CITES

6.3. **Low-dimensional Salvetti complex.** We now turn our attention to Artin groups whose Salvetti complex has a very low dimension. These Artin groups have a completely different flavor from those in the previous subsection.

Definition 6.9 (Large-type Artin groups)**.** An Artin group is said to be of *large-type* if its presentation has no commuting relations, i.e. if every braid relation has length at least 3. We write $\mathcal{L}$ for the class of all large-type Artin groups.

The class $\mathcal{L}$ can be understood using a variation of traditional small cancellation theory. The original article by Appel and Schupp is [AS83] and in [McC10] I give an alternative geometric proof of their key lemma.

Definition 6.10 (2-dimensional Artin groups)**.** An Artin group is said to be 2-*dimensional* if every 3-generator parabolic subgroup is not spherical. This is equivalent to requiring that the corresponding Salvetti complex be at most 2-dimensional. We write $2\mathcal{D}$ to denote this class of Artin groups. Note that this class includes all of the large-type Artin groups. Chermak solved the word problem for the class $2\mathcal{D}$ [Che98].

Definition 6.11 (Charney's extension)**.** In [Cha04] Ruth Charney was able to prove that the $K(\pi, 1)$ conjecture holds for a slight extension of the class of 2-dimensional Artin groups. We write $\mathcal{C}$ for the class of Artin groups covered by Charney's results. Since her proof uses the structure of the non-locally finite Deligne complex rather than the Salvetti complex it is unclear, at least to me, whether or not this implies that the word problem is solvable for the groups in $\mathcal{C}$. The irreducible pieces of the 2-local groups in Charney's class have Salvetti complexes that are at most 3-dimensional.

6.4. **Summary.** The following of a summary of the classes of Artin groups described above. Recall that $\mathcal{Z}$ is the class of free abelian groups, $\textsc{loc}(\mathcal{Z})$ is the class of right-angled Artin groups, $\mathcal{S}$ is the class of all spherical Artin groups, $\textsc{loc}(\mathcal{S})$ is the class of Artin groups of FC-type, $\mathcal{E}$ is class of euclidean Artin groups (in the broad sense that includes the spherical ones) and $\textsc{loc}(\mathcal{E})$ is the new class of all locally euclidean Artin groups. There are obvious inclusions among them.

$$\begin{array}{ccccc} \textsc{loc}(\mathcal{Z}) & \hookrightarrow & \textsc{loc}(\mathcal{S}) & \hookrightarrow & \textsc{loc}(\mathcal{E}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{Z} & \hookrightarrow & \mathcal{S} & \hookrightarrow & \mathcal{E} \end{array}$$

In addition $\mathcal{L}$ is the class of Artin groups of large-type, $2\mathcal{D}$ is the class of 2-dimensional Artin groups and $\mathcal{C}$ is Charney's extension. All three of these classes is already locally closed. The inclusions among them are as follows.

$$\mathcal{L} \hookrightarrow 2\mathcal{D} \hookrightarrow \mathcal{C}$$

The word problem has been solved for the Artin groups in $\textsc{loc}(\mathcal{E} \cup 2\mathcal{D})$, i.e. the Artin groups where every 2-local piece is either euclidean of 2-dimensional and, to the best of my knowledge, these are the only Artin groups where the word problem has been solved. A similar statement can be made for the $K(\pi,1)$ conjecture. To the best of my knowledge, the $K(\pi,1)$ conjecture has only been shown to hold for those Artin groups where every 2-local piece is in Charney's extension $\mathcal{C}$ or is a spherical Artin group in $\mathcal{S}$ or it is one of a small number of euclidean Artin groups where the conjecture has been solved. See for example [CMS10].

Remark 6.12 (Cubulating Artin groups). Given the spectacular successes that Wise, Agol and their coauthors have achieved by reducing questions about hyperbolic 3-manifolds to questions about right-angled Artin groups, it is natural to ask whether a similar reduction is possible for arbitrary Artin groups. This reduction process is called *cubulating a group*, so the question becomes can all Artin groups be cubulated? In particular does every Artin group virtually embed into a right-angled Artin group in a particulary nice way? The answer is no even for some 2-dimensional Artin groups and for the 4-string braid group. See [HJP16] for details.

7. Known parts

In this section we shift our focus to the parts of Artin groups whose structure is known. We begin with the Artin monoids, which turn out to be much easier to work with than Artin groups.

Definition 7.1 (Artin monoids)**.** An *Artin monoid* is the monoid defined by the same presentation as the corresponding Artin group. The word problem for an Artin **monoid** is trivially solvable because the all of the relations preserve length and there are only finitely many words of a fixed length.

Although the brute force algorithm is not practical, it does mean that Artin monoids can be easily investigated, and there do exist good algorithms for working with Artin monoids. As in the case of a parabolic subgroup, there is a natural monoid homomorphism from the Artin monoid to the Artin group with the same presentation, but it is not immediately obvious whether or not this natural map is injective. This question remained open for a number of years before it was finally resolved by Luis Paris in 2002 [Par02].

Theorem 7.2 (Artin monoids inject). *Every Artin monoid injects into the corresponding Artin group.*

Somewhat surprisingly, the proof of this theorem is derived from the proof that braid groups are linear, a result proved independently by Daan Krammer and Stephen Bigelow around 2000 [Kra00, Big01, Kra02]. Bigelow's proof was more topological while Krammer's proof was more algebraic. François Digne and independently Arjeh Cohen and David Wales extended Krammer's algebraic proof to show that all spherical Artin groups are linear [CW02, Dig03]. Luis Paris extended this representation further to arbitrary Artin groups, but in the general case the representation only shows that the positive monoid has a faithful linear representation. And as a consequence, the map from the monoid to the group must be injective. The next portion of an Artin group that we want to consider is the subgroup generated by the squares of the standard generators.

Definition 7.3 (Tits conjecture). Let $A = \textsc{Art}(\Gamma)$ be an Artin group with standard generating set S and let T be the subgroup of A generated by the squares of the generators in S. There are some obvious relations satisfied by the generators of T. For example, if a and b in S commute in A, then a^2 and b^2 commute in T. Jacques Tits conjectured that these relations are sufficient to define the group structure of T. In particular, he conjectured that T is always a right-angled Artin group. This is called the *Tits conjecture*.

There is a natural map from a right-angled Artin group to the Artin group in question where the generators of the domain are sent to the squares of the standard generators in the range. As in the previous case what is not clear is whether or not this map is injective. In [CP01] John Crisp and Luis Paris found a way to represent an arbitrary Artin group inside a mapping class group of a surface and they showed that the image of the right-angled Artin group injected into the mapping class group. Therefore the map into the Artin group must also be injective and this is enough to prove the Tits conjecture.

And this concludes our quick tour of the known positive results about the word problem for Artin groups and for portions of Artin groups. There are many more concrete results that have been shown about those classes of Artin groups where the word problem or the $K(\pi, 1)$ conjecture has been solved, but I am passing over those in silence in order to highlight the vast void at the center of the field.

8. Unknown groups

After completing the survey of the positive results about Artin groups with a solvable word problem, it is now time to examine what Artin groups are left. As noted at the beginning of Section 6 it is sufficient to study Artin groups that are 2-local since the extension to groups that have missing relations is straight-forward. There is also a second reduction that remains merely conjectural at present.

Definition 8.1 (Small-type)**.** Let $A = \textsc{Art}(\Gamma)$ be an Artin group with standard generating set S. The Artin group A is said to be *small-type* or *simply laced* if for every pair of distinct generators a and b in S they either commute or they satisfy the classic braid relation of length 3, i.e. either $ab = ba$ or $aba = bab$. In the classic notation small-type Artin groups are precisely those Artin groups defined by unlabeled graphs.

Remark 8.2 (Small-type). John Crisp created a general method that can be used to prove that one Artin monoid injects into another [Cri99] and this can be used to prove that every Artin monoid injects into an Artin monoid of small type. Luis Paris uses this result as part of his proof that every Artin monoid injects into its Artin group [Par02]. Crisp and Paris also use this as part of their proof of the Tits conjecture [CP01].

It is natural to conjecture that these injections on the monoid level extend to the group level but this is currently an open question. If this is true then small-type Artin groups are universal Artin groups in that every other Artin group can be realized as a (non-parabolic) subgroup of a small-type Artin group. As a consequence the remainder of this section focuses only on Artin groups that are small-type. Unfortunately, there are very few small-type Artin groups that we understand. Let Γ be a connected simple graph and let $G = \textsc{Art}(\Gamma)$ be the corresponding irreducible small-type Artin group. If we search through the classes of Artin groups whose word problem we know how to solve and restrict our attention to those that are small-type, the list of examples is very short.

Theorem 8.3 (Small-type)**.** *Let $A = \textsc{Art}(\Gamma)$ be a small-type Artin group defined by a simple connected unlabeled graph Γ. The only cases where we know how to solve the word problem for group A is when*

1. *Γ is a complete graph and A is large-type,*
2. *Γ is an ADE Dynkin diagram and A is spherical, or*
3. *Γ is an extended ADE Dynkin diagram and A is euclidean.*

And this is the complete list! In other words, every small-type Artin group defined by any connected graph that is not complete, not a Dynkin diagram and not an extended Dynkin diagram has a word problem that we do not know how to solve. The small presentation given in Example 1.3 is a border case since it happens to be contained in Charney's extension so the $K(\pi,1)$ conjecture is known to hold for this group, but we do not know how to solve its word problem. For almost all small-type Artin groups our ignornance is more extreme. The highly restricted nature of the list given in Theorem 8.3 becomes clear once we restrict our attention to special classes of graphs.

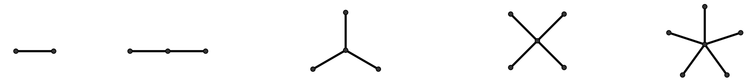
Definition 8.4 (Bipartite)**.** A *complete bipartite graph* Γ is a graph whose vertices can be split into two non-empty subsets so that Γ contains an edge between two vertices if and only if one of these vertices belong to one subset and the other belongs to the other subset. We write $K_{m,n}$ to denote the corresponding complete bipartite graph with m vertices on one subset and n vertices in the other. The corresponding small-type Artin group has the following presentation.

$$\textsc{Art}(K_{m,n}) = \left\langle \begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_m \end{array} \middle| \begin{array}{ll} a_i a_{i'} = a_{i'} a_i, & a_i b_j a_i = b_j a_i b_j \\ b_j b_{j'} = b_{j'} b_j & \end{array} \right\rangle$$

Remark 8.5 (Bipartite). There are only 5 cases where we know how to solve the word problem for the bipartite Artin group $\textsc{Art}(K_{m,n})$ and these are when $mn \leq 4$. The Artin group is spherical for $mn < 4$ and euclidean for $mn = 4$. Concretely $K_{1,1} = A_2$, $K_{1,2} = A_3$, $K_{1,3} = D_4$, $K_{1,4} = \widetilde{D}_4$ and $K_{2,2} = \widetilde{A}_3$. The rest of these groups have word problems that we do not know how to solve.

An even more restricted class is the collection of graphs known as stars or claws. A *star* is a connected graph of the form $K_{1,n}$ in which all of its edges have a vertex in common. The progress in understanding the Artin groups defined by stars has been extremely slow.

Example 8.6 (Stars)**.** Our progress in understand the word problem for Artin groups defined by stars is shown in Figure 12. Only the first four examples have word problems that we know how to solve and the positive results about these four examples are roughly thirty to



Braid_3	Braid_4	$\mathrm{Art}(D_4)$	$\mathrm{Art}(\widetilde{D}_4)$	$\mathrm{Art}(K_{1,5})$
Dehn	Artin	Br-Sa/Deligne	M-Sulway	???
1910s	1940s	1970s	2010s	???

Figure 12. The slow progress in understanding the word problem for Artin groups defined by stars.

forty years apart. Our understanding of the first example comes from the work of Max Dehn on the fundamental group of the trefoil knot complement in the 1910s, our understanding of the second example comes from Emil Artin's solution to the word problem for all braid groups in the 1940s, our understanding of the third example comes from the detailed investigations of the structure of spherical Artin groups by Deligne and, independently Brieskorn and Saito in the 1970s and our understanding of the fourth example comes from my recent work with Rob Sulway on the structure of euclidean Artin groups. And we do not yet know how to solve the word problem for the Artin group $\mathrm{Art}(K_{1,5})$.

Once one realizes that the small-type Artin groups are the key class of Artin groups that one needs to understand in order to have a good understanding of all Artin groups, it becomes clear that most Artin groups remain fundamentally mysterious.

Part 3. Over the horizon

In Part 3 the focus shifts to those collections of groups that are just "over the horizon" in the following sense. They are not currently well understood but they are the ones that are (in my opinion) likely to be better understood in the near future. The collections of groups that I focus on are complex euclidean braid groups, extended affine Artin groups and Lorenzian Artin groups. The first two classes are merely close relatives of Artin groups, but the similarities and differences are interesting. The third class is the obvious next step when trying to understand Artin groups using the signature of their associated bilinear form.

9. Extended affine Artin groups

Despite the wide-spread assumption that all Artin groups should have good algorithmic properties, we saw in Part 2 that the only ones where we truly understand their algebraic structure are those constructed from 2-local Artin groups where the associated Coxeter matrix is positive semi-definite or the associated Salvetti complex is very low dimensional. It could easily be the case that these are the only Artin groups with a nice algebraic structure. If this is indeed the case then it might make sense to start looking at classes of groups that retain some features of the Artin groups we understand while broadening the definition to include other types of groups. In this section we look at groups generated by reflections that preserve a positive semi-definite quadratic form that are not necessarily Coxeter groups.

For every Coxeter group we can find a set of discrete vectors in a vector space with a symmetric bilinear form that form a *root system* for the Coxeter group. In particular, these vectors determine reflections that preserves the corresponding quadratic form that generate the original Coxeter group and they are precisely the conjugacy class of reflections in this Coxeter group. It is tempting to think that all vector arrangements that look like root systems in a vector space with a symmetric bilinear form generate Coxeter groups, but this intuition is only true in the presence of additional contraints.

This section describes a concrete example where this intuition fails. In particular, I want to describe a set of vectors that satisfy almost all of the properties of being a root system in the classical sense. These vectors are in the six-dimension vector space $V = \mathbb{R}^{4,0,2}$ whose positive semi-definite form has 4 positive eigenvalues and 2 zeros. One portion of the root system is closely related to the classical D_4 root system in $\mathbb{R}^4$.

Definition 9.1 (The D_4 root system)**.** The classical D_4 *root system* consists of the 24 vectors $\Phi_{D_4} = \{\pm e_i \pm e_j \mid i,j \in \{1,2,3,4\}\}$ inside $\mathbb{R}^4$ with the standard positive definite inner product where the vectors e_i are the standard unit basis vectors. Note that there are exactly 3 ways to partition $\{1,2,3,4\}$ into two sets of two: $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$. Using this we can partition the 24 vectors in Φ_{D_4} into 3 sets with 8 vectors each. Let $\Phi_{kl} = \{\pm e_i \pm e_j\}$ with $i \neq j \in \{k,l\}$ and define $\Phi_A = \Phi_{12} \cup \Phi_{34}$, $\Phi_B = \Phi_{13} \cup \Phi_{24}$, and $\Phi_C = \Phi_{14} \cup \Phi_{23}$. Then $\Phi_{D_4} = \Phi_A \cup \Phi_B \cup \Phi_C$. Each of these 3 subsets containing 8 vectors is an orthogonal frame, i.e. a set of vectors that pairwise orthogonal or parallel.

The other portion is closely related to a discrete subset of the complex plane called the Eisenstein integers.

Definition 9.2 (Eisenstein integers)**.** If $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ denotes a primitive cube root of unity, then the ring $\mathcal{E} = \mathbb{Z}[\omega] \subset \mathbb{C}$ is called the set of *Eisenstein integers*.

The *radical* of a symmetric bilinear form is the set of vector that are orthogonal to all other vectors and its dimension corresponds to the multiplicity of 0 as a eigenvalue. Let R be the 2-dimensional radical of the symmetric bilinear form on $V = \mathbb{R}^{4,0,2}$ and let a, b and c be any three pairwise linearly independent vectors in R subject to the constraint that $a + b + c = 0$. It is straightforward to find an identification of R with the complex plane $\mathbb{C}$ so that the $\mathbb{Z}$-span of a, b and c is identified with the Eisenstein integers $\mathcal{E}$ and a, b and c are identified with the three cube roots of unity. The subring $2\mathcal{E}$ is a maximal ideal insie $\mathcal{E}$ and the quotient $\mathcal{E}/2\mathcal{E}$ is isomorphic to $\mathbb{F}_4$, the field with 4 elements. The three nonzero cosets of $2\mathcal{E}$ inside $\mathcal{E}$ are represented by a, b and c. Let $\Phi_a = a + 2\mathcal{E}$, $\Phi_b = b + 2\mathcal{E}$ and $\Phi_c = c + 2\mathcal{E}$.

Definition 9.3 (An extended affine root system)**.** We define a "root system" in the space $V = \mathbb{R}^{4,0,2}$ as follows. Let $\Phi_{Aa} = \{u + v \mid u \in \Phi_A, v \in \Phi_a\}$, $\Phi_{Bb} = \{u + v \mid u \in \Phi_B, v \in \Phi_b\}$ and $\Phi_{Cc} = \{u + v \mid u \in \Phi_C, v \in \Phi_c\}$. And then let $\Phi = \Phi_{Aa} \cup \Phi_{Bb} \cup \Phi_{Cc}$. This is known in the literature as the *extended affine root system* of type $\widetilde{D}_4^{(1,1)}$. Under the quotient map from V to V/R, the roots in Φ are sent to the D_4 root system Φ_{D_4}.

It is easy to check that the vectors in Φ define reflections preserving the bilinear form which preserves Φ as a set. The discrete group that these reflection generate is called an *extended affine Coxeter group* but it is not a Coxeter group in the traditional sense since it does not have a Coxeter presentation. There is an official definition of an extended affine root system and corresponding notions of extended affine Coxeter groups, extended affine Artin groups and extended affine Lie algebras. These root systems have been classified and presentations are known for the extended affine Coxeter groups and extended affine Artin groups. The study of these objects was initiated by Saito in the 1980s and 1990s in a series of long papers [Sai74, Sai85, Sai90, ST97, SY00, Sai01] and more recently there is an entire community dedicated to their study. See the text [AAB+97] and the articles that have cited it.

I would like to propose that geometric group theorists who study Coxeter groups and Artin groups should pay more attention to the similarities and differences between ordinary Coxeter and Artin groups and these extended affine Coxeter and Artin groups. The two communities do not appear to be interacting as much as they should and I think that both would benefit from conversations with the other. As geometric group theorists, the natural questions about extended affine Coxeter groups and Artin groups are to what extend the standard machinery extends to this expanded context. Is there a natural Davis complex for the extended affine Coxeter groups? Is there a natural Salvetti complex for the extended affine Artin groups? Are these extended affine Artin groups torsion-free? Do they have a decidable word problem? Do they have a non-trivial center? Is the natural (conjectural) space used to define them a $K(\pi, 1)$? It appears, for the most part, that research on these questions for these groups is just beginning.

10. Complex euclidean braid groups

In this section we consider a second class of groups that are closely related to Artin groups with a positive semi-definite Coxeter matrix. Rather than expanding the notion of a root system to include groups with a larger radical, we consider the case where we expand the notion of reflection to include complex reflections. The complex spherical reflection groups are the complex analog of the spherical Coxeter groups and they have been well studied. The complex euclidean reflection groups are the analog of the euclidean or affine Coxeter groups and these have received much less attention.

10.1. **Complex Spherical Reflection Groups.** We begin with the notion of a complex spherical reflection group and the corresponding complex spherical braid groups that are defined as the braid groups of these group actions.

Definition 10.1 (Complex spherical reflections)**.** A *complex spherical space* is a complex vector space that comes equipped with a positive definite hermitian form that is linear in the second coordinate, and a *complex spherical reflection* is an operator on a complex spherical space that fixes a codimensional one subspace and multiplying some vector by a root of unity. A *complex spherical reflection group*, also known as a *finite complex reflection group* or as a *unitary reflection group*, is a finite group generated by complex spherical reflections acting on some complex spherical space. Such groups have been classified ever since they were introduced. There is a single triply-indexed infinite family together with 34 exceptional groups. The original classification was done by Shephard and Todd [ST54] and more modern proofs can be found in [Coh76] and [LT09].

The simplest exceptional example of a complex spherical reflection is the 24 element group of type G_4 in the Shephard-Todd classication. It has 4 complex reflections of order 3 and it acts on the 4-dimensional

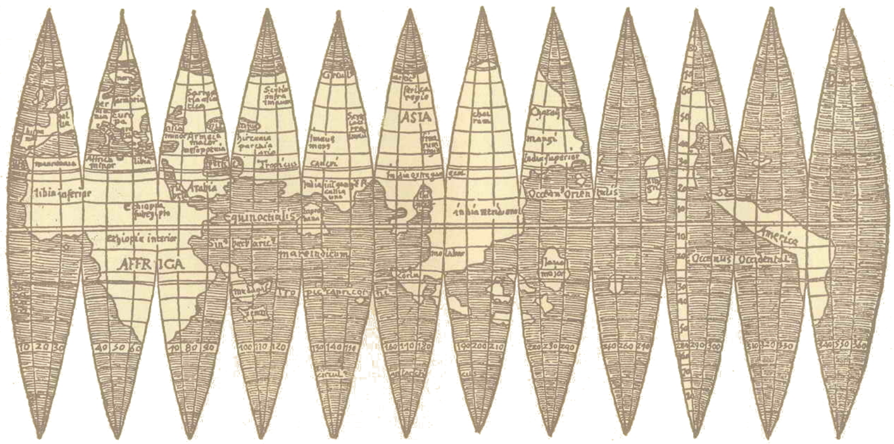


FIGURE 13. Martin Waldseemüller's globe gores from 1507.

regular polytope called the 24-cell. In 2007 John Meier and I developed a technique for visualizing the regular 4-dimensional polytopes as a union of spherical lenses that has been very useful for seeing directly how various groups act on these polytopes.

Definition 10.2 (Lunes and Lenses)**.** A *lune* is a portion of a 2-sphere bounded by two semicircular arcs with a common 0-sphere boundary and its shape is completely determined by the angle at this these semicircles meet. Lunes are commonly used to display spherical data in the plane in a way that the distortion is kept to a minimum. See, for example, the globe gores used to display the Walseemüller map of the earth from 1507 shown in Figure 13. A lens is a 3-dimensional analog of a lune. Concretely, a *lens* is a portion of the 3-sphere determined by two hemispheres sharing a common great circle boundary and the shape of a lens is completely determined by the dihedral angle between these hemispheres along the great circle where they meet.

In the same way that lunes can be used to display the map of a 2-sphere such as the earth in $\mathbb{R}^2$ with very little distortion, lenses can be used to display a map of the 3-sphere in $\mathbb{R}^3$ with very little distortion.

Definition 10.3 (6 lenses)**.** To directly visualize the structure of the 24-cell it is useful to use the 6 lenses displayed in Figure 14. In the picture $\mathbb{C}^2$ has been identified with the quaternions and quaternion labels are given for the 24 points. The points used are $\{\pm 1, \pm i, \pm j, \pm k\} \cup \{\frac{\pm 1 \pm i \pm j \pm k}{2}\}$. The element $\zeta = \frac{1+i+j+k}{2}$. Each of the six figures represents one-sixth of the 3-sphere. The outside circle is a great circle in $\mathbf{S}^3$, the solid lines live in the hemisphere that bounds the front of the lens, the

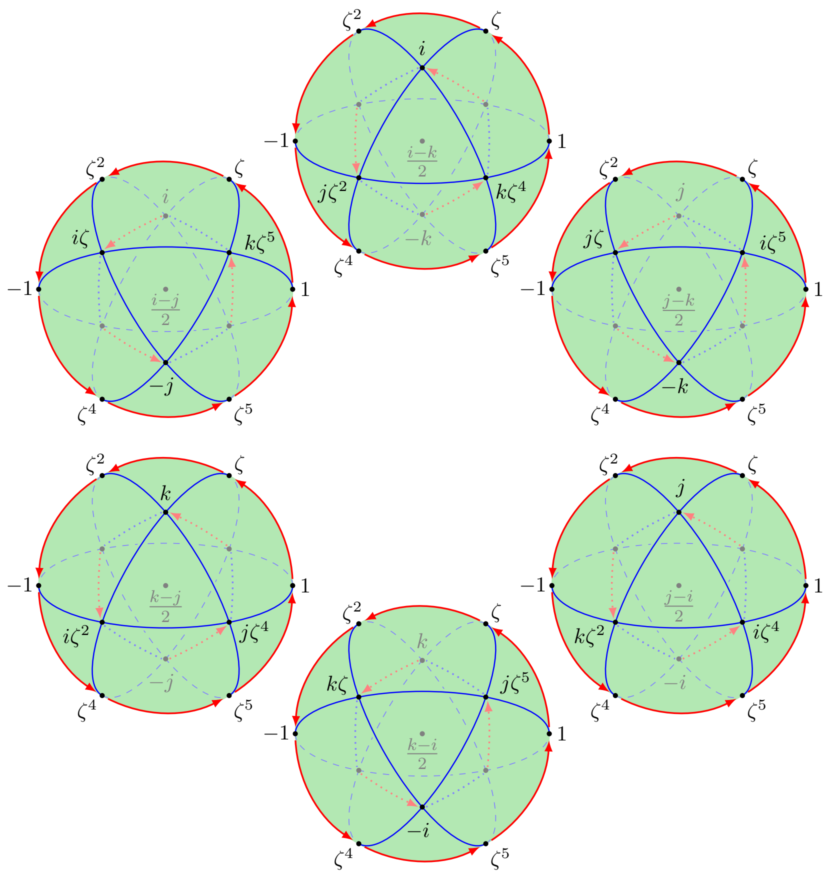


FIGURE 14. Six lenses that together display the structure of the 24-cell. Each figure represents a one-sixth lens in the 3-sphere with dihedral angle $\frac{\pi}{3}$ between its front and back hemispheres. They are arranged so that every front hemisphere is identified with the back hemisphere of the next one when ordered in a counter-clockwise way.

dashed lines live in the hemisphere that bounds the back of the lens and the dotted lines live in the interior of the lens. The dihedral angle between the front and back hemispheres, along the outside boundary circle is $\frac{\pi}{3}$ and all the edges are length $\frac{\pi}{3}$. The six lenses are arranged so that the front hemisphere of each lens is identified with the back hemisphere the next one in counter-clockwise order. Each lens contains one complete octahedral face at its center and six half octahedra, three bottoms halves corresponding to the squares in the front hemisphere and three top halves corresponding to the squares in the back hemisphere.

The label at the center of each lens is the coordinate of the center of the euclidean octahedron spanned by the six nearby vertices.

The arrows in Figure 14 indicate how the 24 vertices move under the map which right multiples the quaternions by ζ. The arrows glue together form four oriented hexagons with vertices $q\langle\zeta\rangle$ that live in the four complex lines $q\mathbb{C}$ where q is 1, i, j or k.

Example 10.4 (Braid(G_4))**.** Using the 6 lens picture Ben Coté and I were able to show that the hyperplane complement of the complex spherical reflection group of type G_4 is homotopy equivalent to a portion of the 2-skeleton of the 24-cell. The quotient by the free action of the G_4 complex spherical reflection group is a 2-complex with a single vertex which encodes the presentation $\langle a, b, c, d \mid abd, bcd, cad\rangle$. Since this is the dual Garside presentation of the 3-string braid group, the corresponding complex spherical braid group in this case, which we denote Braid(G_4), is the 3-string braid group. Although this result is not new, the deformation retraction from the hyperplane complement to a portion of the 2-skeleton of the 24-cell is new, as is this particular method of visualizing the group and its action on $\mathbb{C}^2$. For a more detailed description of this result see [CM].

10.2. **Complex euclidean reflection groups.** Passing from a vector space to an affine space involves forgetting the location of the origin. In this section we discuss complex euclidean reflection groups that act on some complex euclidean space and the corresponding complex euclidean braid groups that are the braid groups of these group actions. These are the complex reflection analogs of euclidean Coxeter groups and euclidean Artin groups.

Definition 10.5 (Complex euclidean reflections)**.** A *complex euclidean space* is a complex spherical space where the location of the origin has been forgotten and a *complex euclidean reflection* is an operator on a complex euclidean space that becomes a complex spherical reflection with an appropriate choice of origin and corresponding choice of coordinate system. A *complex euclidean reflection group* is a discrete group generated by complex euclidean reflections that acts on some complex euclidean space and the corresponding *complex euclidean braid group* is the braid group of this group action.

Example 10.6 (A 1-dimensional example)**.** Let $\mathcal{E}$ be the subring of Eisenstein integers in the complex plane $\mathbb{C}$. For each point z in $\mathcal{E}$ there is a complex reflection of order 3 that fixes z and rotates the remaining points through a positive angle of $\frac{2\pi}{3}$ around the point z. Let G be

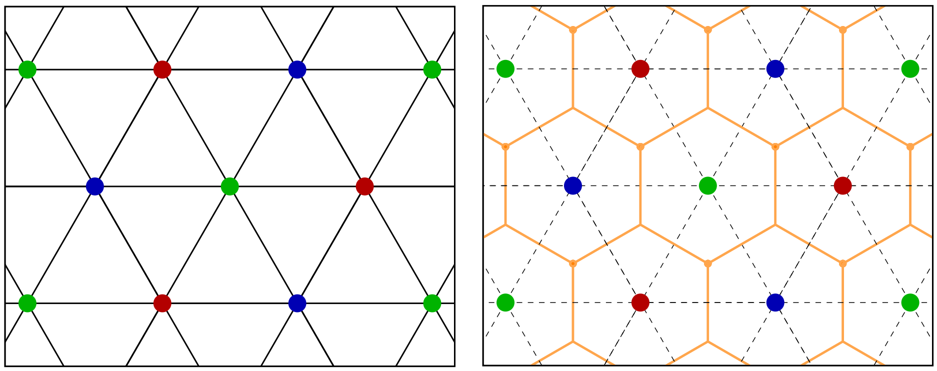


FIGURE 15. A 1-dimensional example of a complex euclidean reflection group and the corresponding Voronoi cell decomposition.

the group generated this discrete set of complex reflections. The result is an example of a 1-dimensional complex euclidean reflection group. The braid group of this example can be computed as follows. The points in $\mathcal{E}$ are the only points with non-trivial stabilizers and once they have been removed the group G acts freely. The quotient of $\mathbb{C} \setminus \mathcal{E}$ by this free G action is a triangular pillowcase with its three corners removed and its fundamental group is $\mathbb{F}_2$, the free group of rank 2. Alternatively, one can form the Vornoi cells around the set of the fixed points of the complex reflections and then deform way the interiors of the hexagons. The reflection group acts freely on this "chicken wire" graph. The quotient graph Γ has 2 vertices, 3 edges and $\pi_1(\Gamma) = \mathbb{F}_2$. See Figure 15.

The discrete groups generated by complex euclidean reflections were essentially classified by Popov in the 1982 [Pop82]. There are complex euclidean notions of reducibility and equivalence and Popov proved many structural results about the inequivalent irreducible complex euclidean reflection groups. In addition he gave algorithms in various subcases and produced a complete list of the resulting groups. The details of the computations themselves were not included and in 2006 Goryunov and Man found an isolated example in dimension 2 that was overlooked in Popov's list [GM06]. It would be a service to the community if someone went through and redid Popov's computations to provide a clean modern proof that the current augmented list of examples is indeed complete.

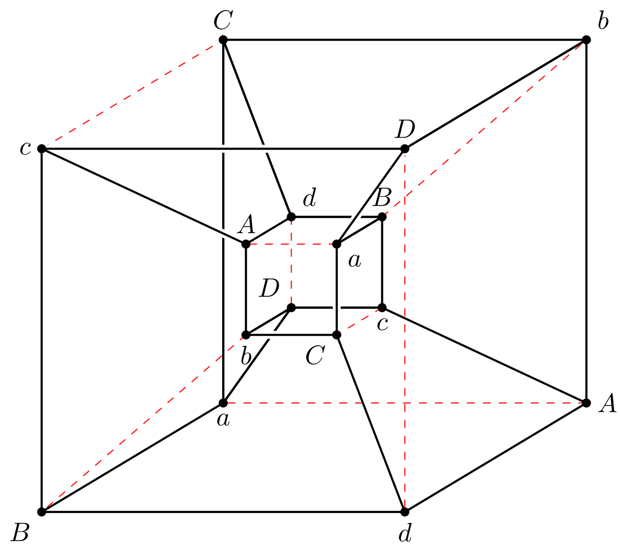


FIGURE 16. The Möbius-Kantor graph.

Remark 10.7 (Popov's classification). Popov's classification of inequivalent irreducible complex euclidean reflection groups contains 30 infinite families and 22 isolated examples. There are 17 infinite families with a continuous parameter and these are closely connected to the complexified versions of the real euclidean reflection groups. Of these there are 7 that also have a discrete parameter. These correspond to the Cartan-Killing types $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$. The remaining 13 infinite families have only a discrete parameter indicating dimension. The 22 isolated examples mostly occur in very low dimensions.

Ben Coté and I have examined one of the smallest isolated examples that acts on $\mathbb{C}^2$. Since it is the only isolated example whose linear part is the complex spherical reflection group of type G_4 in the Shephard-Todd classification, we call this group $\textsc{Refl}(\widetilde{G}_4)$, the complex euclidean reflection group of type $\widetilde{G}_4$. In [CM] we prove the following.

Theorem 10.8 (Complement complex)**.** *The hyperplane complement of* $\textsc{Refl}(\widetilde{G}_4)$ *deformation retracts onto a non-positively curved piecewise euclidean* 2*-complex* K *in which every* 2*-cell is an equilateral triangle and every vertex link is a Möbius-Kantor graph.*

The *Möbius-Kantor graph* is a subgraph of the 1-skeleton of the 4-cube with 8 edges removed. See Figure 16. The key idea behind the proof of Theorem 10.8 is to start with the discrete set of points that arise as 0-dimensional intersections of the fixed hyperplanes of the complex euclidean reflections in $\textsc{Refl}(\widetilde{G}_4)$ and then to deform away the interiors of the corresponding Voronoi cells. For $\textsc{Refl}(\widetilde{G}_4)$ this Voronoi tiling is a tiling by 24-cells and the 3-complex that remains is built out of octahedra. The second step is to see where the hyperplanes themselves intersect this 3-complex and this occurs only in the interiors of the octahedra. In fact, the local situation is exactly the same as the one described in Example 10.4 so we can retract to a union of local 2-complexes that look like the $\textsc{Braid}(G_4)$ case. The corresponding complex euclidean braid group, defined as the braid group of this group action, has some surprising properties.

Theorem 10.9 (Isolated fixed points)**.** *The space of regular points for the complex euclidean reflection group* $\textsc{Refl}(\widetilde{G}_4)$ *acting on* $\mathbb{C}^2$ *is properly contained in its hyperplane complement because of the existence of isolated fixed points.*

The existence of these isolated fixed points that are not contained in the union of the fixed hyperplanes of a complex euclidean reflections of the group is one of the ways in which the complex euclidean situation is very different from the case of Coxeter groups acting on their complexified Tits cones and from the case of complex spherical reflection groups acting on some complex spherical space. In particular, these isolated fixed points leads to the existence of torsion in the corresponding braid group.

Theorem 10.10 (Braid group)**.** *The group* $\textsc{Braid}(\widetilde{G}_4)$ *is a* $\mathrm{CAT}(0)$ *group and it contains elements of order* 2*.*

Proofs of these results can be found in [CM]. Althought there are some special aspects of this particular example that make it difficult to immediately extend it to other complex euclidean reflection groups, it would be interesting to see whether other complex euclidean braid groups have similar properties. In general, almost all questions about complex euclidean braid groups remain completely unexamined. For example, there are not even conjectural presentations for the these types of braid groups.

11. Lorentzian Artin groups

In this final section we look at the prospects for making further progress in our understanding of Artin groups themselves. We begin with a discussion of the signature of a Coxeter matrix.

Definition 11.1 (Signatures and types)**.** Recall that the *signature* of a reall symmetric matrix M such as a Coxeter matrix is (p, n, z) where p, n and z are the number of positive, negative and zero eigenvalues of M. We coarsely divide Coxeter groups and Artin groups into 3 types based on the signature of its Coxeter matrix M. When M has no negative eigenvalues, it is *spherical*, when M has one negative eigenvalue it is *Lorentzian* and when M has more than one negative eigenvalue it is *higher-rank*. The adjective applied to a Coxeter or Artin presentation and the Coxeter group or Artin group it defines is the one for its Coxeter matrix. Finally, when 0 eigenvalues exist we add the adjective *weakly*. In this language, the positive semi-definite matrices that correspond to euclidean Coxeter groups and euclidean Artin groups would be described as weakly spherical.

Remark 11.2 (Lorentzian vs. hyperbolic). The definition of a hyperbolic Coxter group is not a stable definition in the literature. It might refer to a Coxeter group that naturally acts cocompactly by isometries on some hyperbolic space, or to a Coxeter group that naturally acts by isometries on some hyperbolic space with co-finite volume, or even to a Coxeter group that is Gromov hyperbolic. The first class of Coxeter groups is contained in the second while the third is essentially unrelated to either. The definition of a Lorenztian Coxeter group is a generalization of the second definition in the following sense. The Tits representation of a Lorenztian Coxeter group preserves a symmetric bilinear form with a Lorentzian signature and thus the group naturally acts on the hyperboloid model of hyperbolic space, but in general this action is neither cocompact nor co-finite volume.

The spherical and weakly spherical Coxeter groups and Artin groups are large classes that we already understand. If our goal is to understand how to solve the word problem for all Artin groups, the obvious next step is to try and understand those defined by Lorentzian and weakly Lorenztian presentations. Unfortunately, we currently understand essentially none of these groups.

Remark 11.3 (Find one example). Recall from the survey of known results in Part 2 that the only small-type Lorentzian Artin groups where we know how to decide the word problem are the large-type ones based on complete graphs and this result uses a type of small-cancellation theory that is unavailable for the other Lorenztian Artin groups. Thus, it is interesting (and probably very hard) open problem to find at least one small-type Lorentzian Artin group not defined by a complete graph where we know how to solve its word problem.

Small graphs that define Lorenztian Artin groups are very easy to find.

Remark 11.4 (Small graphs). In 2013 Ryan Blair and Ryan Ottman proved that none of the 996 connected graphs with fewer than 8 vertices define Artin groups that are higher rank. Of these graphs 13 are spherical, 9 are weakly spherical or euclidean and 4 are complete graphs that define Lorentzian groups. The remaining 970 small connected graphs define Lorentzian Artin groups whose word problem we do not know how to solve [BO13].

Another family of Lorenztian Artin groups that have simple presentations are those defined by graphs that look like tripods.

Remark 11.5 (Tripods). Let $T_{p,q,r}$ be the graph that is a tree with only one branch point, the branch point has degree 3 and the arms contain p, q and r vertices, respectively where the branch vertex counts as a vertex in each of the arms. This tree defines a spherical Artin group when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, it defines a weakly spherical Artin groups when this sum is equal to 1, and it defines a Lorentzian Artin group when this sum is less than 1. None of these Lorenztian Artin groups have a word problem that we know how to solve.

Finally, if more symmetric presentations seem desirable, many of the highly symmetric graphs studied by graph theorists define Artin groups that are Lorenztian.

Remark 11.6 (Highly symmetric graphs). Several of the highly symmetric graphs studied by graph theorists define Artin groups that are Lorenztian. These include the 10-vertex Petersen graph, the 50-vertex Hoffman-Singleton graph and the 275-vertex McLaughlin graph and the 26-vertex incidence graph of the projective plane over the field with 3 elements. None of the corresponding Artin groups have word problems that we know how to solve. All of these graphs define Lorenztian Coxeter groups that are loosely connected to the study of sporadic finite simple groups, particularly the Monster finite simple group.

We know that these graphs (and many others) define Lorenztian Coxeter groups and Artin groups because the eigenvalues of their Coxeter matrix can be read off from the spectrum of the adjacency matrix of the graph. In particular the spectrum of a graph can be used to immediately determine the type of Coxeter group and Artin group this graph defines. Until we can solve the word problem for at least one of these non-trivial Lorentzian Artin groups, the geometry of most Artin groups will remain fundamentally mysterious.

References

[AAB+97] Bruce N. Allison, Saeid Azam, Stephen Berman, Yun Gao, and Arturo Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. **126** (1997), no. 603, x+122. MR 1376741

[Ago08] Ian Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284. MR 2399130

[Ago13] ———, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087, With an appendix by Agol, Daniel Groves, and Jason Manning. MR 3104553

[AS83] K. I. Appel and P. E. and Schupp, *Artin groups and infinite Coxeter groups*, Invent. Math. **72** (1983), no. 2, 201–220. MR 700768

[BB97] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. MR 1465330

[BB05] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR 2133266

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 2000k:53038

[Big01] Stephen J. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. **14** (2001), no. 2, 471–486. MR 1815219

[BO13] Ryan Blair and Ryan Ottman, *A decomposition theorem for higher rank Coxeter groups*, Comm. Algebra **41** (2013), no. 7, 2508–2518. MR 3169406

[BS72] Egbert Brieskorn and Kyoji Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. **17** (1972), 245–271. MR 48 #2263

[Cha04] Ruth Charney, *The Deligne complex for the four-strand braid group*, Trans. Amer. Math. Soc. **356** (2004), no. 10, 3881–3897. MR 2058510

[Che98] Andrew Chermak, *Locally non-spherical Artin groups*, J. Algebra **200** (1998), no. 1, 56–98. MR 1603264

[CM] Ben Coté and Jon McCammond, *A complex euclidean reflection group with an elegant complement complex*, `arXiv:1707.06624 [math.GR]`.

[CMS10] Filippo Callegaro, Davide Moroni, and Mario Salvetti, *The $K(\pi,1)$ problem for the affine Artin group of type $\tilde{B}_n$ and its cohomology*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 1, 1–22. MR 2578601

[Coh76] Arjeh M. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 3, 379–436. MR 0422448

[CP01] John Crisp and Luis Paris, *The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group*, Invent. Math. **145** (2001), no. 1, 19–36. MR 1839284

[CP14] Ruth Charney and Luis Paris, *Convexity of parabolic subgroups in Artin groups*, Bull. Lond. Math. Soc. **46** (2014), no. 6, 1248–1255. MR 3291260

[Cri99] John Crisp, *Injective maps between Artin groups*, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin, 1999, pp. 119–137. MR 1714842

[CW02] Arjeh M. Cohen and David B. Wales, *Linearity of Artin groups of finite type*, Israel J. Math. **131** (2002), 101–123. MR 1942303

[Dav83] Michael W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), no. 2, 293–324. MR 690848

[Dav08] ______, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR 2360474

[Del72] Pierre Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302. MR 0422673 (54 #10659)

[Dig03] François Digne, *On the linearity of Artin braid groups*, J. Algebra **268** (2003), no. 1, 39–57. MR 2004479

[GM06] Victor Goryunov and Show Han Man, *The complex crystallographic groups and symmetries of J_{10}*, Singularity theory and its applications, Adv. Stud. Pure Math., vol. 43, Math. Soc. Japan, Tokyo, 2006, pp. 55–72. MR 2313408

[GP12] Eddy Godelle and Luis Paris, *Basic questions on Artin-Tits groups*, Configuration spaces, CRM Series, vol. 14, Ed. Norm., Pisa, 2012, pp. 299–311. MR 3203644

[HJP16] Jingyin Huang, Kasia Jankiewicz, and Piotr Przytycki, *Cocompactly cubulated 2-dimensional Artin groups*, Comment. Math. Helv. **91** (2016), no. 3, 519–542. MR 3541719

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460 (92h:20002)

[Kra00] Daan Krammer, *The braid group B_4 is linear*, Invent. Math. **142** (2000), no. 3, 451–486. MR 1804157

[Kra02] ______, *Braid groups are linear*, Ann. of Math. (2) **155** (2002), no. 1, 131–156. MR 1888796

[LT09] Gustav I. Lehrer and Donald E. Taylor, *Unitary reflection groups*, Australian Mathematical Society Lecture Series, vol. 20, Cambridge University Press, Cambridge, 2009. MR 2542964

[McC] Jon McCammond, *The structure of euclidean Artin groups*, Survey article to appear in the Proceedings of the Durham Conference. `arXiv:1312.7781 [math.GR]`.

[McC10] ______, *Combinatorial descriptions of multi-vertex 2-complexes*, Illinois J. Math. **54** (2010), no. 1, 137–154. MR 2776989

[MS] Jon McCammond and Robert Sulway, *Artin groups of euclidean type*, To appear in *Inventiones Mathematicae*.`arXiv:1312.7770 [math.GR]`.

[Par02] Luis Paris, *Artin monoids inject in their groups*, Comment. Math. Helv. **77** (2002), no. 3, 609–637. MR 2003j:20065

[Par14a] ______, *$K(\pi,1)$ conjecture for Artin groups*, Ann. Fac. Sci. Toulouse Math. (6) **23** (2014), no. 2, 361–415. MR 3205598

[Par14b] ______, *Lectures on Artin groups and the $K(\pi,1)$ conjecture*, Groups of exceptional type, Coxeter groups and related geometries, Springer Proc. Math. Stat., vol. 82, Springer, New Delhi, 2014, pp. 239–257. MR 3207280

[Pop82] V. L. Popov, *Discrete complex reflection groups*, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, vol. 15, Rijksuniversiteit Utrecht, Mathematical Institute, Utrecht, 1982. MR 645542

[Sai74] Kyoji Saito, *Einfach-elliptische Singularitäten*, Invent. Math. **23** (1974), 289–325. MR 0354669

[Sai85] ______, *Extended affine root systems. I. Coxeter transformations*, Publ. Res. Inst. Math. Sci. **21** (1985), no. 1, 75–179. MR 780892

[Sai90] ______, *Extended affine root systems. II. Flat invariants*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 1, 15–78. MR 1053908

[Sai01] ______, *Extended affine root systems. V. Elliptic eta-products and their Dirichlet series*, Proceedings on Moonshine and related topics (Montréal, QC, 1999), CRM Proc. Lecture Notes, vol. 30, Amer. Math. Soc., Providence, RI, 2001, pp. 185–222. MR 1881609

[ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canadian J. Math. **6** (1954), 274–304. MR 0059914

[ST97] Kyoji Saito and Tadayoshi Takebayashi, *Extended affine root systems. III. Elliptic Weyl groups*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 2, 301–329. MR 1442503

[SY00] Kyoji Saito and Daigo Yoshii, *Extended affine root system. IV. Simply-laced elliptic Lie algebras*, Publ. Res. Inst. Math. Sci. **36** (2000), no. 3, 385–421. MR 1781435

[Tit13] Jacques Tits, *œuvres/Collected works. Vol. I, II, III, IV*, Heritage of European Mathematics, European Mathematical Society (EMS), Zürich, 2013, Edited by Francis Buekenhout, Bernhard Matthias Mühlherr, Jean-Pierre Tignol and Hendrik Van Maldeghem. MR 3157464

[vdL83] Harm van der Lek, *The homotopy type of complex hyperplane complements*, Ph.D. thesis, University of Nijmegen, 1983.

[Wis12] Daniel T. Wise, *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, vol. 117, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2012. MR 2986461

Contents

Part 1. Among friends 2
1. Coxeter 2
2. Tits 8
3. Davis 11
4. Salvetti 15

Part 2. On the edge 19
5. Big Picture 19
6. Known groups 20
7. Known parts 25
8. Unknown groups 27

Part 3. Over the horizon 29
9. Extended affine Artin groups 29
10. Complex euclidean braid groups 32
11. Lorentzian Artin groups 38
References 41

Dept. of Math., University of California, Santa Barbara, CA 93106
E-mail address: `jon.mccammond@math.ucsb.edu`