Solving Linear Programs

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In this chapter, we present a systematic procedure for solving linear programs. This procedure, called the *simplex method,* proceeds by moving from one feasible solution to another, at each step improving the value of the objective function. Moreover, the method terminates after a finite number of such transitions.

Two characteristics of the simplex method have led to its widespread acceptance as a computational tool. First, the method is robust. It solves *any* linear program; it detects redundant constraints in the problem formulation; it identifies instances when the objective value is unbounded over the feasible region; and it solves problems with one or more optimal solutions. The method is also self-initiating. It uses itself either to generate an appropriate feasible solution, as required, to start the method, or to show that the problem has no feasible solution. Each of these features will be discussed in this chapter.

Second, the simplex method provides much more than just optimal solutions. As byproducts, it indicates how the optimal solution varies as a function of the problem data (cost coefficients, constraint coefficients, and righthand-side data). This information is intimately related to a linear program called the *dual* to the given problem, and the simplex method automatically solves this dual problem along with the given problem. These characteristics of the method are of primary importance for applications, since data rarely is known with certainty and usually is approximated when formulating a problem. These features will be discussed in detail in the chapters to follow.

Before presenting a formal description of the algorithm, we consider some examples. Though elementary, these examples illustrate the essential algebraic and geometric features of the method and motivate the general procedure.

2.1 SIMPLEX METHOD—A PREVIEW

Optimal Solutions

Consider the following linear program:

$$\text{Maximize } z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20, \qquad \text{(Objective 1)}$$

subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6, && (1)\\ x_2 - 8x_3 + 4x_4 &= 4, && (2)\end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Note that as stated the problem has a very special form. It satisfies the following:

1. All decision variables are constrained to be nonnegative.
2. All constraints, except for the nonnegativity of decision variables, are stated as equalities.

3. The righthand-side coefficients are all nonnegative.
4. One decision variable is isolated in each constraint with a $+1$ coefficient (x_1 in constraint (1) and x_2 in constraint (2)). The variable isolated in a given constraint does not appear in any other constraint, and appears with a zero coefficient in the objective function.

A problem with this structure is said to be in *canonical form*. This formulation might appear to be quite limited and restrictive; as we will see later, however, *any* linear programming problem can be transformed so that it is in canonical form. Thus, the following discussion is valid for linear programs in general.

Observe that, given any values for x_3 and x_4, the values of x_1 and x_2 are determined uniquely by the equalities. In fact, setting $x_3 = x_4 = 0$ immediately gives a feasible solution with $x_1 = 6$ and $x_2 = 4$. Solutions such as these will play a central role in the simplex method and are referred to as *basic feasible solutions*. In general, given a canonical form for any linear program, a basic feasible solution is given by setting the variable isolated in constraint j, called the jth *basic-variable*, equal to the righthand side of the jth constraint and by setting the remaining variables, called *nonbasic*, all to zero. Collectively the basic variables are termed a *basis*.*

In the example above, the basic feasible solution $x_1 = 6,\ x_2 = 4,\ x_3 = 0,\ x_4 = 0$, is optimal. For any other feasible solution, x_3 and x_4 must remain nonnegative. Since their coefficients in the objective function are negative, if either x_3 or x_4 is positive, z will be less than 20. Thus the maximum value for z is obtained when $x_3 = x_4 = 0$.

To summarize this observation, we state the:

Optimality Criterion. Suppose that, in a maximization problem, every nonbasic variable has a nonpositive coefficient in the objective function of a canonical form. Then the basic feasible solution given by the canonical form maximizes the objective function over the feasible region.

Unbounded Objective Value

Next consider the example just discussed but with a new objective function:

$$\text{Maximize } z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20, \qquad \text{(Objective 2)}$$

subject to:

$$\begin{aligned} x_1 \qquad - 3x_3 + 3x_4 &= 6, && (1)\\ x_2 - 8x_3 + 4x_4 &= 4, && (2) \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Since x_3 now has a positive coefficient in the objective function, it appears promising to increase the value of x_3 as much as possible. Let us maintain $x_4 = 0$, increase x_3 to a value t to be determined, and update x_1 and x_2 to preserve feasibility. From constraints (1) and (2),

$$\begin{aligned} x_1 \qquad\qquad &= 6 + 3t,\\ x_2 \qquad &= 4 + 8t, \end{aligned}$$

$$z = 20 + 3t.$$

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* We have introduced the new terms *canonical*, *basis*, and *basic variable* at this early point in our discussion because these terms have been firmly established as part of linear-programming vernacular. *Canonical* is a word used in many contexts in mathematics, as it is here, to mean ‘‘a special or standard representation of a problem or concept,’’ usually chosen to facilitate study of the problem or concept. *Basis* and *basic* are concepts in linear algebra; our use of these terms agrees with linear-algebra interpretations of the simplex method that are discussed formally in Appendix A.

No matter how large t becomes, x_1 and x_2 remain nonnegative. In fact, as t approaches $+\infty$, z approaches $+\infty$. In this case, the objective function is unbounded over the feasible region.

The same argument applies to any linear program and provides the:

> ***Unboundedness Criterion.*** Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form. If that variable has negative or zero coefficients in all constraints, then the objective function is unbounded from above over the feasible region.

Improving a Nonoptimal Solution

Finally, let us consider one further version of the previous problem:

$$\text{Maximize } z = 0x_1 + 0x_2 - 3x_3 + \ \ x_4 + 20, \qquad \text{(Objective 3)}$$

subject to:

$$\begin{aligned} x_1 \qquad - 3x_3 + 3x_4 &= 6, & (1)\\ x_2 - 8x_3 + 4x_4 &= 4, & (2) \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Now as x_4 increases, z increases. Maintaining $x_3 = 0$, let us increase x_4 to a value t, and update x_1 and x_2 to preserve feasibility. Then, as before, from constraints (1) and (2),

$$\begin{aligned} x_1 \qquad\qquad &= 6 - 3t,\\ x_2 \qquad &= 4 - 4t,\\ z &= 20 + \ \ t. \end{aligned}$$

If x_1 and x_2 are to remain nonnegative, we require:

$$6 - 3t \geq 0, \qquad \text{that is, } t \leq \tfrac{6}{3} = 2$$

and

$$4 - 4t \geq 0, \qquad \text{that is, } t \leq \tfrac{4}{4} = 1.$$

Therefore, the largest value for t that maintains a feasible solution is $t = 1$. When $t = 1$, the new solution becomes $x_1 = 3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$, which has an associated value of $z = 21$ in the objective function.

Note that, in the new solution, x_4 has a positive value and x_2 has become zero. Since nonbasic variables have been given zero values before, it appears that x_4 has replaced x_2 as a basic variable. In fact, it is fairly simple to manipulate Eqs. (1) and (2) algebraically to produce a new canonical form, where x_1 and x_4 become the basic variables. If x_4 is to become a basic variable, it should appear with coefficient $+1$ in Eq. (2), and with zero coefficients in Eq. (1) and in the objective function. To obtain a $+1$ coefficient in Eq. (2), we divide that equation by 4, changing the constraints to read:

$$\begin{aligned} x_1 \qquad - 3x_3 + 3x_4 &= 6, & (1)\\ \tfrac{1}{4}x_2 - 2x_3 + \ \ x_4 &= 1. & (2') \end{aligned}$$

Now, to eliminate x_4 from the first constraint, we may multiply Eq. (2′) by 3 and subtract it from constraint (1), giving:

$$x_1 - \tfrac{3}{4}x_2 + 3x_3 \qquad = 3, \qquad (1')$$

$$\tfrac{1}{4}x_2 - 2x_3 + \ \ x_4 = \quad 1. \tag{2$'$}$$

Finally, we may rearrange the objective function and write it as:

$$(-z) \qquad\qquad - 3x_3 + \ \ x_4 = -20 \tag{3}$$

and use the same technique to eliminate x_4; that is, multiply (2′) by -1 and add to Eq. (1) giving:

$$(-z) - \tfrac{1}{4}x_2 - \ \ x_3 \qquad\qquad = -21.$$

Collecting these equations, the system becomes:

$$\text{Maximize } z = 0x_1 - \tfrac{1}{4}x_2 - \ \ x_3 + 0x_4 + 21,$$

subject to:

$$x_1 - \tfrac{3}{4}x_2 + 3x_3 \qquad = \ 3, \tag{1$'$}$$

$$\tfrac{1}{4}x_2 - 2x_3 + \quad x_4 = \ 1, \tag{2$'$}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Now the problem is in canonical form with x_1 and x_4 as basic variables, and z has increased from 20 to 21. Consequently, we are in a position to reapply the arguments of this section, beginning with this improved solution. In this case, the new canonical form satisfies the optimality criterion since all nonbasic variables have nonpositive coefficients in the objective function, and thus the basic feasible solution $x_1 = 3,\ x_2 = 0,\ x_3 = 0,\ x_4 = 1$, is optimal.

The procedure that we have just described for generating a new basic variable is called *pivoting*. It is the essential computation of the simplex method. In this case, we say that we have just pivoted on x_4 in the second constraint. To appreciate the simplicity of the pivoting procedure and gain some additional insight, let us see that it corresponds to nothing more than elementary algebraic manipulations to re-express the problem conveniently.

First, let us use constraint (2) to solve for x_4 in terms of x_2 and x_3, giving:

$$x_4 = \tfrac{1}{4}(4 - x_2 + 8x_3) \quad \text{or} \quad x_4 = \mathbf{1 - \tfrac{1}{4}x_2 + 2x_3}. \tag{2$'$}$$

Now we will use this relationship to substitute for x_4 in the objective equation:

$$\begin{aligned} z &= 0x_1 + 0x_2 - 3x_3 + \left(\mathbf{1 - \tfrac{1}{4}x_2 + 2x_3}\right) + 20, \\ z &= 0x_1 - \tfrac{1}{4}x_2 - \ \ x_3 + 0x_4 + 21, \end{aligned}$$

and also in constraint (1)

$$x_1 \qquad - 3x_3 + 3\left(\mathbf{1 - \tfrac{1}{4}x_2 + 2x_3}\right) = 6,$$

or, equivalently,

$$x_1 - \tfrac{3}{4}x_2 + 3x_3 = 3. \tag{1$'$}$$

Note that the equations determined by this procedure for eliminating variables are the same as those given by pivoting. We may interpret pivoting the same way, even in more general situations, as merely rearranging the system by solving for one variable and then substituting for it. We pivot because, for the new basic variable, we want a $+1$ coefficient in the constraint where it replaces a basic variable, and 0 coefficients in all other constraints and in the objective function.

Consequently, after pivoting, the form of the problem has been altered, but the modified equations still represent the original problem and have the same feasible solutions and same objective value when evaluated at any given feasible solution.

Indeed, the substitution is merely the familiar variable-elimination technique from high-school algebra, known more formally as Gauss–Jordan elimination.

In summary, the basic step for generating a canonical form with an improved value for the objective function is described as:

> ***Improvement Criterion.*** Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form. If that variable has a positive coefficient in some constraint, then a new basic feasible solution may be obtained by pivoting.

Recall that we chose the constraint to pivot in (and consequently the variable to drop from the basis) by determining *which basic variable* first goes to zero as we increase the nonbasic variable x_4. The constraint is selected by taking the ratio of the righthand-side coefficients to the coefficients of x_4 in the constraints, i.e., by performing the *ratio test*:

$$\min\left\{\tfrac{6}{3}, \tfrac{4}{4}\right\}.$$

Note, however, that if the coefficient of x_4 in the second constraint were -4 instead of $+4$, the values for x_1 and x_2 would be given by:

$$\begin{aligned} x_1 \qquad &= 6 - 3t, \\ x_2 &= 4 + 4t, \end{aligned}$$

so that as $x_4 = t$ increases from 0, x_2 never becomes zero. In this case, we would increase x_4 to $t = \frac{6}{3} = 2$. This observation applies in general for any number of constraints, so that we need never compute ratios for nonpositive coefficients of the variable that is coming into the basis, and we establish the following criterion:

> ***Ratio and Pivoting Criterion.*** When improving a given canonical form by introducing variable x_s into the basis, pivot in a constraint that gives the minimum ratio of righthand-side coefficient to corresponding x_s coefficient. Compute these ratios only for constraints that have a positive coefficient for x_s.

Observe that the value t of the variable being introduced into the basis is the minimum ratio. This ratio is zero if the righthand side is zero in the pivot row. In this instance, a new basis will be obtained by pivoting, but the values of the decision variables remain unchanged since $t = 0$.

As a final note, we point out that a linear program may have multiple optimal solutions. Suppose that the optimality criterion is satisfied and a nonbasic variable has a zero objective-function coefficient in the final canonical form. Since the value of the objective function remains unchanged for increases in that variable, we obtain an alternative optimal solution whenever we can increase the variable by pivoting.

Geometrical Interpretation

The three simplex criteria just introduced algebraically may be interpreted geometrically. In order to represent the problem conveniently, we have plotted the feasible region in Figs. 2.1(a) and 2.1(b) in terms of only the nonbasic variables x_3 and x_4. The values of x_3 and x_4 contained in the feasible regions of these figures satisfy the equality constraints and ensure nonnegativity of the basic and nonbasic variables:

$$\begin{aligned} x_1 \qquad &= 6 + 3x_3 - 3x_4 \geq 0, && (1) \\ x_2 &= 4 + 8x_3 - 4x_4 \geq 0, && (2) \end{aligned}$$

$$x_3 \geq 0, \qquad x_4 \geq 0.$$

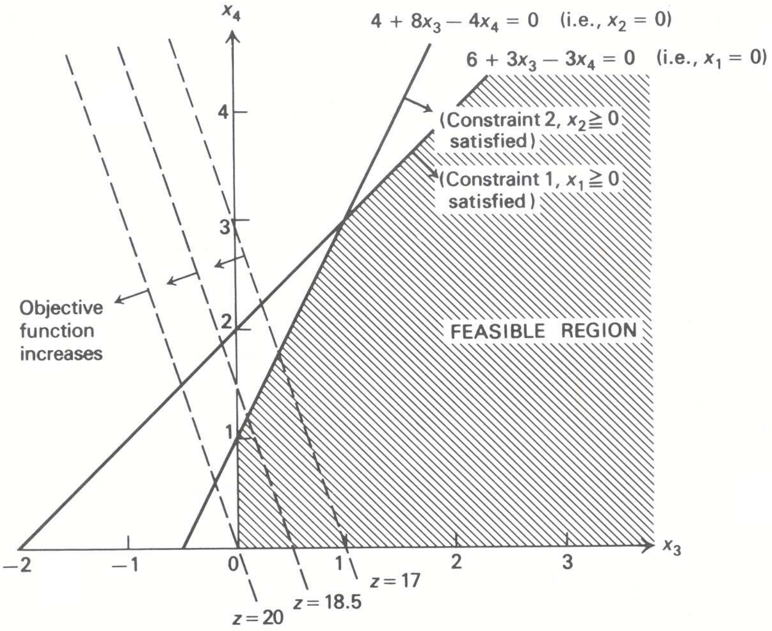

Figure 2.1(a)

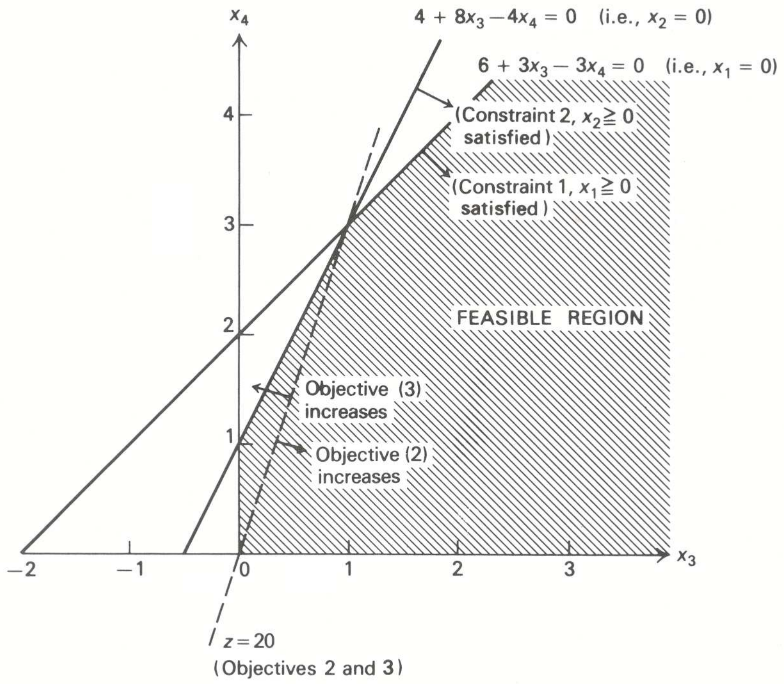

Figure 2.1(b)

Consider the objective function that we used to illustrate the optimality criterion,

$$z = -3x_3 - x_4 + 20. \qquad \text{(Objective 1)}$$

For any value of z, say $z = 17$, the objective function is represented by a straight line in Fig. 2.1(a). As z increases to 20, the line corresponding to the objective function moves parallel to itself across the feasible region. At $z = 20$, it meets the feasible region only at the point $x_3 = x_4 = 0$; and, for $z > 20$, it no longer touches the feasible region. Consequently, $z = 20$ is optimal.

The unboundedness criterion was illustrated with the objective function:

$$z = 3x_3 - x_4 + 20, \qquad \text{(Objective 2)}$$

which is depicted in Fig.2.1(b). Increasing x_3 while holding $x_4 = 0$ corresponds to moving outward from the origin (i.e., the point $x_3 = x_4 = 0$) along the x_3-axis. As we move along the axis, we never meet either constraint (1) or (2). Also, as we move along the x_3-axis, the value of the objective function is increasing to $+\infty$.

The improvement criterion was illustrated with the objective function

$$z = -3x_3 + x_4 + 20, \qquad \text{(Objective 3)}$$

which also is shown in Fig. 2.1(b). Starting from $x_3 = 0,\ x_4 = 0$, and increasing x_4 corresponds to moving from the origin along the x_4-axis. In this case, however, we encounter constraint (2) at $x_4 = t = 1$ and constraint (3) at$x_4 = t = 2$. Consequently, to maintain feasibility in accordance with the ratio test, we move to the intersection of the x_4-axis and constraint (2), which is the optimal solution.

2.2 REDUCTION TO CANONICAL FORM

To this point we have been solving linear programs posed in canonical form with (1) nonnegative variables, (2) equality constraints, (3) nonnegative righthand-side coefficients, and (4) one basic variable isolated in each constraint. Here we complete this preliminary discussion by showing how to transform any linear program to this canonical form.

1. *Inequality constraints*

In Chapter 1, the blast-furnace example contained the two constraints:

$$\begin{aligned} 40x_1 + 10x_2 + 6x_3 &\leq 55.0, \\ 40x_1 + 10x_2 + 6x_3 &\geq 32.5. \end{aligned}$$

The lefthand side in these constraints is the silicon content of the 1000-pound casting being produced. The constraints specify the quality requirement that the silicon content must be between 32.5 and 55.0 pounds. To convert these constraints to equality form, introduce two new nonnegative variables (the blast-furnace example already includes a variable denoted x_4) defined as:

$$\begin{aligned} x_5 &= 55.0 - 40x_1 - 10x_2 - 6x_3, \\ x_6 &= \phantom{55.0 -{}} 40x_1 + 10x_2 + 6x_3 - 32.5. \end{aligned}$$

Variable x_5 measures the amount that the actual silicon content falls *short* of the maximum content that can be added to the casting, and is called a *slack variable*; x_6 is the amount of silicon in *excess* of the minimum requirement and is called a *surplus variable*. The constraints become:

$$\begin{aligned} 40x_1 + 10x_2 + 6x_3 \quad + x_5 \quad\quad &= 55.0, \\ 40x_1 + 10x_2 + 6x_3 \quad\quad\quad - x_6 &= 32.5. \end{aligned}$$

Slack or surplus variables can be used in this way to convert any inequality to equality form.

2. *Free variables*

To see how to treat free variables, or variables unconstrained in sign, consider the basic balance equation of inventory models:

$$\underset{\begin{pmatrix}\text{Production}\\ \text{in period } t\end{pmatrix}}{x_t} \quad + \quad \underset{\begin{pmatrix}\text{Inventory}\\ \text{from period } (t-1)\end{pmatrix}}{I_{t-1}} \quad = \quad \underset{\begin{pmatrix}\text{Demand in}\\ \text{period } t\end{pmatrix}}{d_t} \quad + \quad \underset{\begin{pmatrix}\text{Inventory at}\\ \text{end of period } t\end{pmatrix}}{I_t}.$$

In many applications, we may assume that demand is known and that production x_t must be nonnegative. Inventory I_t may be positive or negative, however, indicating either that there is a surplus of goods to be stored or that there is a shortage of goods and some must be produced later. For instance, if $d_t - x_t - I_{t-1} = 3$, then $I_t = -3$ units must be produced later to satisfy current demand. To formulate models with free variables, we introduce two nonnegative variables I_t^+ and I_t^-, and write

$$I_t = I_t^+ - I_t^-$$

as a substitute for I_t everywhere in the model. The variable I_t^+ represents positive inventory on hand and I_t^- represents backorders (i.e., unfilled demand). Whenever $I_t \geq 0$, we set $I_t^+ = I_t$ and $I_t^- = 0$, and when $I_t < 0$, we set $I_t^+ = 0$ and $I_t^- = -I_t$. The same technique converts any free variable into the difference between two nonnegative variables. The above equation, for example, is expressed with nonnegative variables as:

$$x_t + I_{t-1}^+ - I_{t-1}^- - I_t^+ + I_t^- = d_t.$$

Using these transformations, any linear program can be transformed into a linear program with nonnegative variables and equality constraints. Further, the model can be stated with only nonnegative righthand-side values by multiplying by -1 any constraint with a negative righthand side. Then, to obtain a canonical form, we must make sure that, in each constraint, one basic variable can be isolated with a $+1$ coefficient. Some constraints already will have this form. For example, the slack variable x_5 introduced previously into the silicon equation,

$$40x_1 + 10x_2 + 6x_3 + x_5 = 55.0,$$

appears in no other equation in the model. It can function as an intial basic variable for this constraint. Note, however, that the surplus variable x_6 in the constraint

$$40x_1 + 10x_2 + 6x_3 - x_6 = 32.5$$

does not serve this purpose, since its coefficient is -1.

3. *Artificial variables*

There are several ways to isolate basic variables in the constraints where one is not readily apparent. One particularly simple method is just to add a new variable to any equation that requires one. For instance, the last constraint can be written as:

$$40x_1 + 10x_2 + 6x_3 - x_6 + x_7 = 32.5,$$

with nonnegative basic variable x_7. This new variable is completely fictitious and is called an *artificial variable*. Any solution with $x_7 = 0$ is feasible for the original problem, but those with $x_7 > 0$ are not feasible. Consequently, we should attempt to drive the artificial variable to zero. In a minimization problem, this can be accomplished by attaching a high unit cost M (>0) to x_7 in th objective function (for maximization, add the penalty $-Mx_7$ to the objective function). For M sufficiently large, x_7 will be zero in the final linear programming solution, so that the solution satisfies the original problem constraint without the artificial variable. If $x_7 > 0$ in the final tableau, then there is no solution to the original problem where the artificial variables have been removed; that is, we have shown that the problem is infeasible.

Let us emphasize the distinction between artificial and slack variables. Whereas slack variables have meaning in the problem formulation, artificial variables have no significance; they are merely a mathematical convenience useful for initiating the simplex algorithm.

This procedure for penalizing artificial variables, called the *big M method,* is straightforward conceptually and has been incorporated in some linear programming systems. There are, however, two serious drawbacks to its use. First, we don't know *a priori* how large M must be for a given problem to ensure that all artificial variables are driven to zero. Second, using large numbers for M may lead to numerical difficulties on a computer. Hence, other methods are used more commonly in practice.

An alternative to the big M method that is often used for initiating linear programs is called the *phase I–phase II procedure* and works in two stages. Phase I determines a canonical form for the problem by solving a linear program related to the original problem formulation. The second phase starts with this canonical form to solve the original problem.

To illustrate the technique, consider the linear program:

$$\text{Maximize } z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$$

subject to:

$$\begin{array}{rrrrrrrrrrrcl}
x_1 & - x_2 & + x_3 & - x_4 & - 4x_5 & + 2x_6 & - x_7 & & + x_9 & & & = & 4,\\
-3x_1 & + 3x_2 & + x_3 & - x_4 & - 2x_5 & & & + x_8 & & & & = & 6,\\
& & - x_3 & + x_4 & & + x_6 & & & & + x_{10} & & = & 1,\\
x_1 & - x_2 & + x_3 & - x_4 & - x_5 & & & & & & \underbrace{+ x_{11}}_{\substack{\text{Artificial variables}\\ \text{added}}} & = & 0,\\
\end{array}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 11).$$

Assume that x_8 is a slack variable, and that the problem has been augmented by the introduction of artificial variables $x_9,\ x_{10}$, and x_{11} in the first, third and fourth constraints, so that $x_8,\ x_9,\ x_{10}$, and x_{11} form a basis. The following elementary, yet important, fact will be useful:

Any feasible solution to the augmented system with all artificial variables equal to zero provides a feasible solution to the original problem. Conversely, every feasible solution to the original problem provides a feasible solution to the augmented system by setting all artificial variables to zero.

Next, observe that since the artificial variables $x_9,\ x_{10}$, and x_{11} are all nonnegative, they are all zero only when their sum $x_9 + x_{10} + x_{11}$ is zero. For the basic feasible solution just derived, this sum is 5. Consequently, the artificial variables can be eliminated by ignoring the original objective function for the time being and minimizing $x_9 + x_{10} + x_{11}$ (i.e., minimizing the sum of all artificial variables). Since the artificial variables are all nonnegative, minimizing their sum means driving their sum towards zero. If the minimum sum is 0, then the artificial variables are all zero and a feasible, but not necessarily optimal, solution to the original problem has been obtained. If the minimum is greater than zero, then every solution to the augmented system has $x_9 + x_{10} + x_{11} > 0$, so that *some* artificial variable is still positive. In this case, the original problem has no feasible solution.

The essential point to note is that minimizing the infeasibility in the augmented system is a linear program. Moreover, adding the artificial variables has isolated one basic variable in each constraint. To complete the canonical form of the phase I linear program, we need to eliminate the basic variables from the phase I objective function. Since we have presented the simplex method in terms of maximizing an objective function, for the phase I linear program we will maximize w defined to be *minus* the sum of the artificial variables, rather than minimizing their sum directly. The canonical form for the phase I linear program is then determined simply by adding the artificial variables to the w equation. That is, we add the first, third, and fourth constraints in the previous problem formulation to:

$$(-w) - x_9 - x_{10} - x_{11} = 0,$$

and express this equation as:

$$w = 2x_1 - 2x_2 + x_3 - x_4 - 5x_5 + 3x_6 - x_7 + 0x_9 + 0x_{10} + 0x_{11} - 5.$$

The artificial variables now have zero coefficients in the phase I objective.

Note that the initial coefficients for the nonartificial variable x_j in the w equation is the sum of the coefficients of x_j from the equations with an artificial variable (see Fig. 2.2).

If $w = 0$ is the solution to the phase I problem, then all artificial variables are zero. If, in addition, every artificial variable is nonbasic in this optimal solution, then basic variables have been determined from the original variables, so that a canonical form has been constructed to initiate the original optimization problem. (Some artificial variables may be basic at value zero. This case will be treated in Section 2.5.) Observe that the unboundedness condition is unnecessary. Since the artificial variables are nonnegative, w is bounded from above by zero (for example, $w = -x_9 - x_{10} - x_{11} \leq 0$) so that the unboundedness condition will never apply.

To recap, artificial variables are added to place the linear program in canonical form. Maximizing w either

i) gives max $w < 0$. The original problem is infeasible and the optimization terminates; or
ii) gives max $w = 0$. Then a canonical form has been determined to initiate the original problem. Apply the optimality, unboundedness, and improvement criteria to the original objective function z, starting with this canonical form.

In order to reduce a general linear-programming problem to canonical form, it is convenient to perform the necessary transformations according to the following sequence:

1. Replace each decision variable unconstrained in sign by a difference between two nonnegative variables. This replacement applies to all equations including the objective function.
2. Change inequalities to equalities by the introduction of slack and surplus variables. For $\geq$ inequalities, let the nonnegative *surplus variable* represent the amount by which the lefthand side exceeds the righthand side; for $\leq$ inequalities, let the nonnegative *slack variable* represent the amount by which the righthand side exceeds the lefthand side.
3. Multiply equations with a negative righthand side coefficient by -1.
4. Add a (nonnegative) artificial variable to any equation that does not have an isolated variable readily apparent, and construct the phase I objective function.

To illustrate the orderly application of these rules we provide, in Fig. 2.2, a full example of reduction to canonical form. The succeeding sets of equations in this table represent the stages of problem transformation as we apply each of the steps indicated above. We should emphasize that at each stage the form of the given problem is exactly equivalent to the original problem.

2.3 SIMPLEX METHOD—A FULL EXAMPLE

The simplex method for solving linear programs is but one of a number of methods, or algorithms, for solving optimization problems. By an algorithm, we mean a systematic procedure, usually iterative, for solving a class of problems. The simplex method, for example, is an algorithm for solving the class of linear-programming problems. Any finite optimization algorithm should terminate in one, and only one, of the following possible situations:

1. by demonstrating that there is no feasible solution;
2. by determining an optimal solution; or
3. by demonstrating that the objective function is unbounded over the feasible region.

Problem

Maximize $z = -3y_1 + 2y_2 - y_3 + 4y_4,$

subject to:

$$\begin{aligned} y_1 + y_2 - 4y_3 + 2y_4 &\geqq 4, \\ -3y_1 + y_2 - 2y_3 &\leqq 6, \\ y_2 - y_4 &= -1, \\ y_1 + y_2 - y_3 &= 0, \\ y_3 \geqq 0, \quad y_4 &\geqq 0. \end{aligned}$$

STEP 1 REDUCTION

COMMENTS

Maximize $z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$

subject to:

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 - 4x_5 + 2x_6 &\geqq 4, \\ -3x_1 + 3x_2 + x_3 - x_4 - 2x_5 &\leqq 6, \\ x_3 - x_4 - x_6 &= -1, \\ x_1 - x_2 + x_3 - x_4 - x_5 &= 0, \\ x_j \geqq 0 \quad (j = 1, 2, \ldots, 6). \end{aligned}$$

Substitute

$x_1 - x_2 = y_1, \quad x_5 = y_3$

$x_3 - x_4 = y_2, \quad x_6 = y_4.$

STEP 2 REDUCTION

Maximize $z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$

subject to:

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 - 4x_5 + 2x_6 - x_7 &= 4, \\ -3x_1 + 3x_2 + x_3 - x_4 - 2x_5 + x_8 &= 6, \\ x_3 - x_4 - x_6 &= -1, \\ x_1 - x_2 + x_3 - x_4 - x_5 &= 0, \\ x_j \geqq 0 \quad (j = 1, 2, \ldots, 8). \end{aligned}$$

Introduce surplus variable, x_7.

Introduce slack variable, x_8.

STEP 3 REDUCTION

Maximize $z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$

subject to:

$$
\begin{array}{rrrrrrrrl}
x_1 & - x_2 & + x_3 & - x_4 & - 4x_5 & + 2x_6 & - x_7 & & = 4, \\
-3x_1 & + 3x_2 & + x_3 & - x_4 & - 2x_5 & & & + x_8 & = 6, \\
 & & - x_3 & + x_4 & & + x_6 & & & = 1, \\
x_1 & - x_2 & + x_3 & - x_4 & - x_5 & & & & = 0,
\end{array}
$$

$$x_j \geqq 0 \qquad (j = 1, 2, \ldots, 8).$$

The third constraint was multiplied by -1.

STEP 4 REDUCTION

Maximize $z = -3x_1 + 3x_2 + 2x_3 - 2x_4 - x_5 + 4x_6,$

subject to:

$$
\begin{array}{rrrrrrrrrrrl}
x_1 & - x_2 & + x_3 & - x_4 & - 4x_5 & + 2x_6 & - x_7 & & + x_9 & & & = 4, \\
-3x_1 & + 3x_2 & + x_3 & - x_4 & - 2x_5 & & & + x_8 & & & & = 6, \\
 & & - x_3 & + x_4 & & + x_6 & & & & + x_{10} & & = 1, \\
x_1 & - x_2 & + x_3 & - x_4 & - x_5 & & & & & & + x_{11} & = 0,
\end{array}
$$

$$x_j \geqq 0 \qquad (j = 1, 2, \ldots, 11).$$

Artificial variables, x_9, x_{10}, x_{11}, added.

PHASE I OBJECTIVE

$$
\begin{array}{rl}
\text{Maximize } w = & \qquad\qquad\qquad\qquad\qquad\qquad - x_9 - x_{10} - x_{11} \\
= & +2x_1 - 2x_2 + x_3 - x_4 - 5x_5 + 3x_6 - x_7 \qquad - 5.
\end{array}
$$

Constraints 1, 3, and 4 added to w objective.

Fig. 2.2 Reduction to canonical form.

We will say that an algorithm solves a problem if it always satisfies one of these three conditions. As we shall see, a major feature of the simplex method is that it solves any linear-programming problem.

Most of the algorithms that we are going to consider are iterative, in the sense that they move from one decision point $x_1, x_2, \ldots, x_n$ to another. For these algorithms, we need:

i) a starting point to initiate the procedure;
ii) a termination criterion to indicate when a solution has been obtained; and
iii) an improvement mechanism for moving from a point that is not a solution to a better point.

Every algorithm that we develop should be analyzed with respect to these three requirements.

In the previous section, we discussed most of these criteria for a sample linear-programming problem. Now we must extend that discussion to give a formal and general version of the simplex algorithm. Before doing so, let us first use the improvement criterion of the previous section iteratively to solve a complete problem. To avoid needless complications at this point, we select a problem that does not require artificial variables.

Simple Example.* The owner of a shop producing automobile trailers wishes to determine the best mix for his three products: flat-bed trailers, economy trailers, and luxury trailers. His shop is limited to working 24 days/month on metalworking and 60 days/month on woodworking for these products. The following table indicates production data for the trailers.

	Usage per unit of trailer			*Resources availabilities*
	Flat-bed	*Economy*	*Luxury*	
Metalworking days	$\frac{1}{2}$	2	1	24
Woodworking days	1	2	4	60
Contribution ($ × 100)	6	14	13	

Let the decision variables of the problem be:

x_1 = Number of flat-bed trailers produced per month,
x_2 = Number of economy trailers produced per month,
x_3 = Number of luxury trailers produced per month.

Assuming that the costs for metalworking and woodworking capacity are fixed, the problem becomes:

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + 2x_2 + x_3 &\le 24, \\ x_1 + 2x_2 + 4x_3 &\le 60, \\ x_1 \ge 0, \qquad x_2 \ge 0, \qquad x_3 &\ge 0. \end{aligned}$$

Letting x_4 and x_5 be slack variables corresponding to unused hours of metalworking and woodworking capacity, the problem above is equivalent to the linear program:

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + 2x_2 + x_3 + x_4 &= 24, \\ x_1 + 2x_2 + 4x_3 + x_5 &= 60, \end{aligned}$$

---

* Excel spreadsheet available at `http://web.mit.edu/15.053/www/Sect2.3_Simple_Example.xls`

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 5).$$

This linear program is in canonical form with basic variables x_4 and x_5. To simplify our exposition and to more nearly parallel the way in which a computer might be used to solve problems, let us adopt a tabular representation of the equations instead of writing them out in detail. Tableau 1 corresponds to the given canonical form. The first two rows in the tableau are self-explanatory; they simply represent the constraints, but with the variables detached. The third row represents the z-equation, which may be rewritten as:

$$(-z) + 6x_1 + 14x_2 + 13x_3 = 0.$$

By convention, we say that $(-z)$ is the basic variable associated with this equation. Note that no formal column has been added to the tableau for the $(-z)$-variable.

Tableau 1

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
← x_4	24	$\frac{1}{2}$	$\textcircled{2}$	1	1		$\boxed{1}$	24/2
x_5	60	1	2	4		1	$\boxed{2}$	60/2
$(-z)$	0	+6	+14	+13			$\boxed{3}$	
			↑					

The data to the right of the tableau is not required for the solution. It simply identifies the rows and summarizes calculations. The arrow below the tableau indicates the variable being introduced into the basis; the circled element of the tableau indicates the pivot element; and the arrow to the left of the tableau indicates the variable being removed from the basis.

By the improvement criterion, introducing either x_1, x_2, or x_3 into the basis will improve the solution. The simplex method selects the variable with best payoff per unit (largest objective coefficient), in this case x_2. By the ratio test, as x_2 is increased, x_4 goes to zero before x_5 does; we should pivot in the first constraint. After pivoting, x_2 replaces x_4 in the basis and the new canonical form is as given in Tableau 2.

Tableau 2

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
x_2	12	$\frac{1}{4}$	1	$\frac{1}{2}$	$\frac{1}{2}$		$\boxed{4} = \frac{1}{2}\boxed{1}$	12/(1/2)
← x_5	36	$\frac{1}{2}$		$\textcircled{3}$	-1	1	$\boxed{5} = \boxed{2} - 2\boxed{4}$	36/3
$(-z)$	-168	$+\frac{5}{2}$		+6	-7		$\boxed{6} = \boxed{3} - 14\boxed{4}$	
				↑				

Next, x_3 is introduced in place of x_5 (Tableau 3).

Tableau 3

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
← x_2	6	$\textcircled{\frac{1}{6}}$	1		$\frac{2}{3}$	$-\frac{1}{6}$	$\boxed{7} = \boxed{4} - \frac{1}{2}\boxed{8}$	6/(1/6)
x_3	12	$\frac{1}{6}$		1	$-\frac{1}{3}$	$\frac{1}{3}$	$\boxed{8} = \frac{1}{3}\boxed{5}$	12/(1/6)
$(-z)$	-240	$+\frac{3}{2}$			-5	-2	$\boxed{9} = \boxed{6} - 6\boxed{8}$	
		↑						

Finally, x_1 is introduced in place of x_2 (Tableau 4).

Tableau 4 satisfies the optimality criterion, giving an optimal contribution of \$29,400 with a monthly production of 36 flat-bed trailers and 6 luxury trailers.

Note that in this example, x_2 entered the basis at the first iteration, but does not appear in the optimal basis. In general, a variable might be introduced into (and dropped from) the basis several times. In fact, it is possible for a variable to enter the basis at one iteration and drop from the basis at the very next iteration.

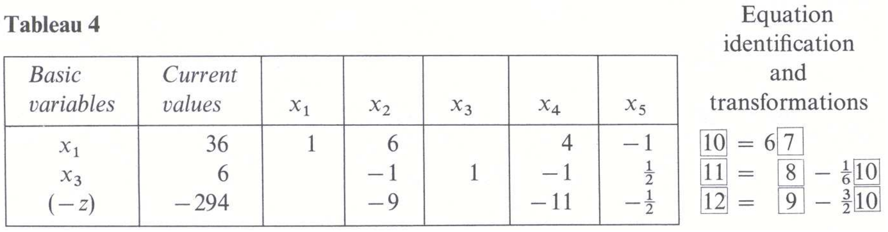

Tableau 4

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations
x_1	36	1	6		4	-1	$\boxed{10} = 6\boxed{7}$
x_3	6		-1	1	-1	$\frac{1}{2}$	$\boxed{11} = \boxed{8} - \frac{1}{6}\boxed{10}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$	$\boxed{12} = \boxed{9} - \frac{3}{2}\boxed{10}$

Variations

The simplex method changes in minor ways if the canonical form is written differently. Since these modifications appear frequently in the management-science literature, let us briefly discuss these variations. We have chosen to consider the maximizing form of the linear program (max z) and have written the objective function in the canonical form as:

$$(-z) + c_1x_1 + c_2x_2 + \cdots + c_nx_n = -z_0,$$

so that the current solution has $z = z_0$. We argued that, if all $c_j \leq 0$, then $z = z_0+c_1x_1+c_2x_2+\cdots+c_nx_n \geq z_0$ for any feasible solution, so that the current solution is optimal. If instead, the objective equation is written as:

$$(z) + c'_1x_1 + c'_2x_2 + \cdots + c'_nx_n = z_0,$$

where $c'_j = -c_j$, then z is maximized if each coefficient $c'_j \geq 0$. In this case, the variable with the most negative coefficient $c'_j < 0$ is chosen to enter the basis. All other steps in the method are unaltered.

The same type of association occurs for the minimizing objective:

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n.$$

If we write the objective function as:

$$(-z) + c_1x_1 + c_2x_2 + \cdots + c_nx_n = -z_0,$$

then, since $z = z_0 + c_1x_1 + \cdots + c_nx_n$, the current solution is optimal if every $c_j \geq 0$. The variable x_s to be introduced is selected from $c_s = \min c_j < 0$, and every other step of the method is the same as for the maximizing problem. Similarly, if the objective function is written as:

$$(z) + c'_1x_1 + c'_2x_2 + \cdots + c'_nx_n = z_0,$$

where $c'_j = -c_j$, then, for a minimization problem, we introduce the variable with the most positive c_j into the basis.

Note that these modifications affect only the way in which the variable entering the basis is determined. The pivoting computations are not altered.

Given these variations in the selection rule for incoming variables, we should be wary of memorizing formulas for the simplex algorithm. Instead, we should be able to argue as in the previous example and as in the simplex preview. In this way, we maintain flexibility for almost any application and will not succumb to the rigidity of a ''formula trap.''

2.4 FORMAL PROCEDURE

Figure 2.3 summarizes the simplex method in flow-chart form. It illustrates both the computational steps of the algorithm and the interface between phase I and phase II. The flow chart indicates how the algorithm is used to show that the problem is infeasible, to find an optimal solution, or to show that the objective function is unbounded over the feasible region. Figure 2.4 illustrates this algorithm for a phase I–phase II example by solving the problem introduced in Section 2.2 for reducing a problem to canonical form.* The remainder of this section specifies the computational steps of the flow chart in algebraic terms.

At any intermediate step during phase II of the simplex algorithm, the problem is posed in the following canonical form:

$$
\begin{array}{llllllllllll}
x_1 & & & & & + \bar{a}_{1,\, m+1}x_{m+1} & + \cdots & + \bar{a}_{1s}x_s & + \cdots & + \bar{a}_{1n}x_n & = & \bar{b}_1, \\
 & x_2 & & & & + \bar{a}_{2,\, m+1}x_{m+1} & + \cdots & & & + \bar{a}_{2n}x_n & = & \bar{b}_2, \\
 & & \ddots & & & \vdots & & & & \vdots & & \vdots \\
 & & & x_r & & + \bar{a}_{r,\, m+1}x_{m+1} & + \cdots & + \boxed{\bar{a}_{rs}}\, x_s & + \cdots & + \bar{a}_{rn}x_n & = & \bar{b}_r, \\
 & & & & \ddots & \vdots & & & & \vdots & & \vdots \\
 & & & & x_m & + \bar{a}_{m,\, m+1}x_{m+1} & + \cdots & + \bar{a}_{ms}x_s & + \cdots & + \bar{a}_{mn}x_n & = & \bar{b}_m, \\
(-z) & & & & & + \bar{c}_{m+1}x_{m+1} & + \cdots & + \bar{c}_s x_s & + \cdots & + \bar{c}_n x_n & = & -\bar{z}_0,
\end{array}
$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

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[*] Excel spreadsheet available at `http://web.mit.edu/15.053/www/Fig2.4_Pivoting.xls`

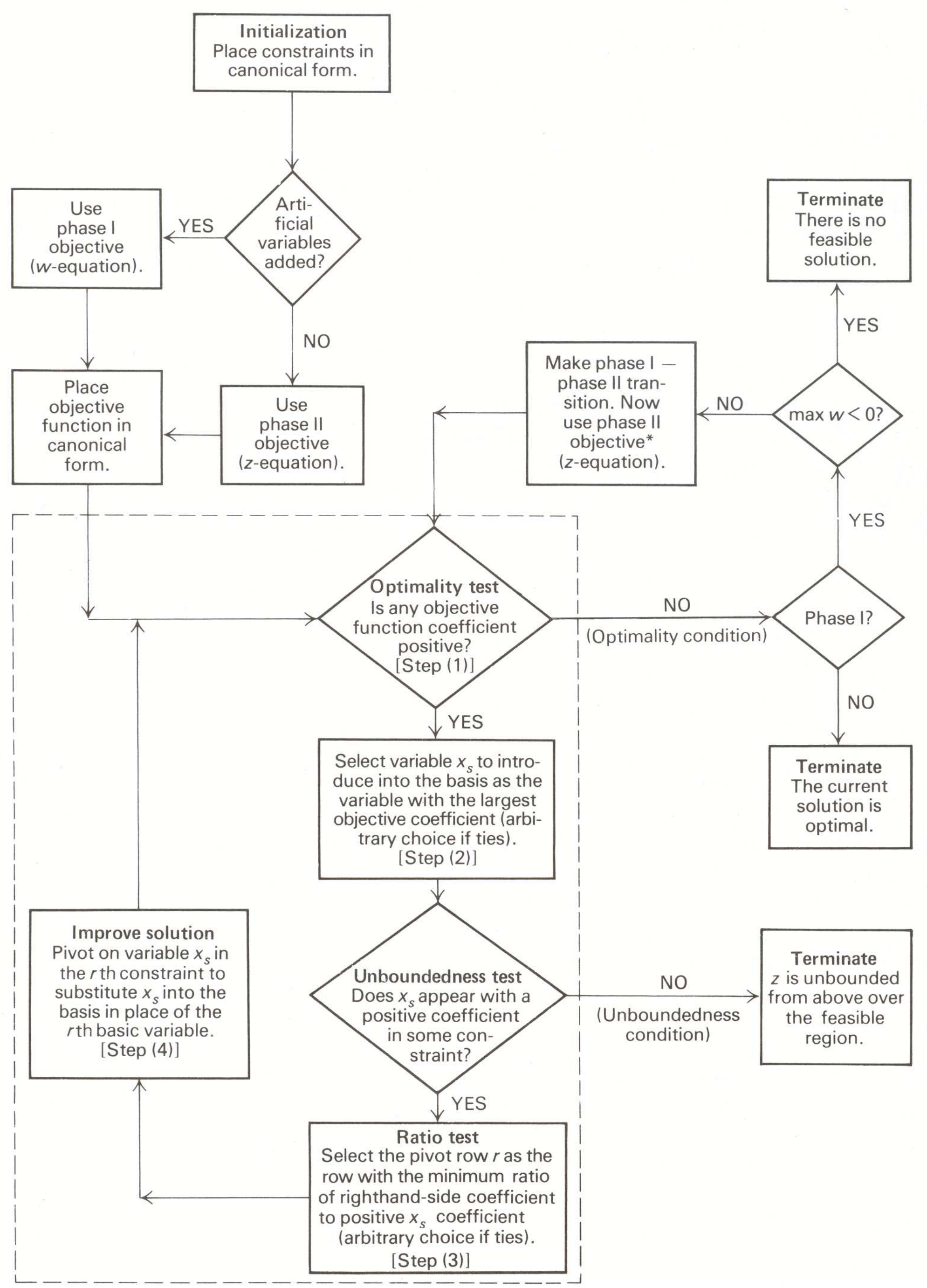

Figure 2.1 Simplex phase I–phase II maximization procedure.

Initial tableau

										Artificial variables				
Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	Equation identification and transformations	Ratio test
x_9	4	1	-1	1	-1	-4	2	-1		1			$\boxed{1}$	4/2
x_8	6	-3	3	1	-1	-2	0	0	1				$\boxed{2}$	
← x_{10}	1	0	0	-1	1	0	①	0			1		$\boxed{3}$	1/1
x_{11}	0	1	-1	1	-1	-1	0	0				1	$\boxed{4}$	
$(-z)$	0	-3	3	2	-2	-1	4	0					$\boxed{5}$	
$(-w)$	5	2	-2	1	-1	-5	3	-1					$\boxed{6}$	
							↑							

Tableau 2

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}		
x_9	2	1	-1	3	-3	-4		-1		1	-2		$\boxed{7} = \boxed{1} - 2\boxed{3}$	2/3
x_8	6	-3	3	1	-1	-2		0	1		0		$\boxed{8} = \boxed{2}$	6/1
x_6	1	0	0	-1	1	0	1	0			1		$\boxed{9} = \boxed{3}$	
← x_{11}	0	1	-1	①	-1	-1		0			0	1	$\boxed{10} = \boxed{4}$	0/1
$(-z)$	-4	-3	3	6	-6	-1		0			-4		$\boxed{11} = \boxed{5} - 4\boxed{3}$	
$(-w)$	2	2	-2	4	-4	-5		-1			-3		$\boxed{12} = \boxed{6} - 3\boxed{3}$	
				↑										

Tableau 3

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}		
← x_9	2	-2	②		0	-1		-1		1	-2	-3	$\boxed{13} = \boxed{7} - 3\boxed{10}$	2/2
x_8	6	-4	4		0	-1		0	1		0	-1	$\boxed{14} = \boxed{8} - \boxed{10}$	6/4
x_6	1	1	-1		0	-1	1	0			1	1	$\boxed{15} = \boxed{9} + \boxed{10}$	
x_3	0	1	-1	1	-1	-1		0			0	1	$\boxed{16} = \boxed{10}$	
$(-z)$	-4	-9	9		0	5		0			-4	-6	$\boxed{17} = \boxed{11} - 6\boxed{10}$	
$(-w)$	2	-2	2		0	-1		-1			-3	-4	$\boxed{18} = \boxed{12} - 4\boxed{10}$	
			↑											

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_2	1	-1	1		0	$-\frac{1}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$	-1	$-\frac{3}{2}$
← x_8	2	0			0	①		2	1	-2	4	5
x_6	2	0			0	$-\frac{3}{2}$	1	$-\frac{1}{2}$		$\frac{1}{2}$	0	$-\frac{1}{2}$
x_3	1	0		1	-1	$-\frac{3}{2}$		$-\frac{1}{2}$		$\frac{1}{2}$	-1	$-\frac{1}{2}$
$(-z)$	-13	0			0	$\frac{19}{2}$		$\frac{9}{2}$		$-\frac{9}{2}$	5	$\frac{15}{2}$
$(-w)$	0	0			0	0		0		-1	-1	-1
						↑						

$\boxed{19} = \frac{1}{2}\boxed{13}$

$\boxed{20} = \boxed{14} - 4\boxed{19}$ 2/1

$\boxed{21} = \boxed{15} + \boxed{19}$

$\boxed{22} = \boxed{16} + \boxed{19}$

$\boxed{23} = \boxed{17} - 9\boxed{19}$

$\boxed{24} = \boxed{18} - 2\boxed{19}$

End of phase I. All artificial variables are nonbasic, so proceed with phase II, dropping the w-equation and maintaining $x_9 = x_{10} = x_{11} = 0$ (i.e., never introduce an artificial variable into the basis).

Final tableau

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_2	2	-1	1		0			$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	1
x_5	2	0			0	1		2	1	-2	4	5
x_6	5	0			0		1	$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{5}{2}$	6	7
x_3	4	0		1	-1			$\frac{5}{2}$	$\frac{3}{2}$	$-\frac{5}{2}$	5	7
$(-z)$	-32	0			0			$-\frac{29}{2}$	$-\frac{19}{2}$	$\frac{29}{2}$	-33	-40

$\boxed{25} = \boxed{19} + \frac{1}{2}\boxed{20}$

$\boxed{26} = \boxed{20}$

$\boxed{27} = \boxed{21} + \frac{3}{2}\boxed{20}$

$\boxed{28} = \boxed{22} + \frac{3}{2}\boxed{20}$

$\boxed{29} = \boxed{23} - \frac{19}{2}\boxed{20}$

End of phase II. Substituting for the original variables, the optimal solution is $y_1 = -2$, $y_2 = 4$, $y_3 = 2$, $y_4 = 5$, max $z = 32$.

Fig. 2.4 Phase I–phase II illustration.

Originally, this canonical form is developed by using the procedures of Section 2.2. The data $\overline{a}_{ij}$, $\overline{b}_i$, $\overline{z}_0$, $\overline{w}_0$, and $\overline{c}_j$ are known. They are either the original data (without bars) or that data as updated by previous steps of the algorithm. We have assumed (by reindexing variables if necessary) that $x_1, x_2, \ldots, x_m$ are the basic variables. Also, since this is a canonical form, $\overline{b}_i \geq 0$ for $i = 1, 2, \ldots, m$.

Simplex Algorithm (Maximization Form)

STEP (0) The problem is initially in canonical form and all $\overline{b}_i \geq 0$.

STEP (1) If $\overline{c}_j \leq 0$ for $j = 1, 2, \ldots, n$, then *stop*; we are optimal. If we continue then there exists some $\overline{c}_j > 0$.

STEP (2) Choose the column to pivot in (i.e., the variable to introduce into the basis) by:

$$\overline{c}_s = \max_j \{\overline{c}_j \,|\overline{c}_j > 0\}.^*$$

If $\overline{a}_{is} \leq 0$ for $i = 1, 2, \ldots, m$, then *stop*; the primal problem is unbounded. If we continue, then $\overline{a}_{is} > 0$ for some $i = 1, 2, \ldots, m$.

STEP (3) Choose row r to pivot in (i.e., the variable to drop from the basis) by the ratio test:

$$\frac{\overline{b}_r}{\overline{a}_{rs}} = \min_i \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \,\middle|\, \overline{a}_{is} > 0 \right\}.$$

STEP (4) Replace the basic variable in row r with variable s and re-establish the canonical form (i.e., pivot on the coefficient $\overline{a}_{rs}$).

STEP (5) Go to step (1).

These steps are the essential computations of the simplex method. They apply to either the phase I or phase II problem. For the phase I problem, the coefficients $\overline{c}_j$ are those of the phase I objective function.

The only computation remaining to be specified formally is the effect that pivoting in step (4) has on the problem data. Recall that we pivot on coefficient $\overline{a}_{rs}$ merely to isolate variable x_s with a $+1$ coefficient in constraint r. The pivot can be viewed as being composed of two steps:

i) normalizing the rth constraint so that x_s has a $+1$ coefficient, and
ii) subtracting multiples of the normalized constraint from the order equations in order to eliminate variable x_s.

These steps are summarized pictorially in Fig. 2.5.

The last tableau in Fig. 2.5 specifies the new values for the data. The new righthand-side coefficients, for instance, are given by:

$$\overline{b}_r^{\text{new}} = \frac{\overline{b}_r}{\overline{a}_{rs}} \qquad \text{and} \qquad \overline{b}_i^{\text{new}} = \overline{b}_i - \overline{a}_{is}\left(\frac{\overline{b}_r}{\overline{a}_{rs}}\right) \geq 0 \quad \text{for } i \neq r.$$

Observe that the new coefficients for the variable x_r being removed from the basis summarize the computations. For example, the coefficient of x_r in the first row of the final tableau is obtained from the first tableau by subtracting $\overline{a}_{1s}/\overline{a}_{rs}$ times the rth row from the first row. The coefficients of the other variables in the first row of the third tableau can be obtained from the first tableau by performing this same calculation. This observation can be used to partially streamline the computations of the simplex method. (See Appendix B for details.)

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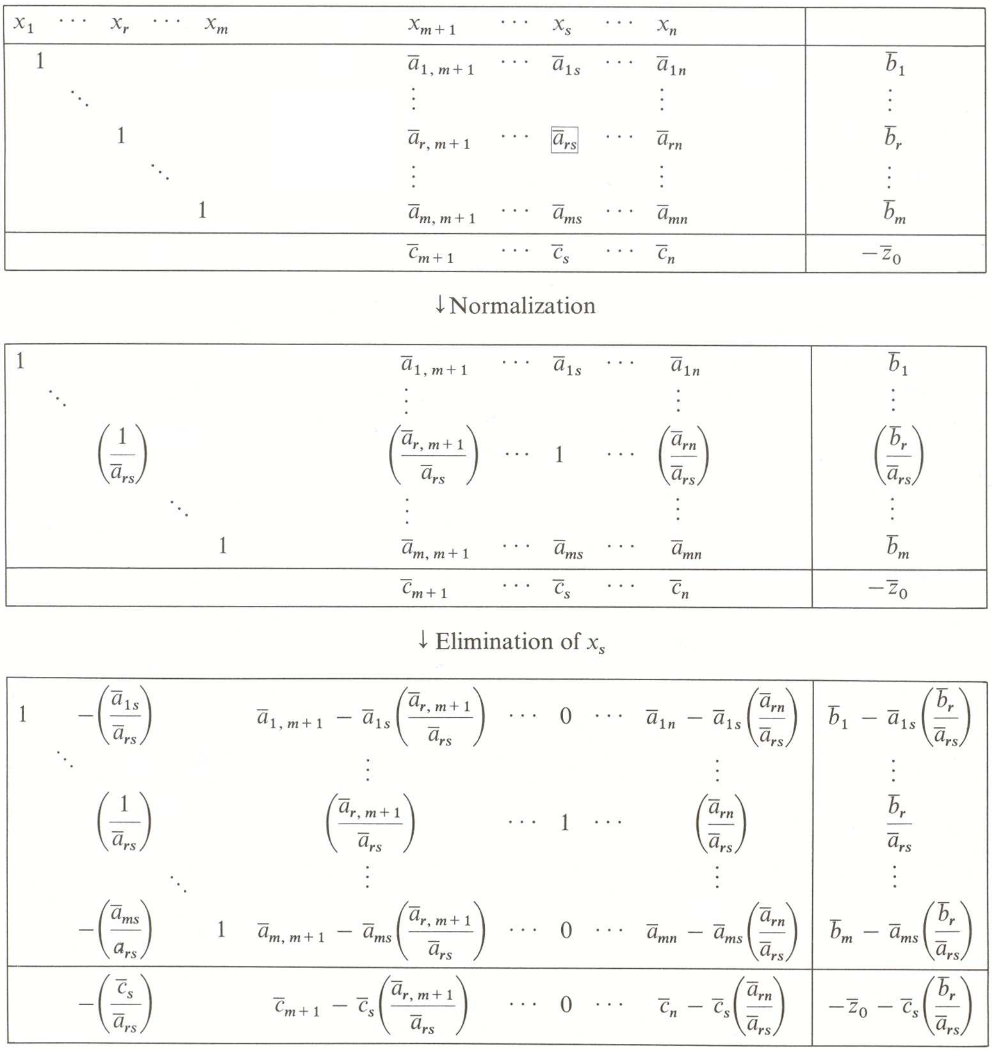
* The vertical bar within braces is an abbreviation for the phrase ‘‘such that.’’



$$\begin{array}{cccccccccc|c}
x_1 & \cdots & x_r & \cdots & x_m & x_{m+1} & \cdots & x_s & \cdots & x_n & \\ \hline
1 & & & & & \overline{a}_{1,m+1} & \cdots & \overline{a}_{1s} & \cdots & \overline{a}_{1n} & \overline{b}_1 \\
 & \ddots & & & & \vdots & & & & \vdots & \vdots \\
 & & 1 & & & \overline{a}_{r,m+1} & \cdots & \boxed{\overline{a}_{rs}} & \cdots & \overline{a}_{rn} & \overline{b}_r \\
 & & & \ddots & & \vdots & & & & \vdots & \vdots \\
 & & & & 1 & \overline{a}_{m,m+1} & \cdots & \overline{a}_{ms} & \cdots & \overline{a}_{mn} & \overline{b}_m \\ \hline
 & & & & & \overline{c}_{m+1} & \cdots & \overline{c}_s & \cdots & \overline{c}_n & -\overline{z}_0
\end{array}$$

↓Normalization

$$\begin{array}{cccccccccc|c}
1 & & & & & \overline{a}_{1,m+1} & \cdots & \overline{a}_{1s} & \cdots & \overline{a}_{1n} & \overline{b}_1 \\
 & \ddots & & & & \vdots & & & & \vdots & \vdots \\
 & & \left(\dfrac{1}{\overline{a}_{rs}}\right) & & & \left(\dfrac{\overline{a}_{r,m+1}}{\overline{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\dfrac{\overline{a}_{rn}}{\overline{a}_{rs}}\right) & \left(\dfrac{\overline{b}_r}{\overline{a}_{rs}}\right) \\
 & & & \ddots & & \vdots & & & & \vdots & \vdots \\
 & & & & 1 & \overline{a}_{m,m+1} & \cdots & \overline{a}_{ms} & \cdots & \overline{a}_{mn} & \overline{b}_m \\ \hline
 & & & & & \overline{c}_{m+1} & \cdots & \overline{c}_s & \cdots & \overline{c}_n & -\overline{z}_0
\end{array}$$

↓ Elimination of x_s

$$\begin{array}{cccccccccc|c}
1 & & -\left(\dfrac{\overline{a}_{1s}}{\overline{a}_{rs}}\right) & & & \overline{a}_{1,m+1} - \overline{a}_{1s}\left(\dfrac{\overline{a}_{r,m+1}}{\overline{a}_{rs}}\right) & \cdots & 0 & \cdots & \overline{a}_{1n} - \overline{a}_{1s}\left(\dfrac{\overline{a}_{rn}}{\overline{a}_{rs}}\right) & \overline{b}_1 - \overline{a}_{1s}\left(\dfrac{\overline{b}_r}{\overline{a}_{rs}}\right) \\
 & \ddots & & & & \vdots & & & & \vdots & \vdots \\
 & & \left(\dfrac{1}{\overline{a}_{rs}}\right) & & & \left(\dfrac{\overline{a}_{r,m+1}}{\overline{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\dfrac{\overline{a}_{rn}}{\overline{a}_{rs}}\right) & \dfrac{\overline{b}_r}{\overline{a}_{rs}} \\
 & & & \ddots & & \vdots & & & & \vdots & \vdots \\
 & & -\left(\dfrac{\overline{a}_{ms}}{a_{rs}}\right) & & 1 & \overline{a}_{m,m+1} - \overline{a}_{ms}\left(\dfrac{\overline{a}_{r,m+1}}{\overline{a}_{rs}}\right) & \cdots & 0 & \cdots & \overline{a}_{mn} - \overline{a}_{ms}\left(\dfrac{\overline{a}_{rn}}{\overline{a}_{rs}}\right) & \overline{b}_m - \overline{a}_{ms}\left(\dfrac{\overline{b}_r}{\overline{a}_{rs}}\right) \\ \hline
 & & -\left(\dfrac{\overline{c}_s}{\overline{a}_{rs}}\right) & & & \overline{c}_{m+1} - \overline{c}_s\left(\dfrac{\overline{a}_{r,m+1}}{\overline{a}_{rs}}\right) & \cdots & 0 & \cdots & \overline{c}_n - \overline{c}_s\left(\dfrac{\overline{a}_{rn}}{\overline{a}_{rs}}\right) & -\overline{z}_0 - \overline{c}_s\left(\dfrac{\overline{b}_r}{\overline{a}_{rs}}\right)
\end{array}$$

Figure 2.5 Algebra for a pivot operation.

Note also that the new value for z will be given by:

$$\overline{z}_0 + \left(\frac{\overline{b}_r}{\overline{a}_{rs}}\right)\overline{c}_s.$$

By our choice of the variable x_s to introduce into the basis, $\overline{c}_s > 0$. Since $\overline{b}_r \geq 0$ and $\overline{a}_{rs} > 0$, this implies that $z^{\text{new}} \geq z^{\text{old}}$. In addition, if $\overline{b}_r > 0$, then z^{new} is strictly greater than z^{old}.

Convergence

Though the simplex algorithm has solved each of our previous examples, we have yet to show that it solves *any* linear program. A formal proof requires results from linear algebra, as well as further technical material that is presented in Appendix B. Let us outline a proof assuming these results. We assume that the linear program has n variables and m equality constraints.

First, note that there are only a finite number of bases for a given problem, since a basis contains m variables (one isolated in each constraint) and there are a finite number of variables to select from. A standard result in linear algebra states that, once the basic variables have been selected, all the entries in the tableau, including the objective value, are determined uniquely. Consequently, there are only a finite number of canonical forms as well. If the objective value *strictly* increases after every pivot, the algorithm never repeats a canonical form and must determine an optimal solution after a *finite* number of pivots (any nonoptimal canonical form is transformed to a new canonical form by the simplex method).

This argument shows that the simplex method solves linear programs as long as the objective value strictly increases after each pivoting operation. As we have just seen, each pivot affects the objective function by adding a multiple of the pivot equation to the objective function. The current value of the z-equation increases by a multiple of the righthand-side coefficient; if this coefficient is positive (not zero), the objective value increases. With this in mind, we introduce the following definition:

> A canonical form is called *nondegenerate* if each righthand-side coefficient is strictly positive. The linear-programming problem is called nondegenerate if, starting with an initial canonical form, every canonical form determined by the algorithm is nondegenerate.

In these terms, we have shown that the simplex method solves every nondegenerate linear program using a finite number of pivoting steps. When a problem is degenerate, it is possible to perturb the data slightly so that every righthand-side coefficient remains positive and again show that the method works. Details are given in Appendix B. A final note is that, empirically, the finite number of iterations mentioned here to solve a problem frequently lies between 1.5 and 2 times the number of constraints (i.e., between $1.5m$ and $2m$).

Applying this perturbation, if required, to both phase I and phase II, we obtain the essential property of the simplex method.

> ***Fundamental Property of the Simplex Method.*** The simplex method (with perturbation if necessary) solves any given linear program in a finite number of iterations. That is, in a finite number of iterations, it shows that there is no feasible solution; finds an optimal solution; or shows that the objective function is unbounded over the feasible region.

Although degeneracy occurs in almost every problem met in practice, it rarely causes any complications. In fact, even without the perturbation analysis, the simplex method never has failed to solve a practical problem, though problems that are highly degenerate with many basic variables at value zero frequently take more computational time than other problems.

Applying this fundamental property to the phase I problem, we see that, if a problem is feasible, the simplex method finds a basic feasible solution. Since these solutions correspond to corner or extreme points of the feasible region, we have the

Fundamental Property of Linear Equations. If a set of linear equations in nonnegative variables is feasible, then there is an extreme-point solution to the equations.

2.5 TRANSITION FROM PHASE I TO PHASE II

We have seen that, if an artificial variable is positive at the end of phase I, then the original problem has no feasible solution. On the other hand, if all artificial variables are nonbasic at value zero at the end of phase I, then a basic feasible solution has been found to initiate the original optimization problem. Section 2.4 furnishes an example of this case. Suppose, though, that when phase I terminates, all artificial variables are zero, but that some artificial variable remains in the basis. The following example illustrates this possibility.

Problem. Find a canonical form for x_1, x_2, and x_3 by solving the phase I problem (x_4, x_5, and x_6 are artificial variables):

$$\text{Maximize } w = \qquad -x_4 - x_5 - x_6,$$

subject to:

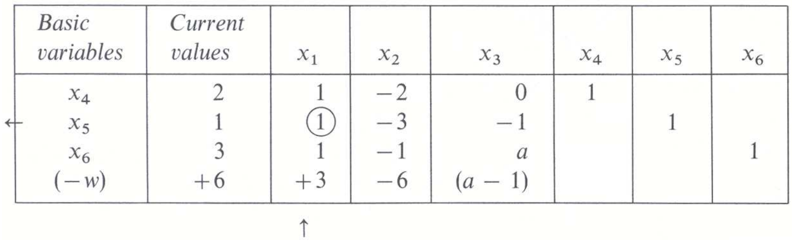
$$\begin{aligned} x_1 - 2x_2 \qquad + x_4 \qquad\qquad &= 2, \\ x_1 - 3x_2 - \ \ x_3 \qquad + x_5 \qquad &= 1, \\ x_1 - \ \ x_2 + ax_3 \qquad\qquad + x_6 &= 3, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 6).$$

To illustrate the various terminal conditions, the coefficient of x_3 is unspecified in the third constraint. Later it will be set to either 0 or 1. In either case, the pivoting sequence will be the same and we shall merely carry the coefficient symbolically.

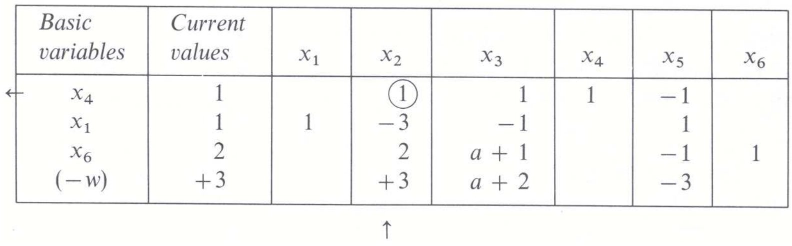
Putting the problem in canonical form by eliminating x_4, x_5, and x_6 from the objective function, the simplex solution to the phase I problem is given in Tableaus 1 through 3.

Tableau 1



Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
x_4	2	1	-2	0	1		
← x_5	1	①	-3	-1		1	
x_6	3	1	-1	a			1
$(-w)$	$+6$	$+3$	-6	$(a-1)$			
		↑					

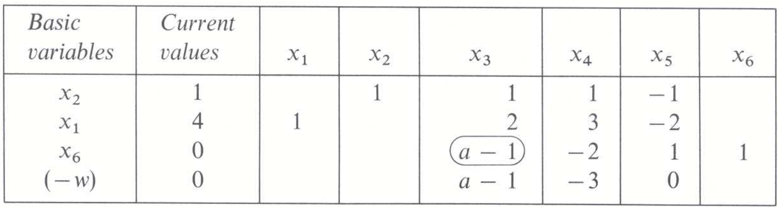
Tableau 2



Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6
← x_4	1		①	1	1	-1	
x_1	1	1	-3	-1		1	
x_6	2		2	$a+1$		-1	1
$(-w)$	$+3$		$+3$	$a+2$		-3	
			↑				

For $a = 0$ or 1, phase I is complete since $\bar{c}_3 = a - 1 \leq 0$, but with x_6 still part of the basis. Note that in Tableau 2, either x_4 or x_6 could be dropped from the basis. We have arbitrarily selected x_4. (A similar argument would apply if x_6 were chosen.)

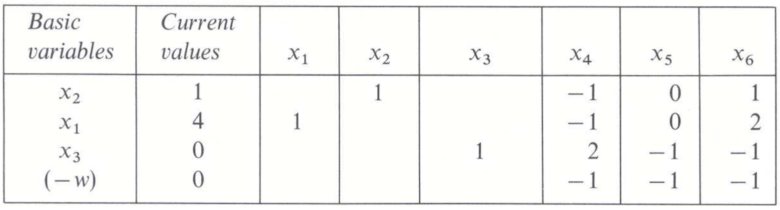
Tableau 3



Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6
x_2	1		1	1	1	-1	
x_1	4	1		2	3	-2	
x_6	0			$\boxed{a - 1}$	-2	1	1
$(-w)$	0			$a - 1$	-3	0	

First, assume $a = 0$. Then we can introduce x_3 into the basis in place of the artificial variable x_6, pivoting on the coefficient $a - 1$ or x_3 in the third constraint, giving Tableau 4.

Tableau 4



Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6
x_2	1		1		-1	0	1
x_1	4	1			-1	0	2
x_3	0			1	2	-1	-1
$(-w)$	0				-1	-1	-1

Note that we have pivoted on a negative coefficient here. Since the righthand-side element of the third equation is zero, dividing by a negative pivot element will not make the resulting righthand-side coefficient negative. Dropping x_4, x_5, and x_6, we obtain the desired canonical form. Note that x_6 is now set to zero and is nonbasic.

Next, suppose that $a = 1$. The coefficient $(a - 1)$ in Tableau 3 is zero, so we cannot pivot x_3 into the basis as above. In this case, however, dropping artificial variables x_4 and x_5 from the system, the third constraint of Tableau 3 reads $x_6 = 0$. Consequently, even though x_6 is a basic variable, in the canonical form for the original problem it will always remain at value zero during phase II. Thus, throughout phase II, a feasible solution to the original problem will be maintained as required. When more than one artificial variable is in the optimal basis for phase I, these techniques can be applied to each variable.

For the general problem, the transition rule from phase I to phase II can be stated as:

> ***Phase I–Phase II Transition Rule.*** Suppose that artificial variable x_i is the ith basic variable at the end of Phase I (at value zero). Let $\overline{a}_{ij}$ be the coefficient of the nonartificial variable x_j in the ith constraint of the final tableau. If some $\overline{a}_{ij} \neq 0$, then pivot on any such $\overline{a}_{ij}$, introducing x_j into the basis in place of x_i. If all $\overline{a}_{ij} = 0$, then maintain x_i in the basis throughout phase II by including the ith constraint, which reads $x_i = 0$.

As a final note, observe that if all $\overline{a}_{ij} = 0$ above, then constraint i is a redundant constraint in the original system, for, by adding multiples of the other equation to constraint i via pivoting, we have produced the equation (ignoring artificial variables):

$$0x_1 + 0x_2 + \cdots + 0x_n = 0.$$

For example, when $a = 1$ for the problem above, (constraint 3) = 2 times (constraint 1)–(constraint 2), and is redundant.

Phase I–Phase II Example

$$\text{Maximize } z = -3x_1 \qquad + x_3,$$

subject to:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4, \\ -2x_1 + x_2 - x_3 &= 1, \\ 3x_2 + x_3 + x_4 &= 9, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4).$$

Adding artificial variables x_5, x_6, and x_7, we first minimize $x_5 + x_6 + x_7$ or, equivalently, maximize $w = -x_5 - x_6 - x_7$. The iterations are shown in Fig. 2.6.* The first tableau is the phase I problem statement. Basic variables x_5, x_6 and x_7 appear in the objective function and, to achieve the initial canonical form, we must add the constraints to the objective function to eliminate these variables.

Tableaus 2 and 3 contain the phase I solution. Tableau 4 gives a feasible solution to the original problem. Artificial variable x_7 remains in the basis and is eliminated by pivoting on the -1 coefficient for x_4. This pivot replaces $x_7 = 0$ in the basis by $x_4 = 0$, and gives a basis from the original variables to initiate phase II.

Tableaus 5 and 6 give the phase II solution.

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* Excel spreadsheet available at `http://web.mit.edu/15.053/www/Fig2.6_Pivoting.xls`

Tableau 1 (Phase I—problem statement)

						Artificial variables				
Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Equation identification and transformations	Ratio test
x_5	4	1	1	1	1	1			$\boxed{1}$	
x_6	1	-2	1	-1	0		1		$\boxed{2}$	
x_7	9	0	3	1	1			1	$\boxed{3}$	
$(-z)$	0	-3	0	1	0				$\boxed{4}$	
$(-w)$	0	0	0	0	0	-1	-1	-1	$\boxed{5}$	

Tableau 2 (Phase I—initial canonical form)

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Equation identification and transformations	Ratio test
x_5	4	1	1	1	1	1			$\boxed{1}$	4/1
← x_6	1	-2	①	-1	0		1		$\boxed{2}$	1/1
x_7	9	0	3	1	1			1	$\boxed{3}$	9/3
$(-z)$	0	-3	0	1	0				$\boxed{4}$	
$(-w)$	14	-1	5	1	2				$\boxed{6} = \boxed{5} + \boxed{1} + \boxed{2} + \boxed{3}$	
			↑							

Tableau 3

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Equation identification and transformations	Ratio test
← x_5	3	③		2	1	1	-1		$\boxed{7} = \boxed{1} - \boxed{2}$	3/3
x_2	1	-2	1	-1	0		1		$\boxed{8} = \boxed{2}$	
x_7	6	6		4	1		-3	1	$\boxed{9} = \boxed{3} - 3\boxed{2}$	6/6
$(-z)$	0	-3		1	0		0		$\boxed{10} = \boxed{4}$	
$(-w)$	9	9		6	2		-5		$\boxed{11} = \boxed{6} - 5\boxed{2}$	
		↑								

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	1		$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	
x_2	3		1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	
← x_7	0			0	$\enclose{circle}{-1}$	-2	-1	1
$(-z)$	3			3	1	1	-1	
$(-w)$	0			0	-1	-3	-2	
					↑			

$\boxed{12} = \frac{1}{3}\boxed{7}$
$\boxed{13} = \boxed{8} + 2\boxed{12}$
$\boxed{14} = \boxed{9} - 6\boxed{12} = \boxed{9} - 2\boxed{7}$
$\boxed{15} = \boxed{10} + 3\boxed{12} = \boxed{10} + \boxed{7}$
$\boxed{16} = \boxed{11} - 9\boxed{12} = \boxed{11} - 3\boxed{7}$
(Pivot to remove artificial variable from the basis.)

Tableau 5 (start phase II)

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
← x_1	1	1		$\enclose{circle}{\frac{2}{3}}$		$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$
x_2	3		1	$\frac{1}{3}$		$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
x_4	0			0	1	2	1	-1
$(-z)$	3			3		-1	-2	1
				↑				

$\boxed{17} = \boxed{12} - \frac{1}{3}\boxed{19}$ 1/(2/3)
$\boxed{18} = \boxed{13} - \frac{2}{3}\boxed{19}$ 3/(1/3)
$\boxed{19} = -1\boxed{14}$
$\boxed{20} = \boxed{15} - \boxed{19}$

Tableau 6

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_3	$\frac{3}{2}$	$\frac{3}{2}$		1		$-\frac{1}{2}$	-1	$\frac{1}{2}$
x_2	$\frac{5}{2}$	$-\frac{1}{2}$	1			$-\frac{1}{2}$	0	$\frac{1}{2}$
x_4	0	0			1	2	1	-1
$(-z)$	$-\frac{3}{2}$	$-\frac{9}{2}$				$\frac{1}{2}$	1	$-\frac{1}{2}$

$\boxed{21} = \frac{3}{2}\boxed{17}$
$\boxed{22} = \boxed{18} - \frac{1}{3}\boxed{21}$
$\boxed{23} = \boxed{19}$
$\boxed{24} = \boxed{20} - 3\boxed{21}$

Optimal solution $x_1 = 0, x_2 = \frac{5}{2}, x_3 = \frac{3}{2}, x_4 = 0, z = \frac{3}{2}$

Fig. 2.6 Phase I–phase II example.

2.6 LINEAR PROGRAMS WITH BOUNDED VARIABLES

In most linear-programming applications, many of the constraints merely specify upper and lower bounds on the decision variables. In a distribution problem, for example, variables x_j representing inventory levels might be constrained by storage capacities u_j and by predetermined safety stock levels ℓ_j so that $\ell_j \leq x_j \leq u_j$. We are led then to consider linear programs with *bounded variables*:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad (i = 1, 2, \ldots, m) \tag{3}$$

$$\ell_j \leq x_j \leq u_j, \qquad (j = 1, 2, \ldots, n). \tag{4}$$

The lower bounds ℓ_j may be $-\infty$ and/or the upper bounds u_j may be $+\infty$, indicating respectively that the decision variable x_j is unbounded from below or from above. Note that when each $\ell_j = 0$ and each $u_j = +\infty$, this problem reduces to the linear-programming form that has been discussed up until now.

The bounded-variable problem can be solved by the simplex method as discussed thus far, by adding slack variables to the upper-bound constraints and surplus variables to the lower-bound constraints, thereby converting them to equalities. This approach handles the bounding constraints explicitly. In contrast, the approach proposed in this section modifies the simplex method to consider the bounded-variable constraints implicitly. In this new approach, pivoting calculations are computed only for the equality constraints (3) rather than for the entire system (3) and (4). In many instances, this reduction in pivoting calculations will provide substantial computational savings. As an extreme illustration, suppose that there is one equality constraint and 1000 nonnegative variables with upper bounds. The simplex method will maintain 1001 constraints in the tableau, whereas the new procedure maintains only the single equality constraint.

We achieve these savings by using a canonical form with one basic variable isolated in each of the equality constraints, as in the usual simplex method. However, basic feasible solutions now are determined by setting nonbasic variables to either their lower or upper bound. This method for defining basic feasible solutions extends our previous practice of setting nonbasic variables to their lower bounds of zero, and permits us to assess optimality and generate improvement procedures much as before.

Suppose, for example, that x_2 and x_4 are nonbasic variables constrained by:

$$\begin{aligned} 4 &\leq x_2 \leq 15, \\ 2 &\leq x_4 \leq 5; \end{aligned}$$

and that

$$\begin{aligned} z &= 4 - \tfrac{1}{4}x_2 + \tfrac{1}{2}x_4, \\ x_2 &= 4, \\ x_4 &= 5, \end{aligned}$$

in the current canonical form. In any feasible solution, $x_2 \geq 4$, so $-\frac{1}{4}x_2 \leq -1$; also, $x_4 \leq 5$, so that $\frac{1}{2}x_4 \leq \frac{1}{2}(5) = 2\frac{1}{2}$. Consequently,

$$z = 4 - \tfrac{1}{4}x_2 + \tfrac{1}{2}x_4 \leq 4 - 1 + 2\tfrac{1}{2} = 5\tfrac{1}{2}$$

for any feasible solution. Since the current solution with $x_2 = 4$ and $x_4 = 5$ attains this upper bound, it must be optimal. In general, the current canonical form represents the optimal solution whenever nonbasic variables at their lower bounds have nonpositive objective coefficients, and nonbasic variables at their upper bound have nonnegative objective coefficients.

> ***Bounded Variable Optimality Condition.*** In a maximization problem in canonical form, if every nonbasic variable at its lower bound has a nonpositive objective coefficient, and every nonbasic variable at its upper bound has a nonnegative objective coefficient, then the basic feasible solution given by that canonical form maximizes the objective function over the feasible region.

Improving a nonoptimal solution becomes slightly more complicated than before. If the objective coefficient $\overline{c}_j$ of nonbasic variable x_j is positive and $x_j = \ell_j$, then we increase x_j; if $\overline{c}_j < 0$ and $x_j = u_j$, we decrease x_j. In either case, the objective value is improving.

When changing the value of a nonbasic variable, we wish to maintain feasibility. As we have seen, for problems with only nonnegative variables, we have to test, via the ratio rule, to see when a basic variable first becomes zero. Here we must consider the following contingencies:

i) the nonbasic variable being changed reaches its upper or lower bound; or
ii) some basic variable reaches either its upper or lower bound.

In the first case, no pivoting is required. The nonbasic variable simply changes from its lower to upper bound, or upper to lower bound, and remains nonbasic. In the second case, pivoting is used to remove the basic variable reaching either its lower or upper bound from the basis.

These ideas can be implemented on a computer in a number of ways. For example, we can keep track of the lower bounds throughout the algorithm; or every lower bound

$$x_j \geq \ell_j$$

can be converted to zero by defining a new variable

$$x''_j = x_j - \ell_j \geq 0,$$

and substituting $x''_j + \ell_j$ for x_j everywhere throughout the model. Also, we can always redefine variables so that every nonbasic variable is at its lower bound. Let x'_j denote the slack variable for the upper-bound constraint $x_j \leq u_j$; that is,

$$x_j + x'_j = u_j.$$

Whenever x_j is nonbasic at its upper bound u_j, the slack variable $x'_j = 0$. Consequently, substituting $u_j - x'_j$ for x_j in the model makes x'_j nonbasic at value zero in place of x_j. If, subsequently in the algorithm, x'_j becomes nonbasic at its upper bound, which is also u_j, we can make the same substitution for x'_j, replacing it with $u_j - x_j$, and x_j will appear nonbasic at value zero. These transformations are usually referred to as the *upper-bounding substitution*.

The computational steps of the upper-bounding algorithm are very simple to describe if both of these transformations are used. Since all nonbasic variables (either x_j or x'_j) are at value zero, we increase a variable for maximization as in the usual simplex method if its objective coefficient is positive. We use the usual ratio rule to determine at what value t_1 for the incoming variable, a basic variable first reaches zero. We also must ensure that variables do not exceed their upper bounds. For example, when increasing nonbasic variable x_s to value t, in a constraint with x_1 basic, such as:

$$x_1 - 2x_s = 3,$$

we require that:

$$x_1 = 3 + 2t \leq u_1 \qquad \left(\text{that is, } t \leq \frac{u_1 - 3}{2}\right).$$

We must perform such a check in every constraint in which the incoming variable has a negative coefficient; thus $x_s \leq t_2$ where:

$$t_2 = \min_i \left\{ \frac{u_k - \overline{b}_i}{-\overline{a}_{is}} \,\middle|\, \overline{a}_{is} < 0 \right\},$$

and u_k is the upper bound for the basic variable x_k in the ith constraint, $\overline{b}_i$ is the current value for this variable, and $\overline{a}_{is}$ are the constraint coefficients for variable x_s. This test might be called the *upper-bounding ratio test*. Note that, in contrast to the usual ratio test, the upper-bounding ratio uses negative coefficients $\overline{a}_{is} < 0$ for the nonbasic variable x_s being increased.

In general, the incoming variable x_s (or x'_s) is set to:

$$x_s = \min\,\{u_s,\ t_1,\ t_2\}.$$

If the minimum is

i) u_s, then the upper bounding substitution is made for x_s (or x'_s);

ii) t_1, then a usual simplex pivot is made to introduce x_s into the basis;

iii) t_2, then the upper bounding substitution is made for the basic variable x_k (or x'_k) reaching its upper bound and x_s is introduced into the basis in place of x'_k (or x_k) by a usual simplex pivot.

The procedure is illustrated in Fig. 2.7. Tableau 1 contains the problem formulation, which is in canonical form with $x_1,\ x_2,\ x_3$, and x_4 as basic variables and x_5 and x_6 as nonbasic variables at value zero. In the first iteration, variable x_5 increases, reaches its upper bound, and the upper bounding substitution $x'_5 = 1 - x_5$ is made. Note that, after making this substitution, the variable x'_5 has coefficients opposite in sign from the coefficients of x_5. Also, in going from Tableau 1 to Tableau 2, we have updated the current value of the basic variables by multiplying the upper bound of x_5, in this case $u_5 = 1$, by the coefficients of x_5 and moving these constants to the righthand side of the equations.

In Tableau 2, variable x_6 increases, basic variable x_2 reaches zero, and a usual simplex pivot is performed. After the pivot, it is attractive to increase x'_5 in Tableau 3. As x'_5 increases basic variable x_4 reaches its upper bound at $x_4 = 5$ and the upper-bounding substitution $x'_4 = 5 - x_4$ is made. Variable x'_4 is isolated as the basic variable in the fourth constraint in Tableau 4 (by multiplying the constraint by -1 after the upper-bounding substitution); variable x'_5 then enters the basis in place of x'_4. Finally, the solution in Tableau 5 is optimal, since the objective coefficients are nonpositive for the nonbasic variables, each at value zero.

Tableau 1

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6	Ratio test
x_1	15	1				4	1	15/4
x_2	8		1			6	2	8/6
x_3	4			1		−7	−2	(15 − 4)/7
x_4	2				1	−1	−1	(5 − 2)/1
$(-z)$	0					2	1	
Upper bounds		15	15	15	5	1	8	
						↑		

$(15 - 4)/7$, $(5 - 2)/1$: Upper bound ratios

$$\begin{cases} t_1 = 8/6 \\ t_2 = 11/7 \\ u_5 = 1 \\ \text{(Upper bound substitution)} \end{cases}$$

Tableau 2

	Basic variables	*Current values*	x_1	x_2	x_3	x_4	x'_5	x_6	
	x_1	11	1				−4	1	11/1
←	x_2	2		1			−6	②	2/2
	x_3	11			1		7	−2	(15 − 11)/2
	x_4	3				1	1	−1	(5 − 3)/1
	$(-z)$	−2					−2	1	
	Upper bounds		15	15	15	5	1	8	
								↑	

$$\begin{cases} t_1 = 1 \\ t_2 = 2 \\ u_6 = 8 \\ \text{(Usual pivot)} \end{cases}$$

Tableau 3

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x'_5	x_6	
x_1	10	1	$-\frac{1}{2}$			−1		(15 − 10)/1
x_6	1		$\frac{1}{2}$			−3	1	(8 − 1)/3
x_3	13		1	1		1		13/1
x_4	4		$\frac{1}{2}$		1	−2		(5 − 4)/2
$(-z)$	−3		$-\frac{1}{2}$			1		
Upper bounds		15	15	15	5	1	8	
						↑		

$$\begin{cases} t_1 = 13 \\ t_2 = \frac{1}{2} \\ u_5 = 1 \\ \text{(Upper bound substitution for } x_4\text{)} \end{cases}$$

Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4'	x_5'	x_6
x_1	10	1	$-\frac{1}{2}$			-1	
x_6	1		$\frac{1}{2}$			-3	1
x_3	13		1	1		1	
← x_4'	1		$-\frac{1}{2}$		1	②	
$(-z)$	-3		$-\frac{1}{2}$			1	
Upper bounds		15	15	15	5	1	8
						↑	

$$\begin{cases} t_1 = \frac{1}{2} \\ t_2 = \frac{8}{3} \\ u_5 = 1 \\ \text{(Usual pivot)} \end{cases}$$

(15 − 10)/1
(8 − 1)/3
13/1
1/2

Tableau 5

Basic variables	Current values	x_1	x_2	x_3	x_4'	x_5'	x_6
x_1	$10\frac{1}{2}$	1	$-\frac{3}{4}$		$\frac{1}{2}$		
x_6	$2\frac{1}{2}$		$-\frac{1}{4}$		$\frac{3}{2}$		1
x_3	$12\frac{1}{2}$		$\frac{5}{4}$	1	$-\frac{1}{2}$		
x_5'	$\frac{1}{2}$		$-\frac{1}{4}$		$\frac{1}{2}$	1	
$(-z)$	$-3\frac{1}{2}$		$-\frac{1}{4}$		$-\frac{1}{2}$		
Upper bounds		15	15	15	5	1	8

Optimal solution: $x_1 = 10\frac{1}{2}, x_2 = 0, x_3 = 12\frac{1}{2}, x_4 = 5 - x_4' = 5, x_5 = 1 - x_5' = \frac{1}{2}, x_6 = 2\frac{1}{2}$.

Fig. 2.7 Simplex method with bounded variables.

EXERCISES

1. Given:

$$\begin{aligned} x_1 \quad\quad\quad\quad &+ 2x_4 = 8, \\ x_2 \quad\quad &+ 3x_4 = 6, \\ x_3 &+ 8x_4 = 24, \\ -z \quad\quad\quad\quad\quad &+ 10x_4 = -32, \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

a) What is the optimal solution of this problem?
b) Change the coefficient of x_4 in the z-equation to -3. What is the optimal solution now?
c) Change the signs on all x_4 coefficients to be negative. What is the optimal solution now?

2. Consider the linear program:

$$\text{Maximize } z = 9x_2 + x_3 - 2x_5 - x_6,$$

subject to:

$$\begin{aligned} 5x_2 + 50x_3 + x_4 + x_5 \quad\quad &= 10, \\ x_1 - 15x_2 + 2x_3 \quad\quad\quad\quad\quad\quad &= 2, \\ x_2 + x_3 \quad\quad + x_5 + x_6 &= 6, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, 6).$$

a) Find an initial basic feasible solution, specify values of the decision variables, and tell which are basic.
b) Transform the system of equations to the canonical form for carrying out the simplex routine.
c) Is your initial basic feasible solution optimal? Why?
d) How would you select a column in which to pivot in carrying out the simplex algorithm?
e) Having chosen a pivot column, now select a row in which to pivot and describe the selection rule. How does this rule guarantee that the new basic solution is feasible? Is it possible that no row meets the criterion of your rule? If this happens, what does this indicate about the original problem?
f) Without carrying out the pivot operation, compute the new basic feasible solution.
g) Perform the pivot operation indicated by (d) and (e) and check your answer to (f). Substitute your basic feasible solution in the original equations as an additional check.
h) Is your solution optimal now? Why?

3. a) Reduce the following system to canonical form. Identify slack, surplus, and artificial variables.

$$\begin{aligned} -2x_1 + x_2 &\leq 4 && (1) \\ 3x_1 + 4x_2 &\geq 2 && (2) \\ 5x_1 + 9x_2 &= 8 && (3) \\ x_1 + x_2 &\geq 0 && (4) \\ 2x_1 + x_2 &\geq -3 && (5) \\ -3x_1 - x_2 &\leq -2 && (6) \\ 3x_1 + 2x_2 &\leq 10 && (7) \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

b) Formulate phase I objective functions for the following systems with $x_1 \geq 0$ and $x_2 \geq 0$:
   i) expressions 2 and 3 above.

ii) expressions 1 and 7 above.
iii) expressions 4 and 5 above.

4. Consider the linear program

$$\text{Maximize } z = x_1,$$

subject to:

$$\begin{aligned} -x_1 + x_2 &\leq 2, \\ x_1 + x_2 &\leq 8, \\ -x_1 + x_2 &\geq -4, \\[1ex] x_1 \geq 0, \qquad x_2 &\geq 0. \end{aligned}$$

a) State the above in canonical form.
b) Solve by the simplex method.
c) Solve geometrically and also trace the simplex procedure steps graphically.
d) Suppose that the objective function is changed to $z = x_1 + cx_2$. Graphically determine the values of c for which the solution found in parts (b) and (c) remains optimal.
e) Graphically determine the shadow price corresponding to the third constraint.

5. The bartender of your local pub has asked you to assist him in finding the combination of mixed drinks that will maximize his revenue. He has the following bottles available:

1 quart (32 oz.) Old Cambridge (a fine whiskey—cost \$8/quart)
1 quart Joy Juice (another fine whiskey—cost \$10/quart)
1 quart Ma's Wicked Vermouth (\$10/quart)
2 quarts Gil-boy's Gin (\$6/quart)

Since he is new to the business, his knowledge is limited to the following drinks:

Whiskey Sour	2 oz. whiskey	Price \$1
Manhattan	2 oz. whiskey 1 oz. vermouth	\$2
Martini	2 oz. gin 1 oz. vermouth	\$2
Pub Special	2 oz. gin 2 oz. whiskey	\$3

Use the simplex method to maximize the bar's profit. (Is the cost of the liquor relevant in this formulation?)

6. A company makes three lines of tires. Its four-ply biased tires produce \$6 in profit per tire, its fiberglass belted line \$4 a tire, and its radials \$8 a tire. Each type of tire passes through three manufacturing stages as a part of the entire production process.

Each of the three process centers has the following hours of available production time per day:

	Process	*Hours*
1	Molding	12
2	Curing	9
3	Assembly	16

The time required in each process to produce one hundred tires of each line is as follows:

	Hours per 100 units		
Tire	*Molding*	*Curing*	*Assembly*
Four-ply	2	3	2
Fiberglass	2	2	1
Radial	2	1	3

Determine the optimum product mix for each day's production, assuming all tires are sold.

7. An electronics firm manufactures printed circuit boards and specialized electronics devices. Final assembly operations are completed by a small group of trained workers who labor simultaneously on the products. Because of limited space available in the plant, no more then ten assemblers can be employed. The standard operating budget in this functional department allows a maximum of $9000 per month as salaries for the workers.

   The existing wage structure in the community requires that workers with two or more years of experience receive $1000 per month, while recent trade-school graduates will work for only $800. Previous studies have shown that experienced assemblers produce $2000 in ‘‘value added" per month while new-hires add only $1800. In order to maximize the value added by the group, how many persons from each group should be employed? Solve graphically and by the simplex method.

8. The processing division of the Sunrise Breakfast Company must produce one ton (2000 pounds) of breakfast flakes per day to meet the demand for its Sugar Sweets cereal. Cost per pound of the three ingredients is:

Ingredient A	$4 per pound
Ingredient B	$3 per pound
Ingredient C	$2 per pound

   Government regulations require that the mix contain at least 10% ingredient A and 20% ingredient B. Use of more than 800 pounds per ton of ingredient C produces an unacceptable taste.

   Determine the minimum-cost mixture that satisfies the daily demand for Sugar Sweets. Can the bounded-variable simplex method be used to solve this problem?

9. Solve the following problem using the two phases of the simplex method:

$$\text{Maximize } z = 2x_1 + x_2 + x_3,$$

subject to:

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &\leq 9, \\ 2x_2 + x_3 &\geq 4, \\ x_1 \qquad + x_3 &= 6, \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Is the optimal solution unique?

10. Consider the linear program:

$$\text{Maximize } z = -3x_1 + 6x_2,$$

subject to:

$$\begin{aligned} 5x_1 + 7x_2 &\leq 35, \\ -x_1 + 2x_2 &\leq \ \ 2, \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

a) Solve this problem by the simplex method. Are there alternative optimal solutions? How can this be determined at the final simplex iteration?
b) Solve the problem graphically to verify your answer to part (a).

11. Solve the following problem using the simplex method:

$$\text{Minimize } z = x_1 - 2x_2 - 4x_3 + 2x_4,$$

subject to:

$$\begin{aligned} x_1 \quad\; - 2x_3 \qquad\quad &\leq 4, \\ x_2 \qquad\; - x_4 &\leq 8, \\ -2x_1 + x_2 + 8x_3 + x_4 &\leq 12, \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

12. a) Set up a linear program that will determine a feasible solution to the following system of equations and inequalities if one exists. *Do not solve* the linear program.

$$\begin{aligned} x_1 - 6x_2 + x_3 - x_4 &= 5, \\ -2x_2 + 2x_3 - 3x_4 &\geq 3, \\ 3x_1 \qquad - 2x_3 + 4x_4 &= -1, \end{aligned}$$

$$x_1 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0.$$

b) Formulate a phase I linear program to find a feasible solution to the system:

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &\leq -3, \\ -x_1 - x_2 + 2x_3 &\leq -1, \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

Show, from the phase I objective function, that the system contains no feasible solution (no pivoting calculations are required).

13. The tableau given below corresponds to a maximization problem in decision variables $x_j \geq 0$ ($j = 1, 2, \ldots, 5$):

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_3	4	-1	a_1	1		
x_4	1	a_2	-4		1	
x_5	b	a_3	3			1
$(-z)$	-10	c	-2			

State conditions on all five unknowns a_1, a_2, a_3, b, and c, such that the following statements are true.

a) The current solution is optimal. There are multiple optimal solutions.
b) The problem is unbounded.
c) The problem is infeasible.
d) The current solution is not optimal (assume that $b \geq 0$). Indicate the variable that enters the basis, the variable that leaves the basis, and what the total change in profit would be for one iteration of the simplex method for all values of the unknowns that are not optimal.

14. Consider the linear program:

$$\text{Maximize } z = \alpha x_1 + 2x_2 + x_3 - 4x_4,$$

subject to:

$$\begin{aligned} x_1 + x_2 \phantom{{}+3x_3} - x_4 &= 4 + 2\Delta && (1)\\ 2x_1 - x_2 + 3x_3 - 2x_4 &= 5 + 7\Delta && (2)\end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0, \qquad x_4 \geq 0,$$

where α and Δ are viewed as parameters.

a) Form two new constraints as $(1') = (1) + (2)$ and $(2') = -2(1) + (2)$. Solve for x_1 and x_2 from $(1')$ and $(2')$, and substitute their values in the objective function. Use these transformations to express the problem in canonical form with x_1 and x_2 as basic variables.
b) Assume $\Delta = 0$ (constant). For what values of α are x_1 and x_2 *optimal basic variables* in the problem?
c) Assume $\alpha = 3$. For what values of Δ do x_1 and x_2 form an optimal basic feasible solution?

15. Let

$$(-w) + d_1x_1 + d_2x_2 + \cdots + d_mx_m = 0 \qquad (*)$$

be the phase I objective function when phase I terminates for maximizing w. Discuss the following two procedures for making the phase I to II transition when an artificial variable remains in the basis at value zero. Show, using either procedure, that every basic solution determined during phase II will be feasible for the *original* problem formulation.

a) Multiply each coefficient in $(*)$ by -1. Initiate phase II with the original objective function, but maintain $(*)$ in the tableau as a new constraint with (w) as the basic variable.
b) Eliminate $(*)$ from the tableau and at the same time eliminate from the problem any variable x_j with $d_j < 0$. Any artificial variable in the optimal phase I basis is now treated as though it were a variable from the original problem.

16. In our discussion of reduction to canonical form, we have replaced variables unconstrained in sign by the difference between two nonnegative variables. This exercise considers an alternative transformation that does not introduce as many new variables, and also a simplex-like procedure for treating free variables directly without any substitutions. For concreteness, suppose that $y_1,\ y_2$, and y_3 are the only unconstrained variables in a linear program.

a) Substitute for y_1, y_2, and y_3 in the model by:

$$\begin{aligned} y_1 &= x_1 - x_0,\\ y_2 &= x_2 - x_0,\\ y_3 &= x_3 - x_0,\end{aligned}$$

with $x_0 \geq 0,\ x_1 \geq 0,\ x_2 \geq 0$, and $x_3 \geq 0$. Show that the models are equivalent before and after these substitutions.
b) Apply the simplex method directly with $y_1,\ y_2$, and y_3. When are these variables introduced into the basis at positive levels? At negative levels? If y_1 is basic, will it ever be removed from the basis? Is the equation containing y_1 as a basic variable used in the ratio test? Would the simplex method be altered in any other way?

17. Apply the phase I simplex method to find a feasible solution to the problem:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2,\\ -x_1 - 3x_2 + x_3 &= 1,\\ 2x_1 - 3x_2 + 4x_3 &= 7,\end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0.$$

Does the termination of phase I show that the system contains a redundant equation? How?

18. Frequently, linear programs are formulated with *interval constraints* of the form:

$$5 \le 6x_1 - x_2 + 3x_3 \le 8.$$

   a) Show that this constraint is equivalent to the constraints

$$\begin{aligned} 6x_1 - x_2 + 3x_3 + x_4 &= 8, \\ 0 \le x_4 &\le 3. \end{aligned}$$

   b) Indicate how the general *interval linear program*

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

   subject to:

$$\begin{aligned} b_i' \le \sum_{j=1}^{n} a_{ij} x_j &\le b_i \quad (i = 1, 2, \dots, m), \\ x_j &\ge 0 \quad (j = 1, 2, \dots, n), \end{aligned}$$

   can be formulated as a bounded-variable linear program with m equality constraints.

19. a) What is the solution to the linear-programming problem:

$$\text{Maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n,$$

   subject to:

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &\le b, \\ 0 \le x_j \le u_j \quad &(j = 1, 2, \dots, n), \end{aligned}$$

   with bounded variables and one additional constraint? Assume that the constants c_j, a_j, and u_j for $j = 1, 2, \dots, n$, and b are all positive and that the problem has been formulated so that:

$$\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \ge \cdots \ge \frac{c_n}{a_n}.$$

   b) What will be the steps of the simplex method for bounded variables when applied to this problem (in what order do the variables enter and leave the basis)?

20. a) Graphically determine the steps of the simplex method for the problem:

$$\text{Maximize } 8x_1 + 6x_2,$$

   subject to:

$$\begin{aligned} 3x_1 + 2x_2 &\le 28, \\ 5x_1 + 2x_2 &\le 42, \\ x_1 \qquad &\le 8, \\ x_2 &\le 8, \\ x_1 \ge 0, \qquad x_2 &\ge 0. \end{aligned}$$

   Indicate on the sketch the basic variables at each iteration of the simplex algorithm in terms of the given variables and the slack variables for the four less-than-or-equal-to constraints.

   b) Suppose that the bounded-variable simplex method is applied to this problem. Specify how the iterations in the solution to part (a) correspond to the bounded-variable simplex method. Which variables from x_1, x_2, and the slack variable for the first two constraints, are basic at each iteration? Which nonbasic variables are at their upper bounds?

   c) Solve the problem algebraically, using the simplex algorithm for bounded variables.