Chapter 36

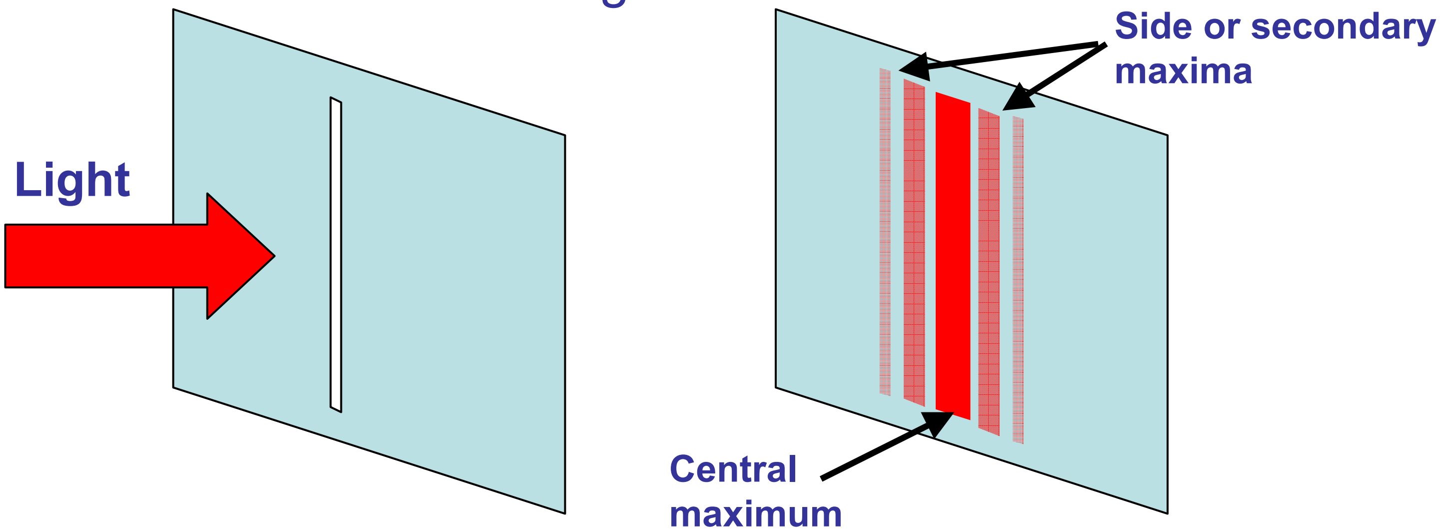
Diffraction

In Chapter 35, we saw how light beams passing through different slits can interfere with each other and how a beam after passing through a single slit flares-diffracts- in Young's experiment. Diffraction through a single slit or past either a narrow obstacle or an edge produces rich interference patterns. The physics of diffraction plays an important role in many scientific and engineering fields.

In this chapter we explain diffraction using the wave nature of light and discuss several applications of diffraction in science and technology.

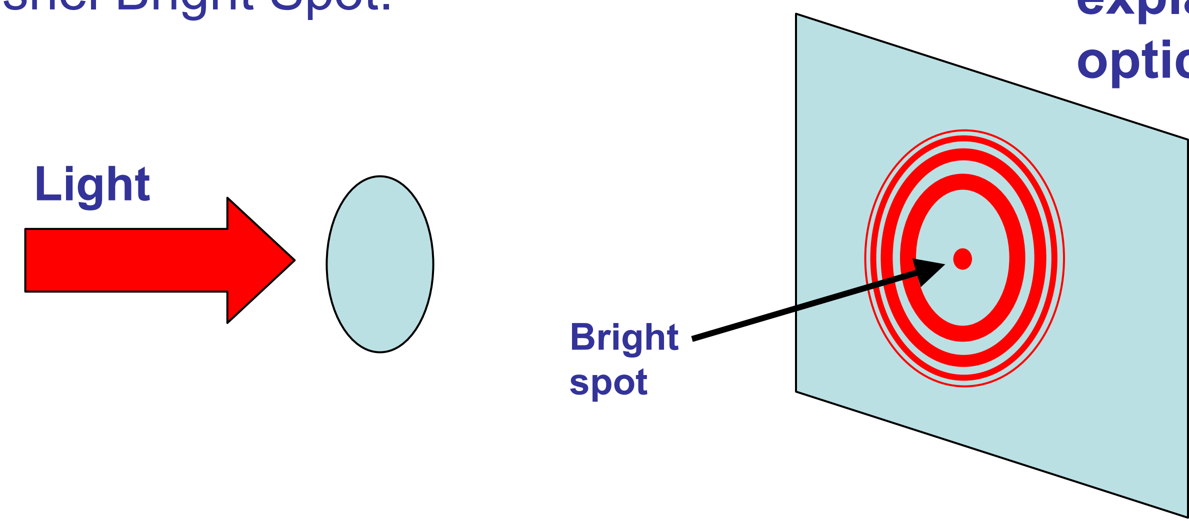
Diffraction and the Wave Theory of Light

Diffraction Pattern from a single narrow slit.

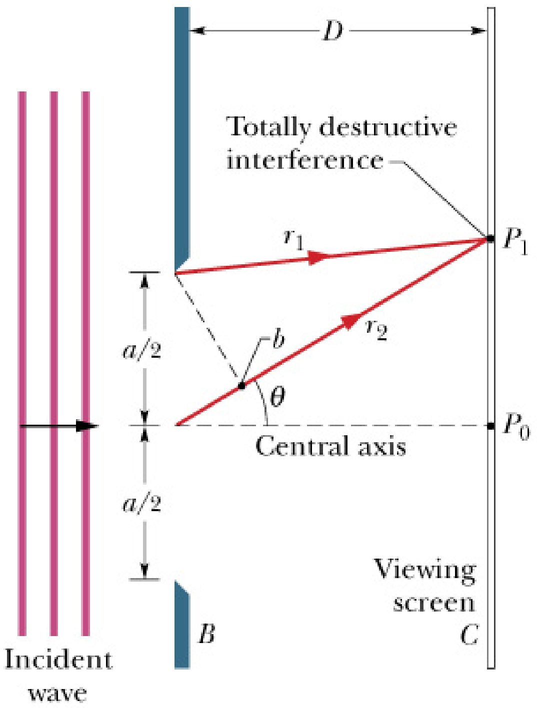


These patterns cannot be explained using geometrical optics (Ch. 34)!

Fresnel Bright Spot.



Diffraction by a Single Slit: Locating the Minima



(a)

When the path length difference between rays r_1 and r_2 is $\lambda/2$, the two rays will be out of phase when they reach P_1 on the screen, resulting in destructive interference at P_1. The path length difference is the distance from the starting point of r_2 at the center of the slit to point b.

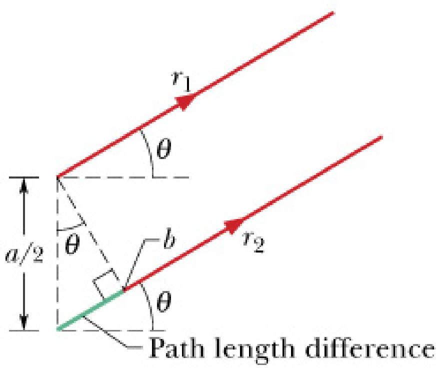
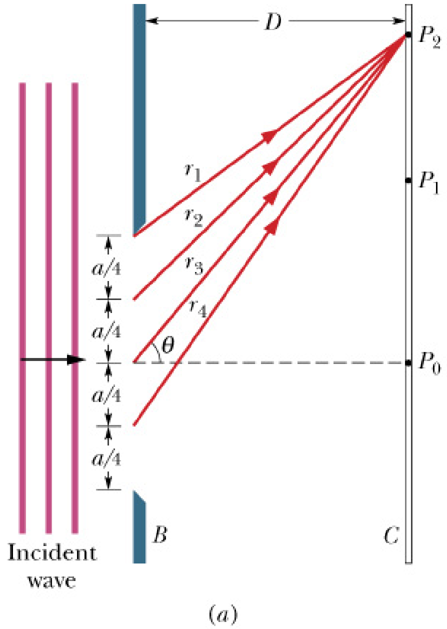


Fig. 36-4 (b)

For $D >> a$, the path length difference between rays r_1 and r_2 is $(a/2) \sin \theta$.

Diffraction by a Single Slit: Locating the Minima, Cont'd



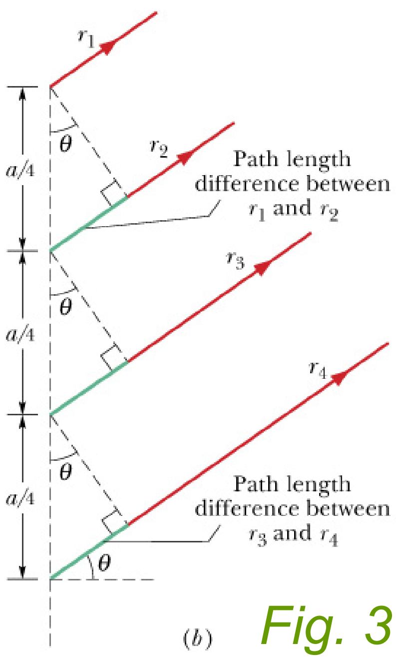


Fig. 36-5

Repeat previous analysis for pairs of rays, each separated by a vertical distance of *a*/2 at the slit.

Setting path length difference to $\lambda/2$ for each pair of rays, we obtain the first dark fringes at:

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2} \rightarrow a\sin\theta = \lambda \quad \text{(first minimum)}$$

For second minimum, divide slit into 4 zones of equal widths *a*/4 (separation between pairs of rays). Destructive interference occurs when the path length difference for each pair is $\lambda/2$.

$$\frac{a}{4}\sin\theta = \frac{\lambda}{2} \rightarrow a\sin\theta = 2\lambda \quad \text{(second minimum)}$$

Dividing the slit into increasingly larger even numbers of zones, we can find higher order minima:

$$a\sin\theta = m\lambda, \quad \text{for } m = 1, 2, 3\ldots \quad \text{(minima-dark fringes)}$$

Intensity in Single-Slit Diffraction, Qualitatively

To obtain the locations of the minima, the slit was equally divided into *N* zones, each with width Δx. Each zone acts as a source of Huygens wavelets. Now these zones can be superimposed at the screen to obtain the intensity as function of θ, the angle to the central axis.

To find the net electric field E_θ (intensity α E_θ^2) at point P on the screen, we need the phase relationships among the wavelets arriving from different zones:

$$\begin{pmatrix}\text{phase}\\ \text{difference}\end{pmatrix} = \left(\frac{2\pi}{\lambda}\right)\begin{pmatrix}\text{path length}\\ \text{difference}\end{pmatrix} \qquad \Delta\phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x \sin\theta)$$

N=18

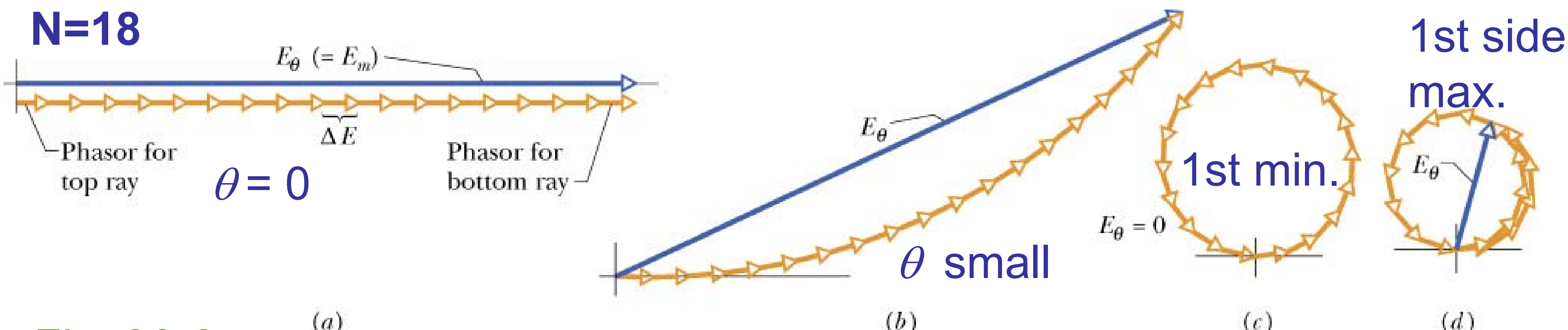


Fig. 36-6

Intensity in Single-Slit Diffraction, Quantitatively

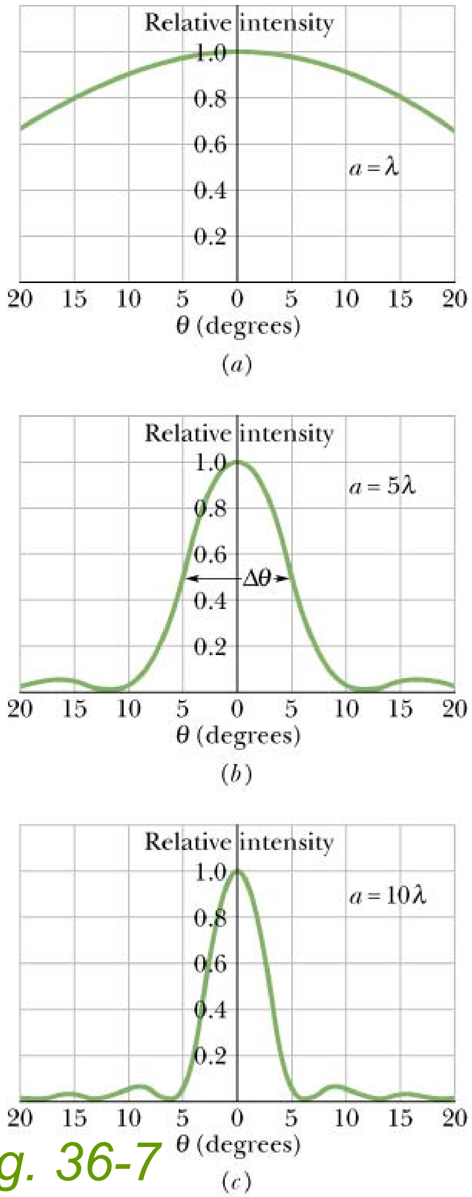


Fig. 36-7

Here we will show that the intensity at the screen due to a single slit is:

$$I(\theta) = I_m \left(\frac{\sin\alpha}{\alpha} \right)^2 \qquad (36\text{-}5)$$

$$\text{where} \quad \alpha = \frac{1}{2}\phi = \frac{\pi a}{\lambda}\sin\theta \qquad (36\text{-}6)$$

In Eq. 36-5, minima occur when:

$$\alpha = m\pi, \qquad \text{for } m = 1, 2, 3\ldots$$

If we put this into Eq. 36-6 we find:

$$m\pi = \frac{\pi a}{\lambda}\sin\theta, \qquad \text{for } m = 1, 2, 3\ldots$$

$$\text{or} \qquad a\sin\theta = m\lambda, \qquad \text{for } m = 1, 2, 3\ldots$$

(minima-dark fringes)

Proof of Eqs. 36-5 and 36-6

If we divide slit into infinitesimally wide zones Δx, the arc of the phasors approaches the arc of a circle. The length of the arc is E_m. ϕ is the difference in phase between the infinitesimal vectors at the left and right ends of the arc. ϕ is also the angle between the 2 radii marked *R*.

The dash line bisecting f forms two triangles, where: $\sin \tfrac{1}{2}\phi = \dfrac{E_\theta}{2R}$

In radian measure: $\phi = \dfrac{E_m}{R}$

Solving the previous 2 equations for E_θ one obtains: $E_\theta = \dfrac{E_m}{\frac{1}{2}\phi}\sin\tfrac{1}{2}\phi$

The intensity at the screen is therefore:

$$\frac{I(\theta)}{I_m} = \frac{E_\theta^2}{E_m^2} \rightarrow I(\theta) = I_m\left(\frac{\sin\alpha}{\alpha}\right)^2$$

ϕ is related to the path length difference across the entire slit:

$$\phi = \left(\frac{2\pi}{\lambda}\right)(a\sin\theta)$$

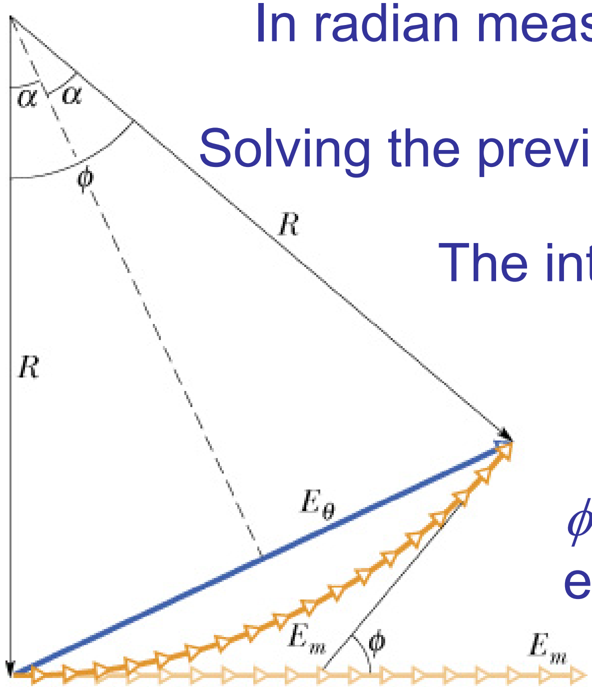


Fig. 36-8

Diffraction by a Circular Aperture

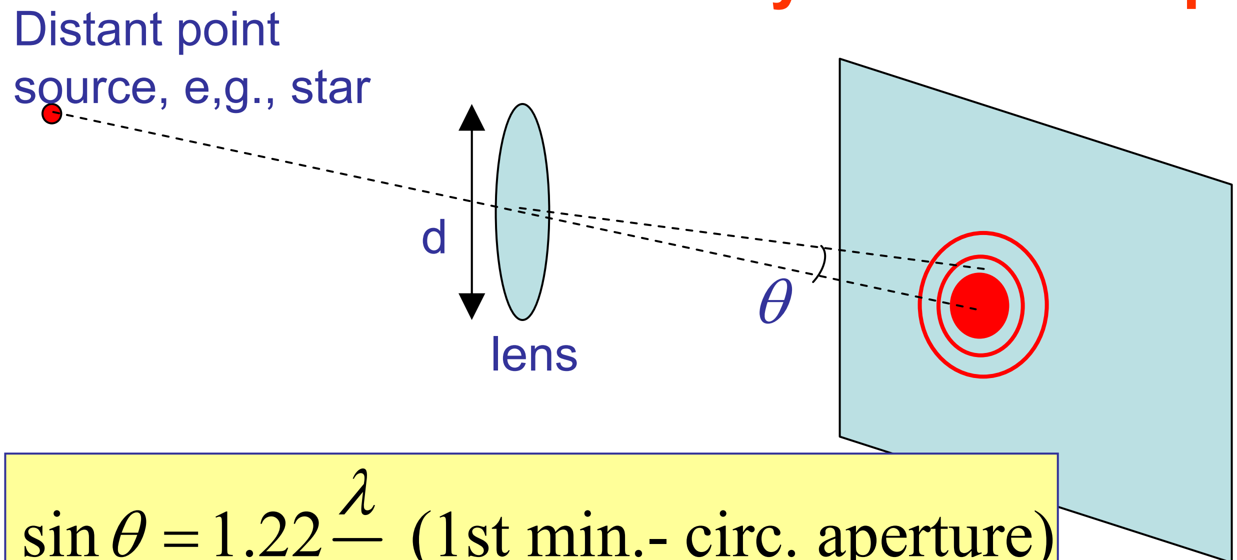
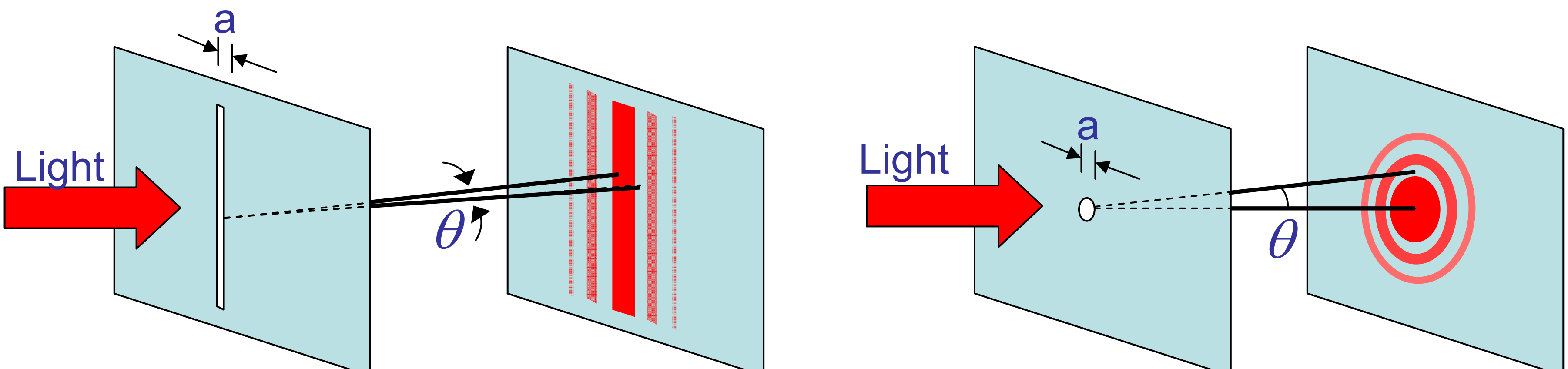


Image is not a point, as expected from geometrical optics! Diffraction is responsible for this image pattern

$$\sin\theta = 1.22\frac{\lambda}{d} \quad \text{(1st min.- circ. aperture)}$$



$$\sin\theta = 1.22\frac{\lambda}{a} \quad \text{(1st min.- single slit)}$$

Resolvability

Rayleigh's Criterion: two point sources are barely resolvable if their angular separation θ_R results in the central maximum of the diffraction pattern of one source's image is centered on the first minimum of the diffraction pattern of the other source's image.

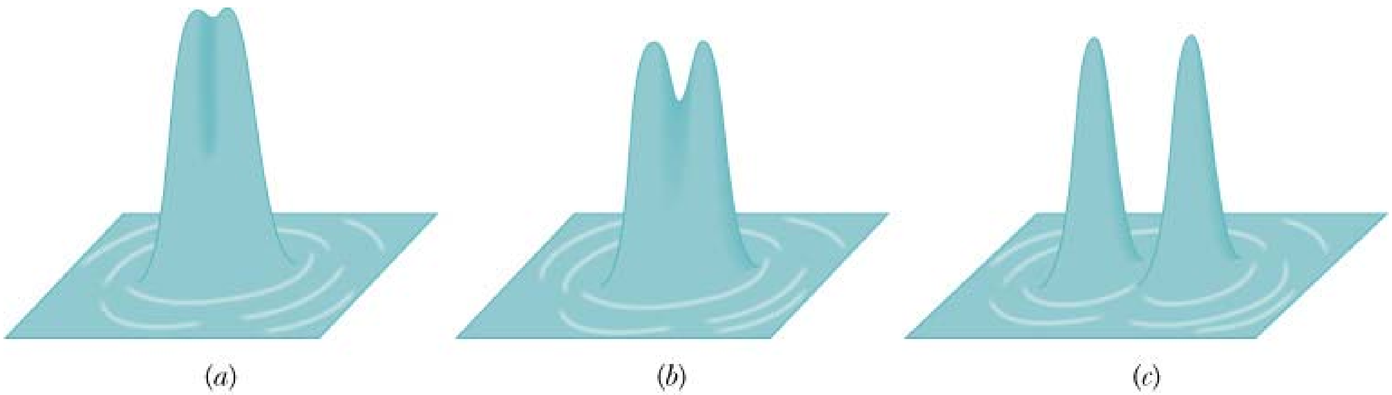


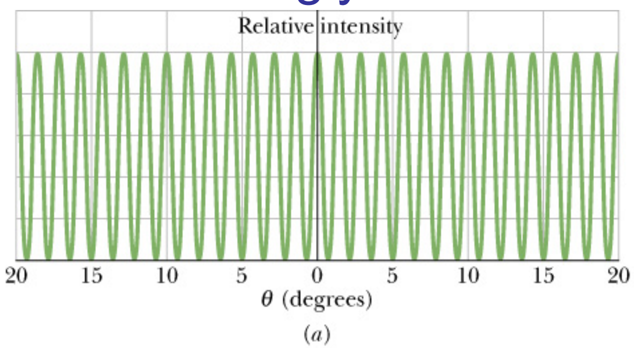
Fig. 36-10

$$\theta_R = \sin^{-1}\left(1.22\frac{\lambda}{d}\right) \overset{\theta_R \text{ small}}{\approx} 1.22\frac{\lambda}{d} \quad \text{(Rayleigh's criterion)}$$

Diffraction by a Double Slit

Double slit experiment described in Ch. 35 where assumed that the slit width $a \ll \lambda$. What if this is not the case?

Two vanishingly narrow slits $a \ll \lambda$



Single slit $a \sim \lambda$

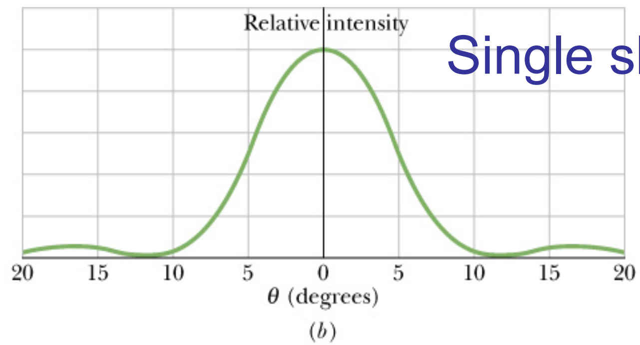
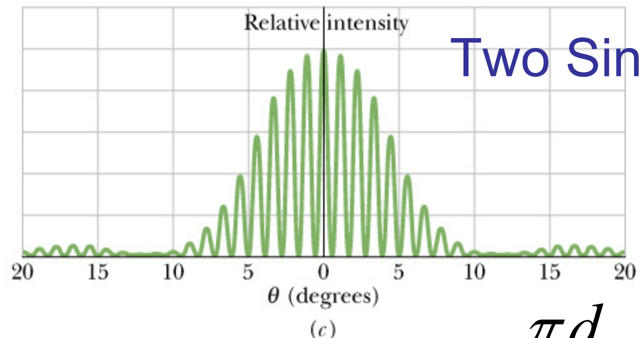


Fig. 36-14

Two Single slits $a \sim \lambda$



$$I(\theta) = I_m \left(\cos^2 \beta\right)\left(\frac{\sin\alpha}{\alpha}\right)^2 \quad \text{(double slit)}$$

$$\beta = \frac{\pi d}{\lambda}\sin\theta$$

$$\alpha = \frac{\pi a}{\lambda}\sin\theta$$

Diffraction Gratings

Device with *N* slits (rulings) can be used to manipulate light, such as separate different wavelengths of light that are contained in a single beam. How does a diffraction grating affect monochromatic light?

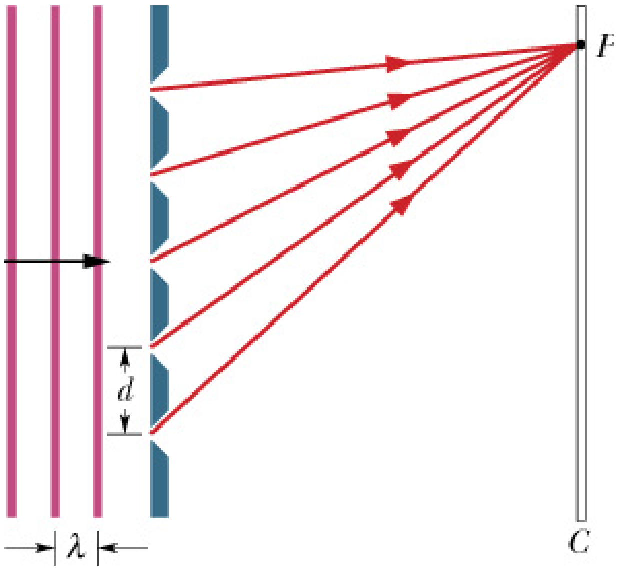


Fig. 36-17

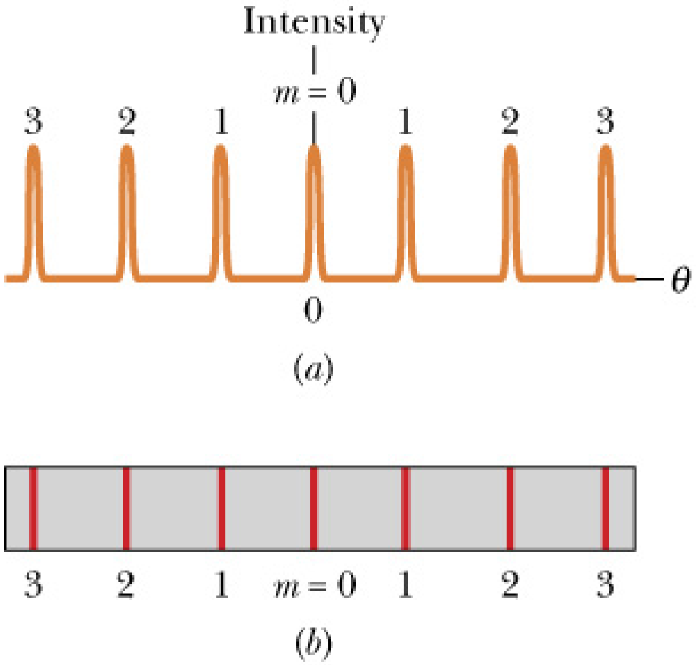


Fig. 36-18

$$d \sin\theta = m\lambda \quad \text{for } m = 0, 1, 2\ldots \quad \text{(maxima-lines)}$$

Width of Lines

The ability of the diffraction grating to resolve (separate) different wavlength depends on the width of the lines (maxima)

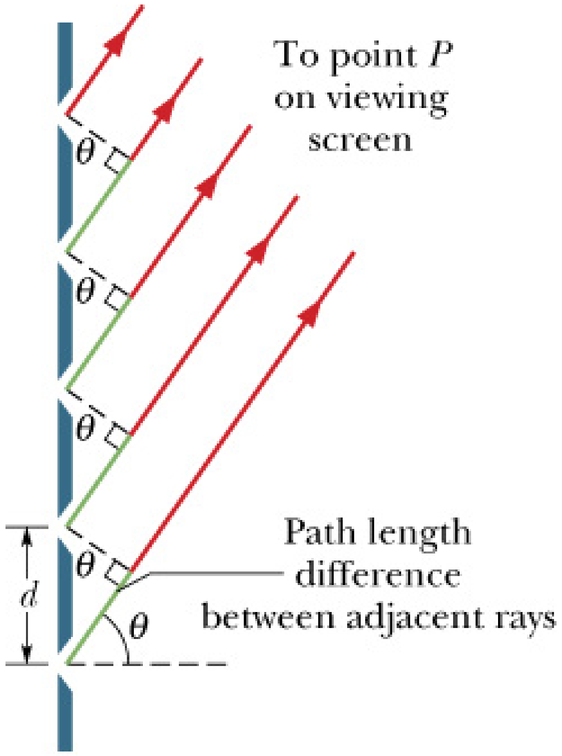


Fig. 36-19

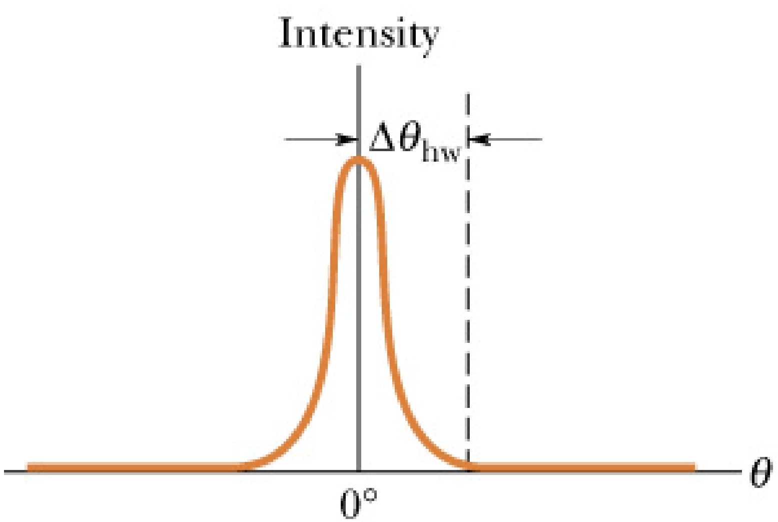


Fig. 36-20

Width of Lines, cont'd

In this course, a sound wave is roughly defined as any longitudinal wave (particles moving along the direction of wave propagation).

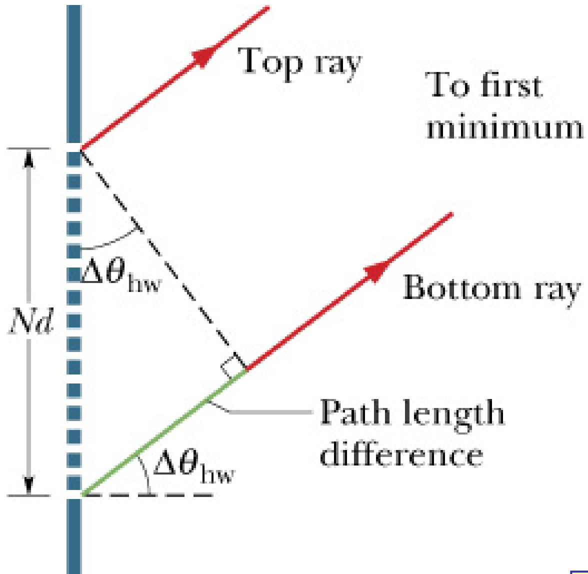


Fig. 36-21

$$Nd\sin\Delta\theta_{\text{hw}} = \lambda \quad , \qquad \sin\Delta\theta_{hw} \approx \Delta\theta_{hw}$$

$$\Delta\theta_{\text{hw}} = \frac{\lambda}{Nd} \quad \text{(half width of central line)}$$

$$\Delta\theta_{\text{hw}} = \frac{\lambda}{Nd\cos\theta} \quad \text{(half width of line at } \theta\text{)}$$

Grating Spectroscope

Separates different wavelengths (colors) of light into distinct diffraction lines

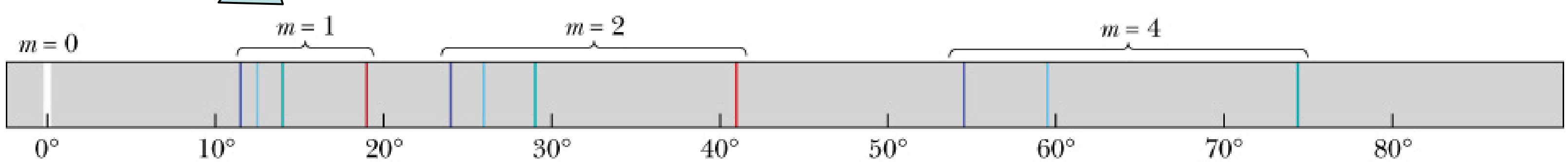


Fig. 36-23

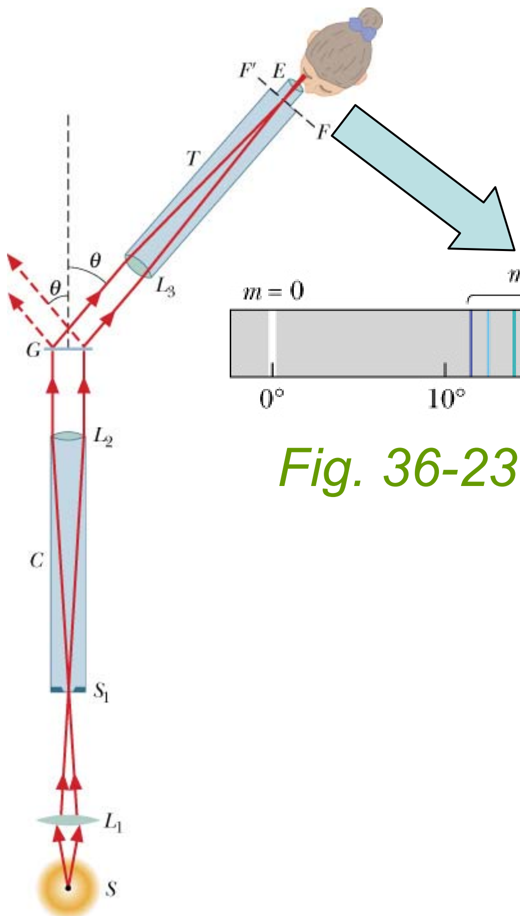


Fig. 36-22

Optically Variable Graphics

Gratings embedded in device send out hundreds or even thousands of diffraction orders to produce virtual images that vary with viewing angle. Complicated to design and extremely difficult to counterfeit, so makes an excellent security graphic.

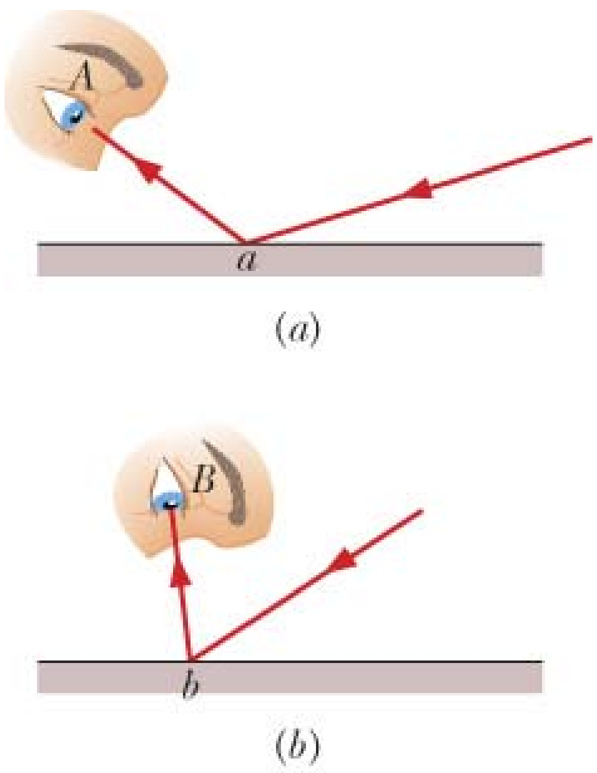


Fig. 36-25

Gratings: Dispersion and Resolving Power

Dispersion: the angular spreading of different wavelengths by a grating

$$D = \frac{\Delta\theta}{\Delta\lambda} \quad \text{(dispersion defined)}$$

$$D = \frac{m}{d\cos\theta} \quad \text{(dispersion of a grating)} \quad \text{(36-30)}$$

Resolving Power

$$R = \frac{\lambda_{\text{avg}}}{\Delta\lambda} \quad \text{(resolving power defined)}$$

$$R = Nm \quad \text{(resolving power of a grating)} \quad \text{(36-32)}$$

Proof of Eq. 36-30

Angular position of maxima

$$d \sin \theta = m\lambda$$

Differential of first equation (what change in angle does a change in wavelength produce?)

$$d\left(\cos \theta\right) d\theta = m d\lambda$$

For small angles

$$d\theta \rightarrow \Delta\theta \quad \text{and} \quad d\lambda \rightarrow \Delta\lambda$$

$$d\left(\cos \theta\right) \Delta\theta = m\Delta\lambda$$

$$\frac{\Delta\theta}{\Delta\lambda} = \frac{m}{d\left(\cos \theta\right)}$$

Proof of Eq. 36-32

Rayleigh's criterion for half-width to resolve two lines

$$\Delta\theta_{\text{hw}} = \frac{\lambda}{Nd\cos\theta}$$

Substituting for $\Delta\theta$ in calculation on previous slide

$$\Delta\theta_{\text{hw}} \rightarrow \Delta\theta$$

$$\rightarrow \frac{\lambda}{N} = m\Delta\lambda$$

$$R = \frac{\lambda}{\Delta\lambda} = Nm$$

Dispersion and Resolving Power Compared

In this course, a sound wave is roughly defined as any longitudinal wave (particles moving along the direction of wave propagation).

Table 36-1

Grating	*N*	*d* (nm)	θ	D (°/μm)	*R*
A	10 000	2540	13.4°	23.2	10 000
B	20 000	2540	13.4°	23.2	20 000
C	10 000	1360	25.5°	46.3	10 000

Data are for λ = 589 nm and *m* = 1

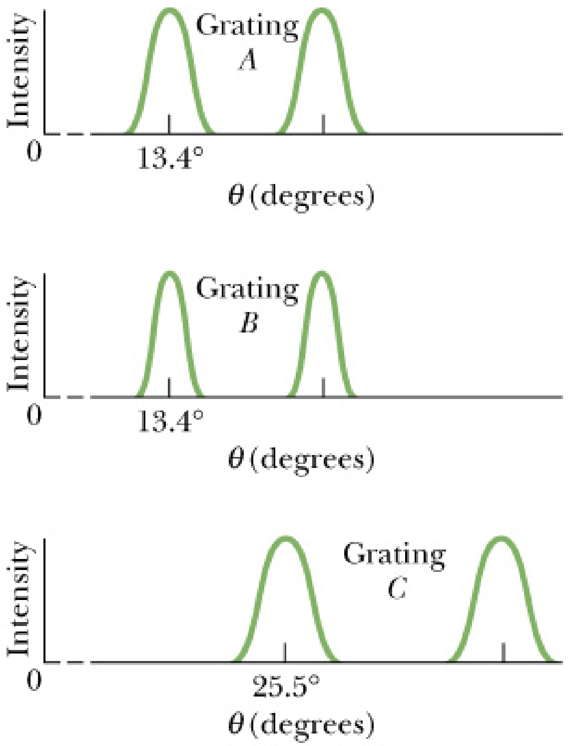


Fig. 36-26

X-Ray Diffraction

X-rays are electromagnetic radiation with wavelength ~1 Å = 10^{-10} m (visible light ~$5.5\text{x}10^{-7}$ m)

X-ray generation

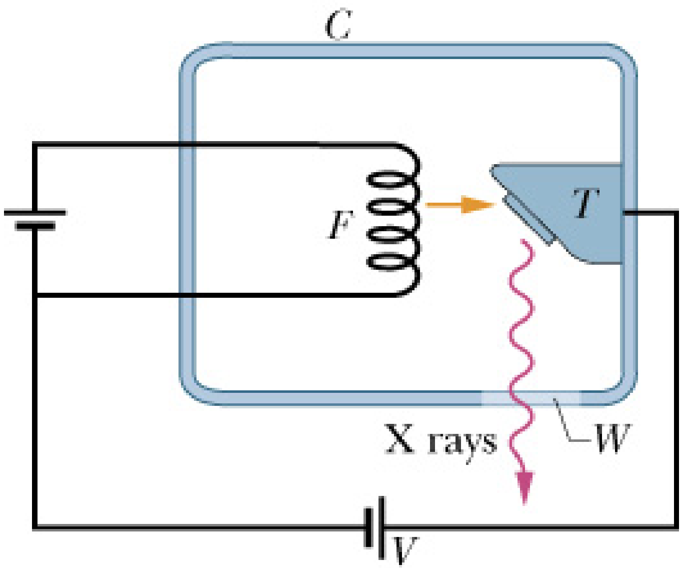
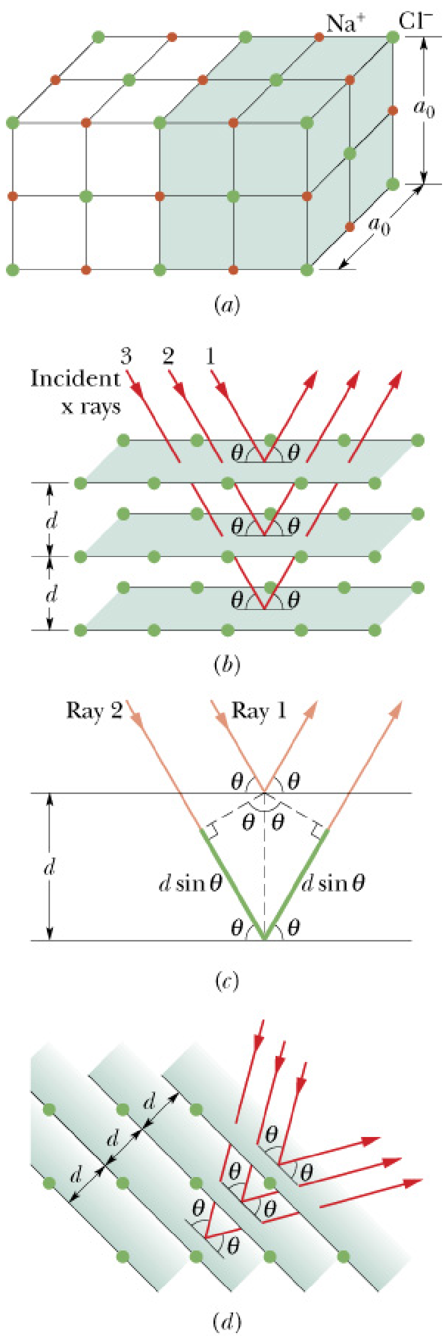


Fig. 36-27

X-ray wavelengths to short to be resolved by a standard optical grating

$$\theta = \sin^{-1}\frac{m\lambda}{d} = \sin^{-1}\frac{(1)(0.1\text{ nm})}{3000\text{ nm}} = 0.0019°$$

X-Ray Diffraction, cont'd



Diffraction of x-rays by crystal: spacing *d* of adjacent crystal planes on the order of 0.1 nm

→ three-dimensional diffraction grating with diffraction maxima along angles where reflections from different planes interfere constructively

$$2d\sin\theta = m\lambda \quad \text{for } m = 0, 1, 2\ldots \quad \text{(Bragg's law)}$$

Fig. 36-28

X-Ray Diffraction, cont'd

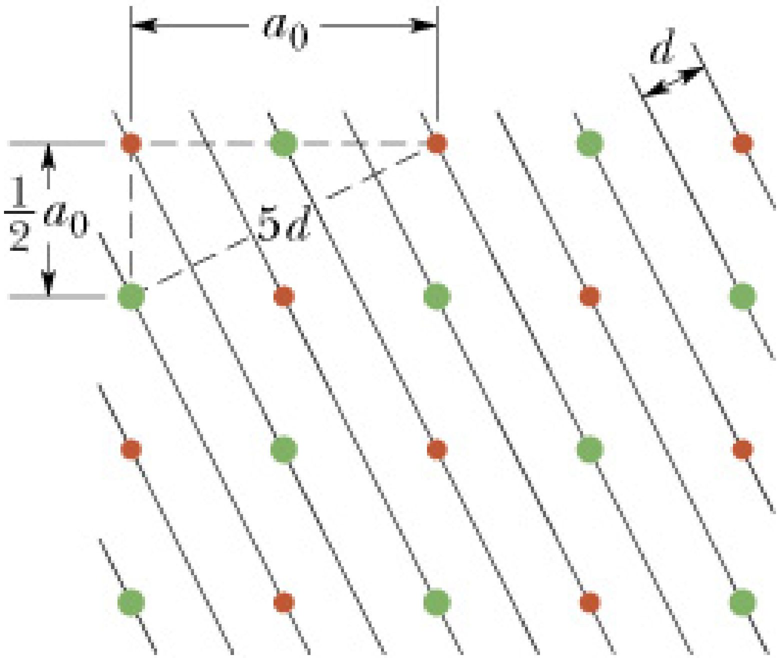


Fig. 36-29

interplanar spacing *d* is relatedto the unit cell dimensaion a_0

$$5d = \sqrt{\tfrac{5}{4}a_0^2} \quad \text{or} \quad d = \frac{a_0}{20} = 0.2236a_0$$

Not only can crystals be used to separate different x-ray wavelengths, but x-rays in turn can be used to study crystals, for example determine the type crystal ordering and a_0