THE EULER CHARACTERISTIC OF A GRAPH

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ABSTRACT. Euler's theorem on the Euler characteristic of planar graphs is a fundamental result, and is usually proved using induction (for example, see Theorem 53.3 in our textbook). Here I present a totally different proof, discovered jointly by Stephanie Mathew (and undergraduate at the time) and Red Burton (a computer program designed by Siemion Fajtlowicz to make conjectures in graph theory and combinatorics). I learned of this proof thanks to the remarkable website of David Eppstein, who has compiled twenty proofs of Euler's formula (as well as many other lovely expository articles).

1. EULERIAN PATHS

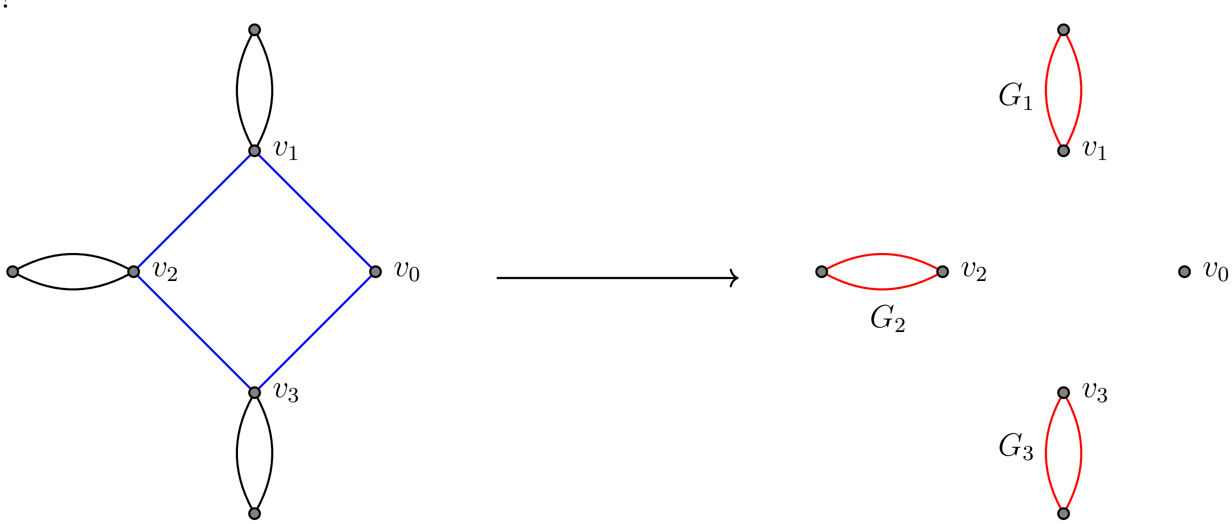
The Königsberg Bridge problem inspired Euler to study what are now called *Eulerian paths* on a graph: a path that traverses each edge precisely once. As we saw in class, an Eulerian path has a rather special feature: all of the intermediate vertices on an Eulerian path (i.e. any vertex that isn't the first or the last one) have even degree. Note that if our Eulerian path happens to return to where it started, then there's nothing that distinguishes an initial vertex from an intermediate one, which proves the following:

Proposition 1.1. *In any Eulerian tour of a graph (i.e. a path that starts and ends at the same vertex and traverses each edge of the graph exactly once), every vertex has even degree.*

This gives us a way to prove that a given graph doesn't have an Eulerian tour – just find some vertex of odd degree. But what if we want to prove that a graph *does* have an Eulerian tour? Does it suffice to prove that every vertex has even degree? Euler discovered that it does!

Theorem 1.2. *Any connected graph whose vertices all have even degree has an Eulerian tour.*

We illustrate the proof before writing it down formally. Given a graph G, pick a random vertex v_0 and take a random walk along the edges of G, labeling each vertex on your walk and deleting any edge you traverse. Eventually you'll get back to v_0 and get stuck. Whatever's left of the original graph consists of a bunch of small connected graphs: G_1 containing v_1, G_2 containing v_2, etc. All the vertices in all the G_i's still have even degree, so by induction, each of these graphs must have an Eulerian tour. This gives us an Eulerian tour of our original graph G: start at v_0, walk to v_1, take an Eulerian tour of G_1, walk to v_2, take an Eulerian tour of G_2, etc!



The original graph G *What's left of G after our random walk*

Proof. We induct on the number of edges of the graph. Consider a connected graph G in which every vertex has even degree, and pick any vertex v_0 of G. Start walking from v_0 along edges at random, labeling v_i the i^{th} vertex you arrive at along the walk. As you walk, delete every edge you traverse. Keep walking until you get stuck somewhere (i.e. until you get to a vertex where there are no edges you can take to leave). This vertex must be v_0, since otherwise it would have odd degree!

Now consider the graph G' that remains after taking the above walk. First observe that since we walked along an Eulerian tour, we've removed an even number of edges from each of the vertices v_i; this means that every vertex in G' still has even degree. For each $j \geq 1$, let G_j denote the connected component of G' that contains v_j, unless v_j is already a vertex in some previously defined G_i with $i < j$, in which case set G_j to be the lone vertex v_j with no edges.

Note that each G_j is a connected graph in which every vertex has even degree. Since G_j has a smaller number of edges than G, our inductive hypothesis guarantees that G_j has an Eulerian tour for every j. We've thus constructed an Eulerian tour of our original graph G: starting at v_0, walk to v_1 and take the Eulerian tour of G_1, then walk to v_2 and take the Eulerian tour of G_2, etc. $\square$

Combined, our results completely characterize graphs that have Eulerian tours:

Theorem 1.3. *A graph has an Eulerian tour if and only if it's connected and every vertex has even degree.*

2. Euler's Formula

Given a graph G, let $V(G)$ denote the number of vertices of G, $E(G)$ the number of edges of G, and $F(G)$ the number of faces of G (i.e. the number of two dimensional pieces G partitions the plane into). When there is no ambiguity, we shall simply write V, E, and F. Our goal is to prove the following fundamental discovery of Euler:

Theorem 2.1 (Euler's Formula)**.** *For any connected planar graph,* $V - E + F = 2$.

Remark. The quantity $V - E + F$ is called the *Euler characteristic* of a graph.

The first insight we'll need to prove this was conjectured by Red Burton, a computer program designed by Siemion Fajtlowicz:

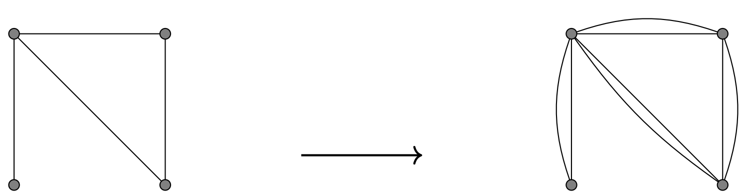
Proposition 2.2. *Euler's formula* $V - E + F = 2$ *holds for any graph that has an Eulerian tour.*

Before proving this, we use it to give a quick Theorem 2.1. This proof was discovered by Stephanie Mathew when she was a sophomore engineering student at the University of Houston.

Proof of Theorem 2.1. Given a connected planar graph G. We form a new graph G' on the same set of vertices as G as follows: for every edge e in G, add an edge parallel to it (see picture below). Note that each edge we add in this way adds precisely one face, whence

$$E(G') = 2E(G) \qquad \text{and} \qquad F(G') = F(G) + E(G).$$

This immediately implies that the Euler characteristic of G' is the same as the Euler characteristic of G. But by Theorem 1.3 we know G' has an Eulerian tour, whence Proposition 2.2 implies that the Euler characteristic of G' is 2. It follows that the Euler characteristic of G is also 2, as claimed. $\square$



The graph G   *The graph G'*

It remains only to prove Red Burton's conjecture.

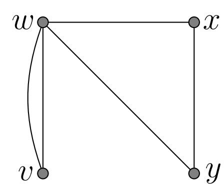
Proof of Proposition 2.2. Let G be a graph that has an Eulerian tour. This Eulerian tour traverses each edge precisely once, but it might visit a vertex multiple times. For any vertex v, let $r(v)$ denote the *redundancy* of v: the number of times the Eulerian tour revisits v (see example below). Since we are considering an Eulerian tour, we immediately see that

$$\sum_{v \in V} r(v) = E + 1 - V.$$

On the other hand, every time you return to a vertex you create a new interior face, whence the left hand side of the above equation counts the number of interior faces:

$$\sum_{v \in V} r(v) = F - 1.$$

Combining the above two identities yields Euler's formula for the case of an Eulerian graph. $\square$



$r(x) = r(y) = 0$, *while* $r(v) = r(w) = 1$

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