16-311-Q Introduction to Robotics

Lecture 6: Degrees of Maneuverability Instantaneous Center of Rotation

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A ROBOT WITH N STANDARD WHEELS OUT OF M WHEELS

Rolling constraints

$$J_1(\beta_s)R(\theta)\dot{\xi}_I - J_2\dot{\varphi} = 0$$

$$\varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}, \quad J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}, \quad J_2 = diag(r_1, r_2, \dots, r_N)$$

$$\varphi(t) = (N_f + N_s) \times 1, \quad J_1 = (N_f + N_s) \times 3$$

J_1 is a matrix with projections for all wheels to their motions along their individual wheel planes, which have a fixed angle for the fixed wheels and a time-varying angle for the steerable wheels

Sliding constraints

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0, \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

$$C_1 = (N_f + N_s) \times 3$$

C_1 is a matrix with projections for all wheels to their motions in the direction orthogonal to wheels' planes

BACK TO THE DEGREE OF MOBILITY δ_m ...

- δ_m quantifies the degrees of controllable freedom based on changes to wheels' velocity, expressed by the equation system from the no lateral slip constraints applied to all standard wheels (fixed and steerable)
- Each row of the projection matrix is a **kinematic constraint**: no-sliding, imposed by one of the wheels of the chassis along the direction orthogonal to the wheel

$$C_{1f} R(\theta)\dot{\xi}_I = 0$$

$$C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

$$C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$ Projection matrix

where

$$\dot{\xi}_R = R(\theta)\dot{\xi}_I$$

Hold simultaneously

Standard fixed wheel 1: $\begin{bmatrix} P^{1f\perp}_{\dot{x}_R} & P^{1f\perp}_{\dot{y}_R} & P^{1f\perp}_{\dot{\theta}_R} \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = 0$

Standard fixed wheel N_f : $\begin{bmatrix} P^{N_f f\perp}_{\dot{x}_R} & P^{N_f f\perp}_{\dot{y}_R} & P^{N_f f\perp}_{\dot{\theta}_R} \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = 0$

Standard steering wheel 1: $\begin{bmatrix} P^{1s\perp}_{\dot{x}_R} & P^{1s\perp}_{\dot{y}_R} & P^{1s\perp}_{\dot{\theta}_R} \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = 0$

Standard steering wheel N_s : $\begin{bmatrix} P^{N_s s\perp}_{\dot{x}_R} & P^{N_s s\perp}_{\dot{y}_R} & P^{N_s s\perp}_{\dot{\theta}_R} \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = 0$

INDEPENDENT # OF KINEMATIC CONSTRAINTS VS. MOBILITY

Is each one of these no-side motion constraint equations imposing an additional *independent* constraint to the robot kinematics?

Independence $\rightarrow$ Are all constraint equations independent from each other? $\rightarrow$ Are the rows in the matrix (linearly) independent?

$\rightarrow$ rank[$C_1(\beta_s)$]$_{N \times 3}$ $\rightarrow$ *The greater the rank, the more constrained the motion is*

- Mathematically, the constraint equations says that the vector $\dot{\xi}_R$ must belong to the **null space** $N(C_1(\beta_s))$ of the projection matrix $C_1(\beta_s)_{N\times3}$

$$N(C_1(\beta_s)) = \{\boldsymbol{x} \in \mathbb{R}^2 \times \mathbb{S} \mid C_1(\beta_s)\boldsymbol{x} = \boldsymbol{0}\}$$

- If matrix $C_1(\beta_s)$ has rank 3 (i.e., all 3 columns are independent), then the *Null space* only contains **0** (no motion) as possible solution to the homogeneous equation: no other motion vector can satisfy the constraints, *robot can't move!*

$$\textbf{rank}[\ C_1(\beta_s)\ ]_{N \times 3} = 3 - \dim N\ [C_1(\beta_s)]$$

- The larger the Null space, the larger the set of motion vectors that can satisfy the no side motion constraints $\rightarrow$ the more motion freedom the robot has

DEGREE OF MOBILITY (NO SIDE MOTION CONSTRAINTS)

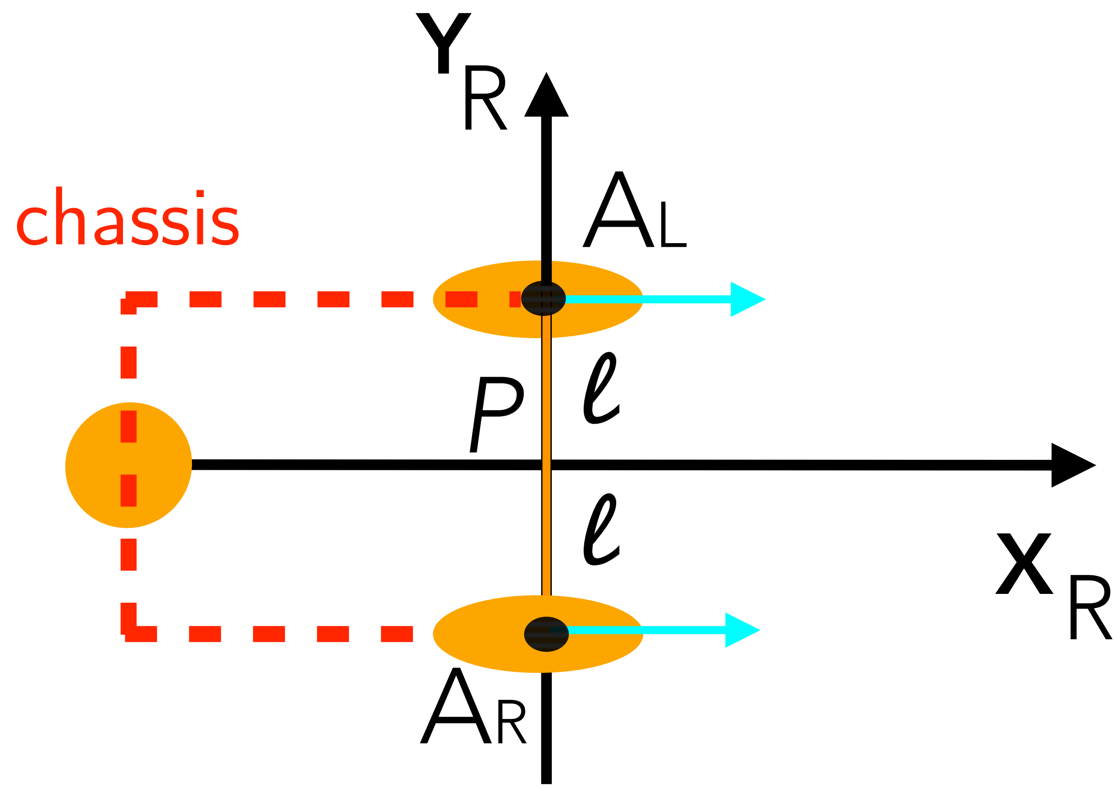
Degree of mobility, δ_m

δ_m = **dim N [$C_1(\beta_s)$] = 3 - rank[$C_1(\beta_s)$]** **0 ≤ rank[$C_1(\beta_s)$]$_{N \times 3}$ ≤ 3**

- no standard wheels (no side-motion constraints): rank[$C_1(\beta_s)$] = 0
- all motion directions constrained: rank[$C_1(\beta_s)$] = 3

- N [$C_1(\beta_s)$] $\subseteq \mathbb{R}^2 \times \mathbb{S}$, the exact dimension of N [$C_1(\beta_s)$] is the same as the dimension of the vector *basis* B(N) that spans the null space of $C_1(\beta_s)$, which cannot be higher than 3, the dimension of $\mathbb{R}^2 \times \mathbb{S}$
- **If dim[N] = 2** → The basis of N is made of two linearly independent vectors B(N) = ($\mathbf{v}_1$, $\mathbf{v}_2$) , which means that all feasible motion vectors **v** (that are in N) can be generated as a linear combination of these two vectors
→ One dimension of velocity is not directly controllable
- **If dim[N] =1** → Two dimensions of velocity are not directly controllable
- **if dim[N] =0** → All dimensions of velocity are not directly controllable
- **if dim[N] =3** → All dimensions of velocity are directly controllable

DIMENSION OF NULL SPACE AND CONTROLLABLE DEGREES OF VELOCITY



$$\mathbf{C_1(\beta_s) \rightarrow C_{1f}}$$

$$\textbf{rank}[\ \mathbf{C_{1f}}\ ]_{2 \times 3} = \mathbf{3 - dim\ N}[\ \mathbf{C_{1f}}\ ]$$

$$C_{1f} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} (\text{wheel } A_L) \\ (\text{wheel } A_R) \end{matrix}$$

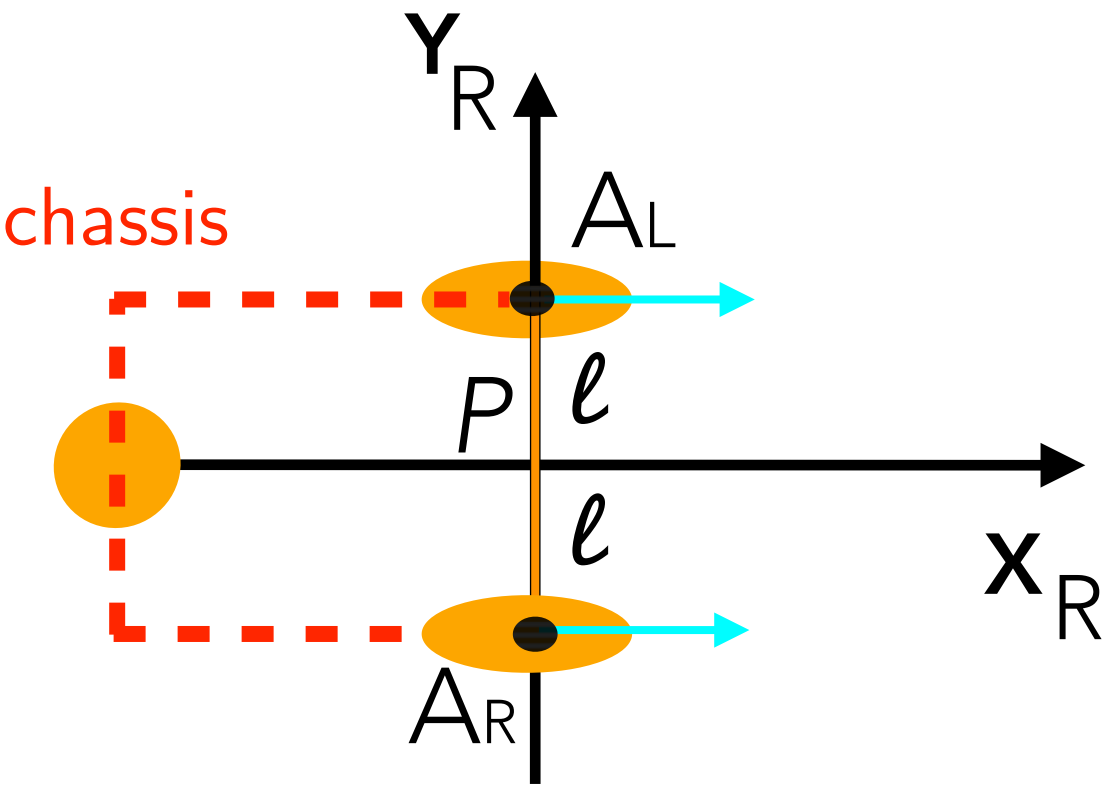
$$N(C_{1f}) = \{\dot{\xi} \mid C_{1f}\dot{\xi}_R = 0\}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} (r\dot{\varphi}\ \text{wheel} A_L) \\ (r\dot{\varphi}\ \text{wheel} A_R) \end{matrix} \quad \begin{cases} \dot{y}_R = 0 \\ \dot{y}_R = 0 \end{cases} \Rightarrow \text{Solution set: } \begin{cases} \dot{x}_R \in \mathbb{R}, & \textit{free variable} \\ \dot{y}_R = 0 & \\ \dot{\theta}_R \in \mathbb{S}, & \textit{free variable} \end{cases}$$

All (feasible velocity) vectors in the null space are generated by the *linear combination*:

$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = \dot{x}_R \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \dot{\theta}_R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{such that the basis } B(N(C_{1f})) \text{ of } N \text{ is: } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

DIMENSION OF NULL SPACE AND CONTROLLABLE DEGREES OF MOTION



$$C_1(\beta_s) \rightarrow C_{1f}$$

$$\text{rank}[\ C_{1f}\ ]_{2 \times 3} = 3 - \dim N[\ C_{1f}\ ]$$

$$C_{1f} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{(wheel } A_L\text{)} \\ \text{(wheel } A_R\text{)} \end{matrix}$$

$$N(C_{1f}) = \{\dot{\xi} \mid C_{1f}\dot{\xi}_R = 0\}$$

$$B(N(C_{1f})) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\dim N[\ C_{1f}\ ] = \dim B(\ C_{1f}\ ) = 2$
- $\text{rank}[\ C_{1f}\ ] = 3 - 2 = 1$

- Directly controllable degrees of freedom in velocity: $\dot{x}$ and $\dot{\theta}$
- Control in orientation is obtained by acting upon wheels' speeds, no steering control is required

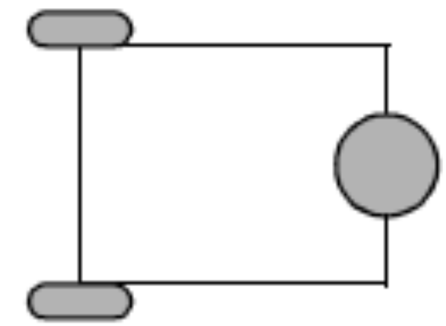
DEGREE OF MOBILITY: EXAMPLES

Differential drive robot with two standard fixed wheels

1. ***Two wheels on the same axle:*** only one independent kinematic constraint (the second wheel is constrained by the axle) $\rightarrow \text{rank}(C_1) = 1 \rightarrow \delta_m = \mathbf{2}$

   A. Control of the rate of change in orientation

   B. Control of forward/backward speed

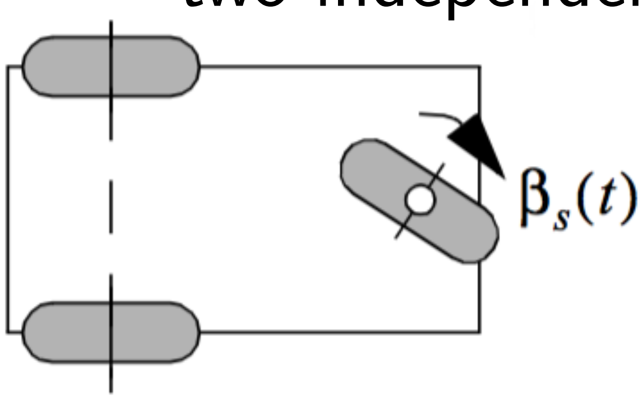


Tricycle chassis: one steerable + two fixed standard wheels

1. ***Two wheels on same axles + one on a different one:***

   two independent kinematic constraints $\rightarrow$ $\text{rank}(C_1) = 2 \rightarrow \delta_m = \mathbf{1}$

   A. Control of forward/backward speed



An explicit steering control is required to change orientation of front wheel, the two rear wheels get powered with same speed

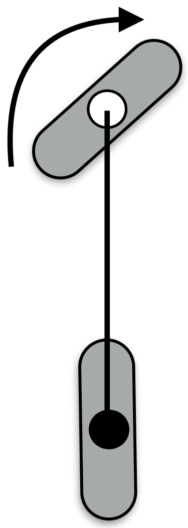
DEGREE OF MOBILITY: EXAMPLES

Bicycle chassis: one fixed and one steerable wheel

1. ***Two wheels on different axles:*** two independent kinematic constraints

$\rightarrow$ $\text{rank}(C_1) = 2 \rightarrow \boldsymbol{\delta_m = 1}$

A. Control of forward/backward speed

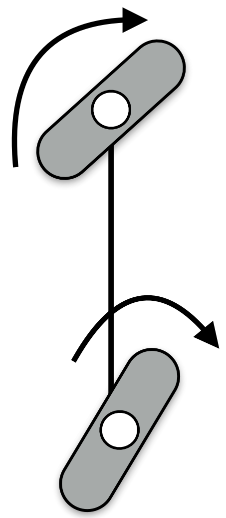


An explicit steering control is required to change orientation (it can't be done by <u>velocity commands to the wheels</u>)

Pseudo-Bicycle chassis: two steerable wheels

1. ***Two steering wheels on different axles:*** two independent kinematic constraints $\rightarrow$ $\text{rank}(C_1) = 2 \rightarrow \boldsymbol{\delta_m = 1}$

A. Instantaneous control of forward/backward speed



An explicit steering control is required to change orientation of both wheels (in a consistent manner)

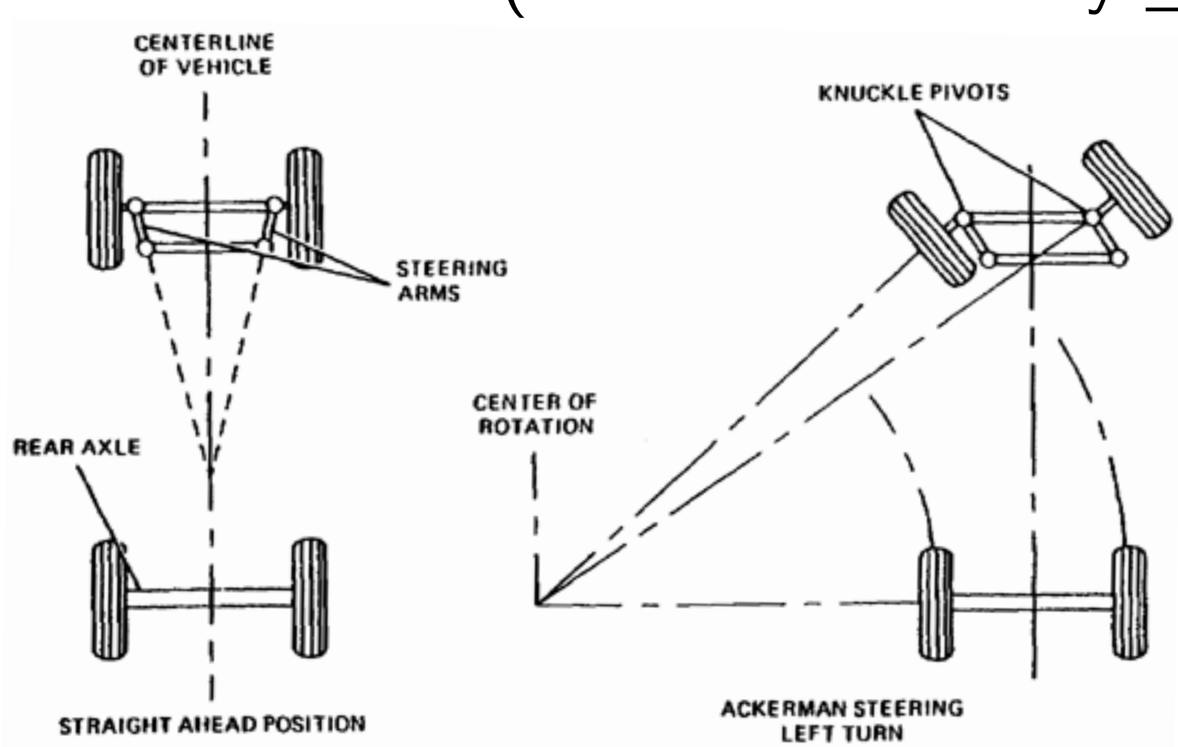
DEGREE OF MOBILITY: EXAMPLES

Car-like Ackerman vehicle two standard fixed wheels + two steerable (but jointly constrained) wheels

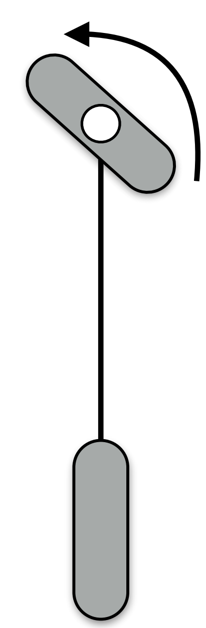
1. ***Two rear wheels on the same axle + Two steering wheels connected by steering arms:*** four kinematic constraints, only two are linearly independent $\rightarrow C_1 = C_{1f}$ and $\text{rank}(C_{1f}) = 2 \rightarrow \delta_m = \mathbf{1}$

    A. Control of forward/backward speed

An explicit steering control (onto the front wheels) is required to change orientation (it can't be done by velocity commands to the wheels)



Kinematically equivalent to bicycle model



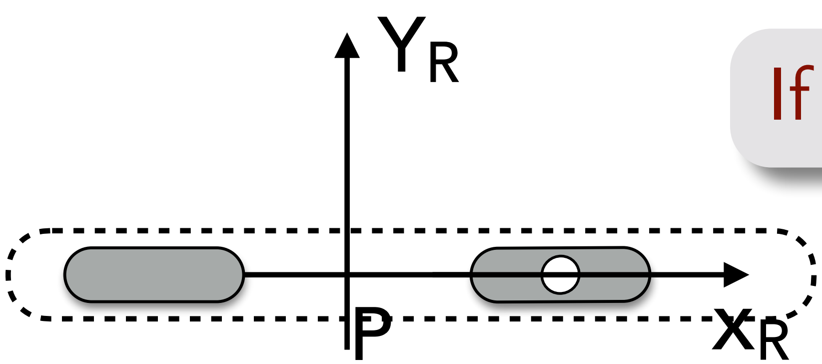
DEGREE OF MOBILITY: EXAMPLES

Locked Bicycle: one fixed (steerable before) wheel + one fixed standard wheel

1. ***The front wheel in the same plane of rear wheel + locked in forward position:*** two independent kinematic constraints $\rightarrow$ rank(C_1) = 2 $\rightarrow$ $\boldsymbol{\delta_m = 1}$
   A. Control of forward/backward speed

An explicit steering control is required to change orientation (it can't be done by velocity commands to the wheels)

$(l_1 = l_2), (\beta_1 = \beta_2 = \pi/2), (\alpha_1 = 0, \alpha_2 = \pi)$



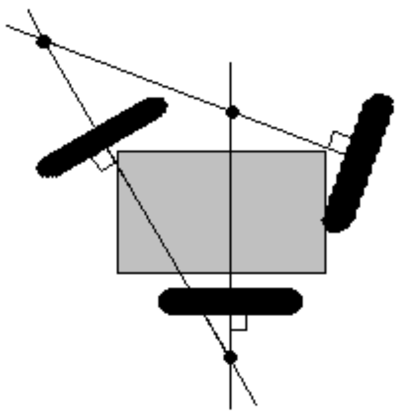
If rank(C_{1f}) > 1 $\Rightarrow$ Motion is constrained on a line/circle

Linearly independent!
→ rank(C_{1f}) = 2

$$C_1(\beta_s)) = C_{1f} = \begin{bmatrix} \cos(\pi/2) & \sin(\pi/2) & l_1 \sin(\pi/2) \\ \cos(3\pi/2) & \sin(3\pi/2) & l_1 \sin(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & l_1 \\ 0 & -1 & l_1 \end{bmatrix}$$

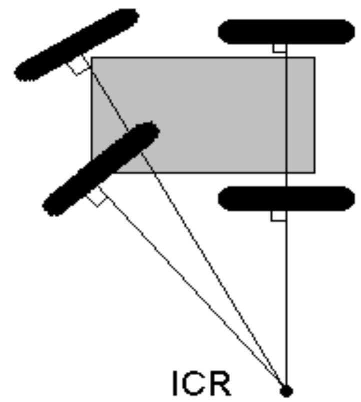
EXAMPLES FOR DEGREE OF MOBILITY

Don't focus on ICR (for now) …



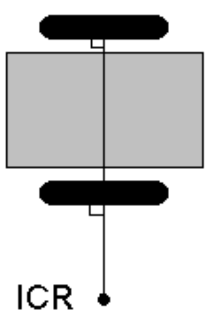
Cannot move anywhere (No ICR)

- Degree of mobility : 0



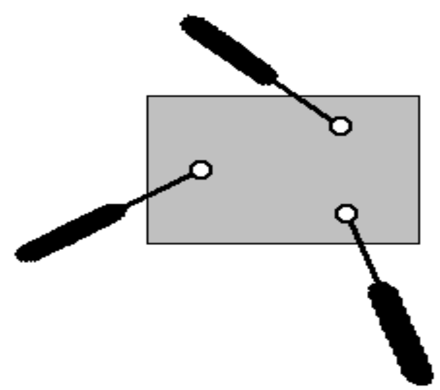
Fixed arc motion (Only one ICR)

- Degree of mobility : 1



Variable arc motion (line of ICRs)

- Degree of mobility : 2



Fully free motion

(ICR can be located at any position)

- Degree of mobility : 3

DEGREE OF STEERABILITY

- Degree of mobility: controllable freedom based on changes on wheel velocity
  - **Degree of steerability δ_s: controllable freedom based on changes in wheel orientation $\rightarrow$ number of independently controllable steering parameters (~steering wheels)**

$$\delta_s = rank\left[C_{1s}(\beta_s)\right] \qquad 0 \leq \delta_s \leq 2$$

The *higher the rank*, the more ***degrees of steering freedom***,

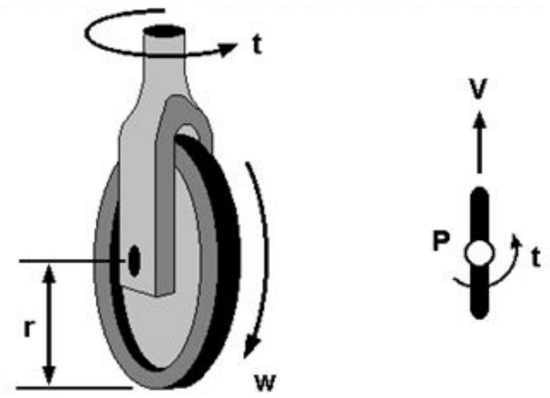
$\Rightarrow$ greater maneuverability

- ✦ 0 corresponds to no steering wheels
- ✦ 1 corresponds to one (or more, dependent) steering wheels (e.g., Ackermann)
- ✦ 2 corresponds to two independent steering wheels
- ✦ 3 would correspond to have instantaneously 3 linearly independent “fixed” wheels which means no motion (two of them must steer in a dependent way, in order to allow for motion)

DEGREE OF STEERABILITY VS. DEGREE OF MOBILITY

- Degree of mobility: controllable freedom based on changes on wheel velocity, decreases with the increase in rank$[C_1(\beta_s)]$
- Degree of steerability: controllable freedom based on changes in wheel orientation, increases with the increase in rank$[C_{1s}(\beta_s)]$

At any instant, the orientation imposes a kinematic constraint (perpendicular to the orientation) and reduces mobility

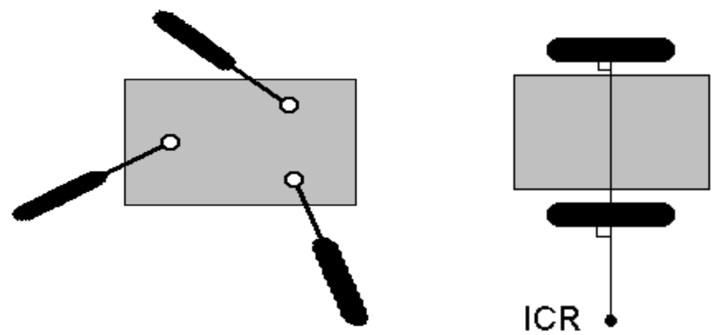


The ability to change orientation allows more freedom choosing trajectories

~conflicting instantaneous degrees of maneuverability

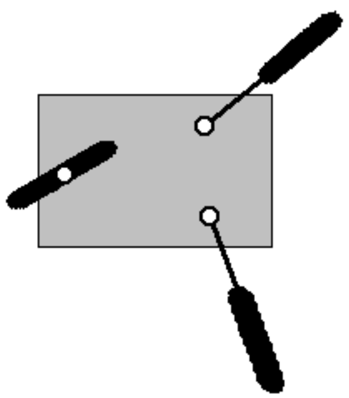
Note: Steering has an *indirect* impact on a robot chassis pose: robot must first move for the change in steering angle to have impact on pose

EXAMPLES FOR DEGREE OF STEERABILITY

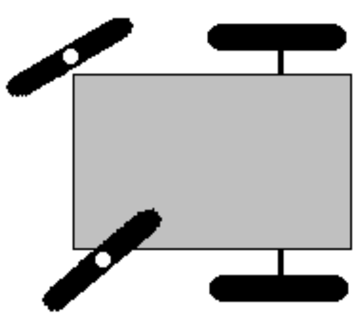


No centered orientable wheels

- Degree of steerability : 0

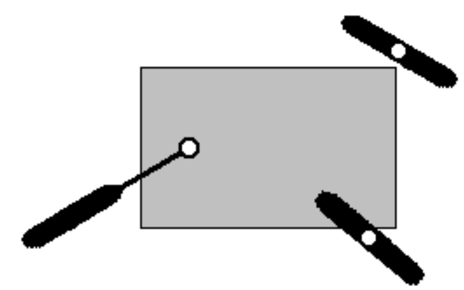


One centered orientable wheel



Two mutually dependent centered orientable wheels

- Degree of steerability : 1



Two mutually independent centered orientable wheels

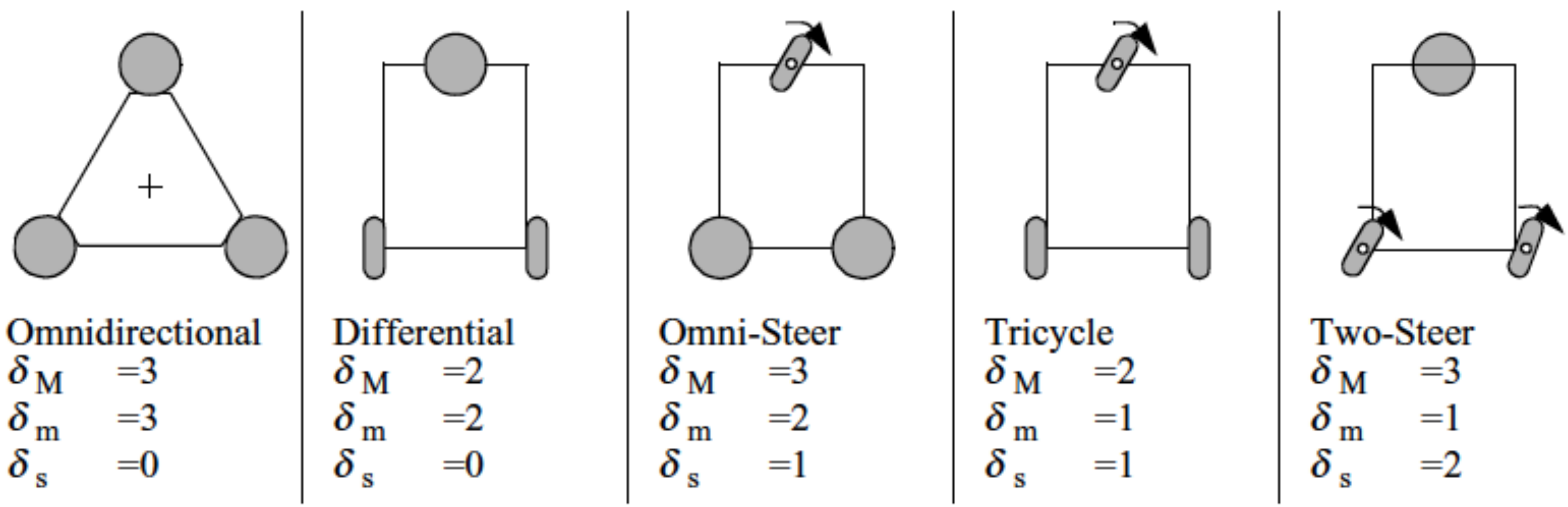
- Degree of steerability : 2

DEGREE OF MANEUVERABILITY

Degree of maneuverability δ_M: overall degrees of freedom that a robot can manipulate by changing wheels' speed (direct mobility) and wheels' orientation (indirect mobility)

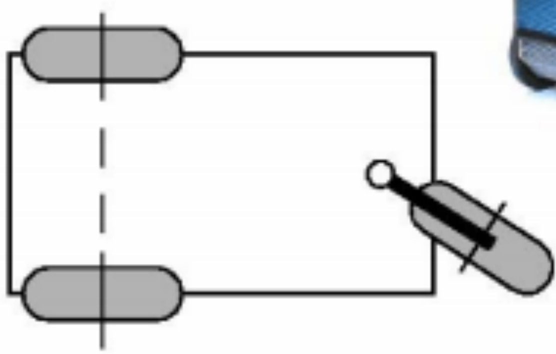
$$\delta_M = \delta_m + \delta_s$$

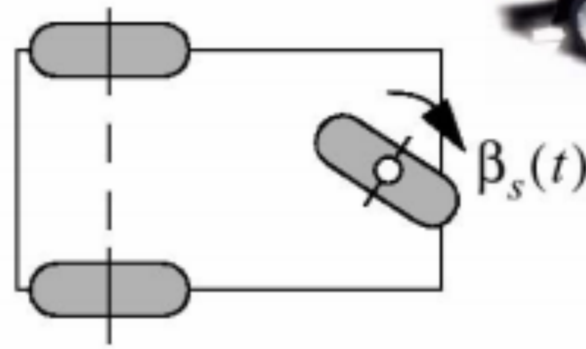
Two robots with the same value of degree of maneuverability are *not maneuverable in the same way,* since contributions can come from different kinematic aspects (either mobility or steerability)



DEGREE OF MANEUVERABILITY

$\delta_M = (2+0) = 2$

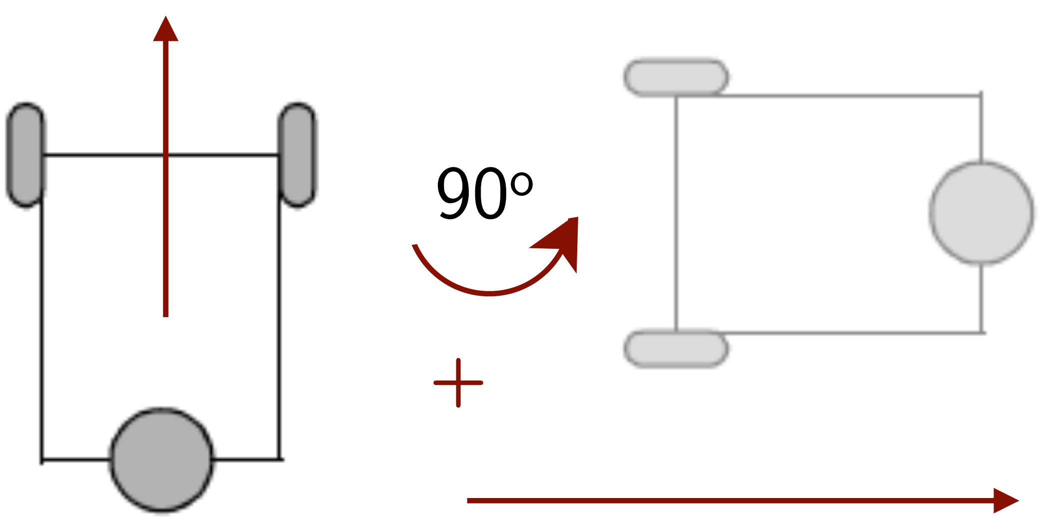


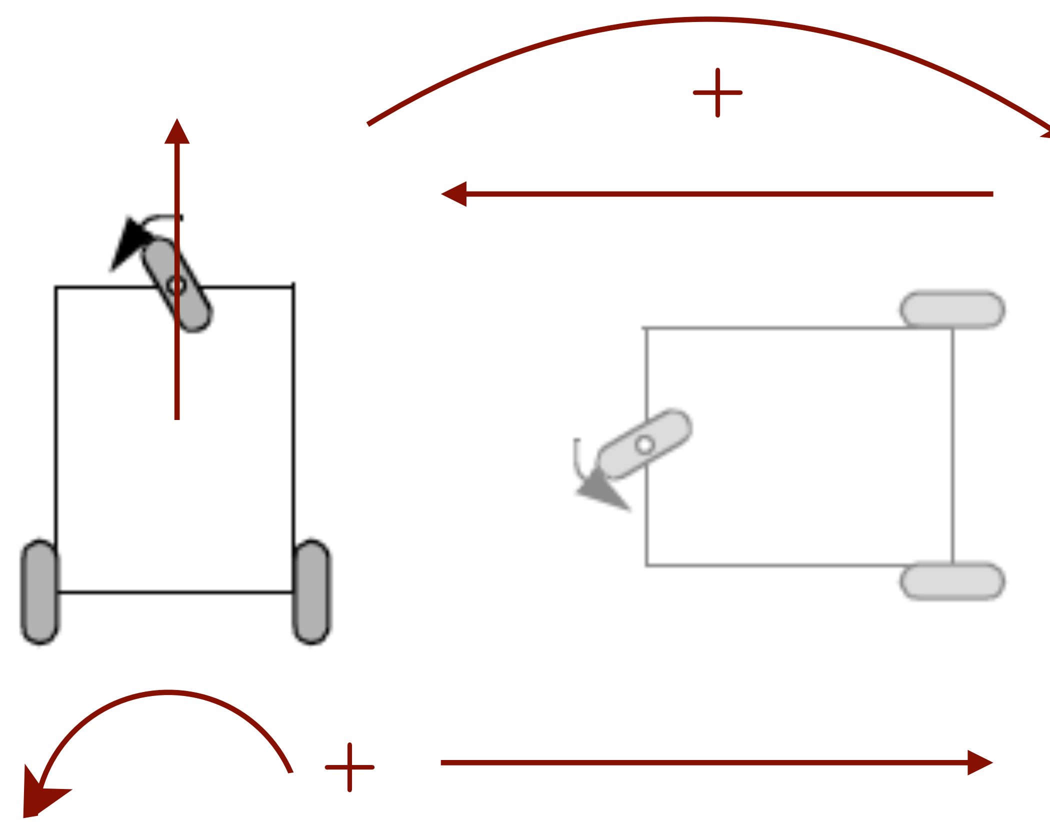


$\delta_M = (1+1) = 2$

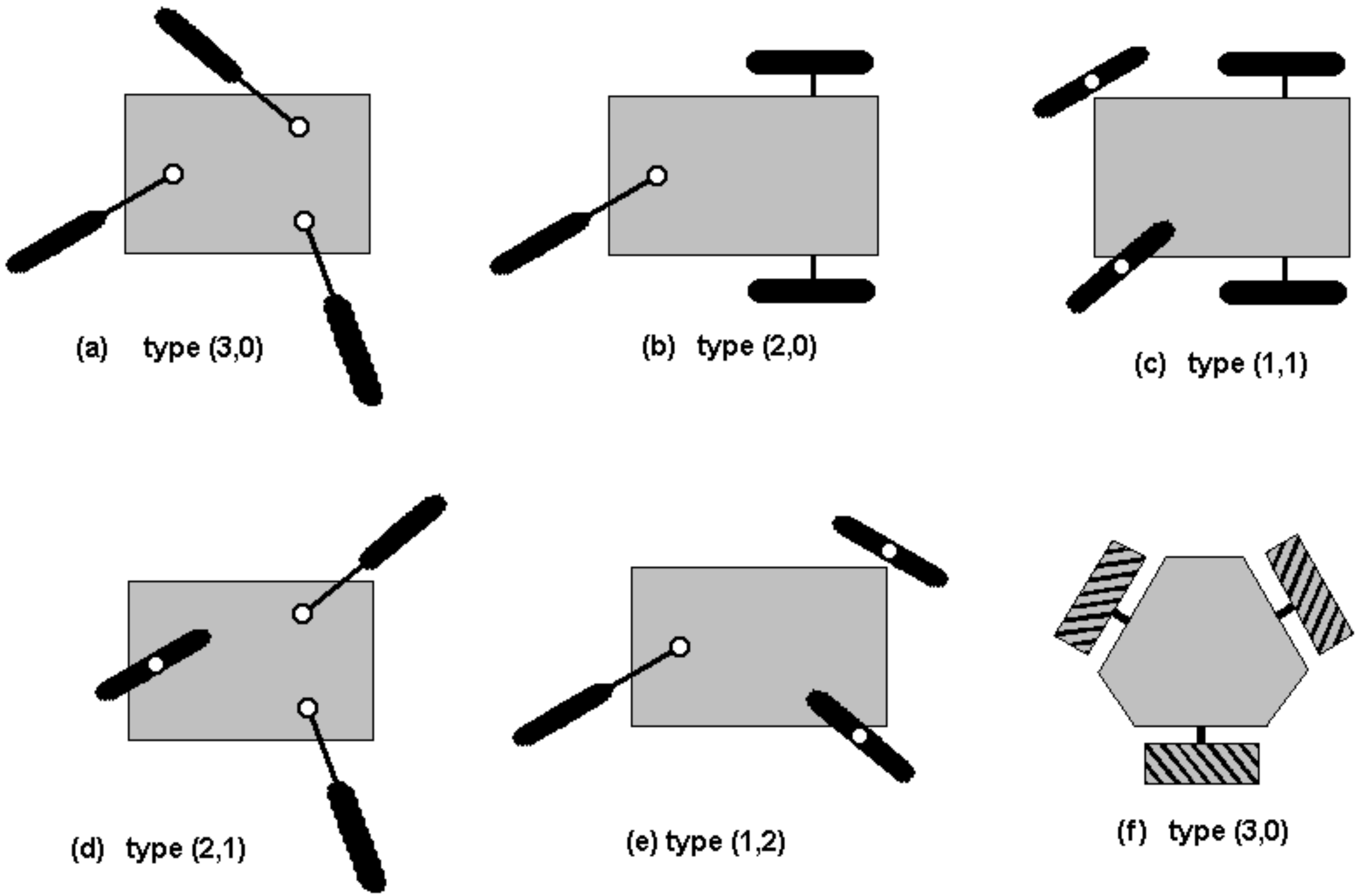
Differential drive robot with two standard fixed wheels + castor/omni

Tricycle chassis: one steerable + two fixed standard wheels (~same speed)





OTHER EXAMPLES OF VEHICLES WITH DIFFERENT MANEUVERABILITY

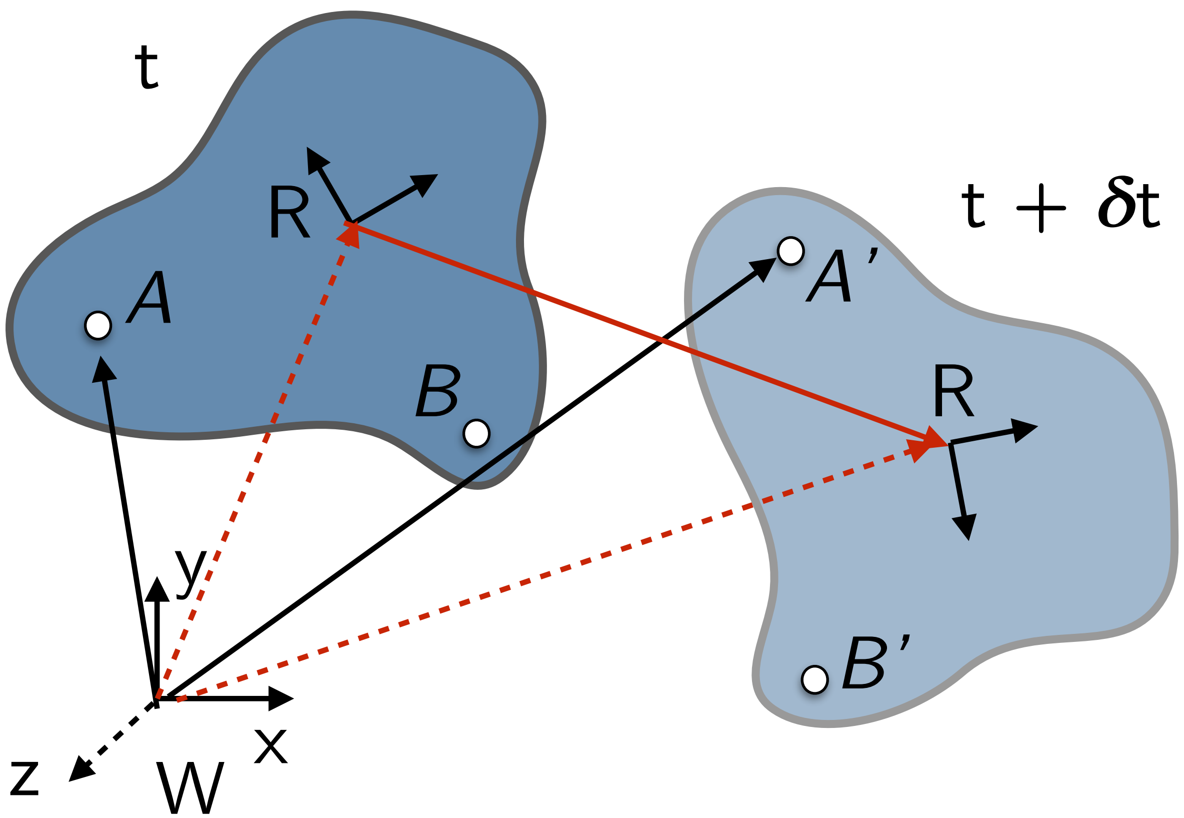


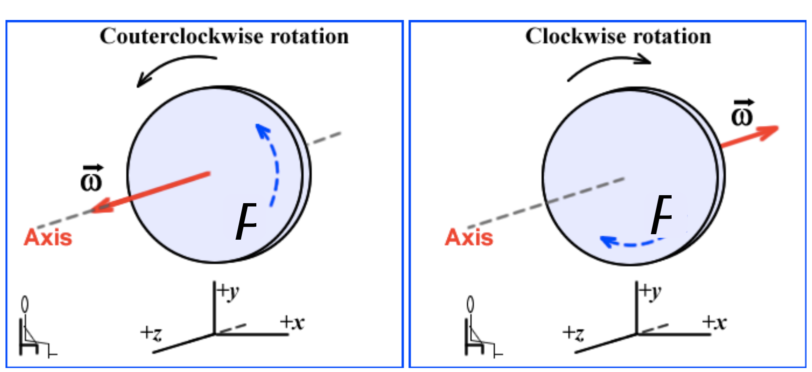
$$(\delta_m, \delta_s)$$

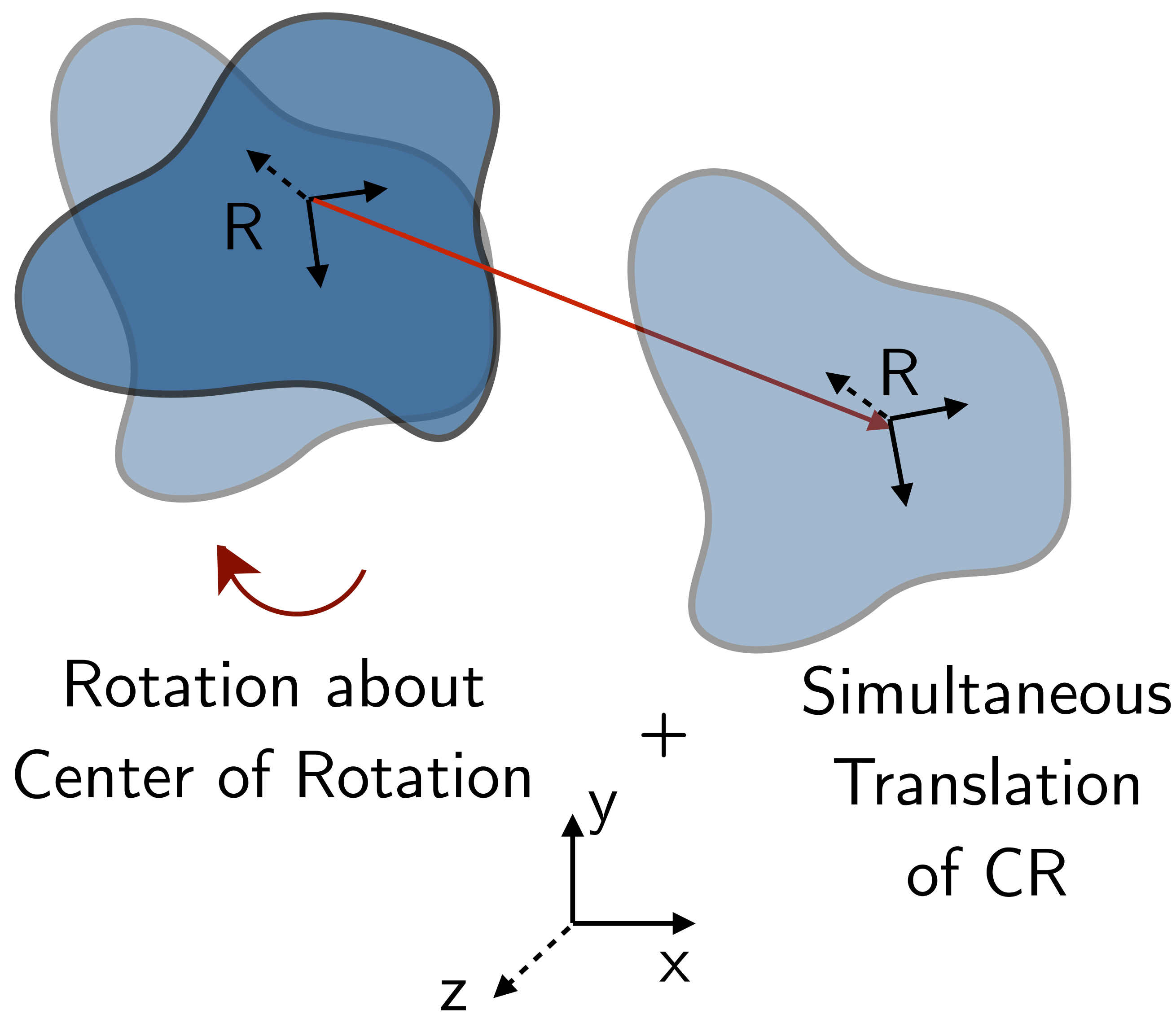
Let's look at the same problem(s) from a **geometric point of view** ....

Instantaneous center of rotation (ICR) / Instantaneous center of curvature (ICC)

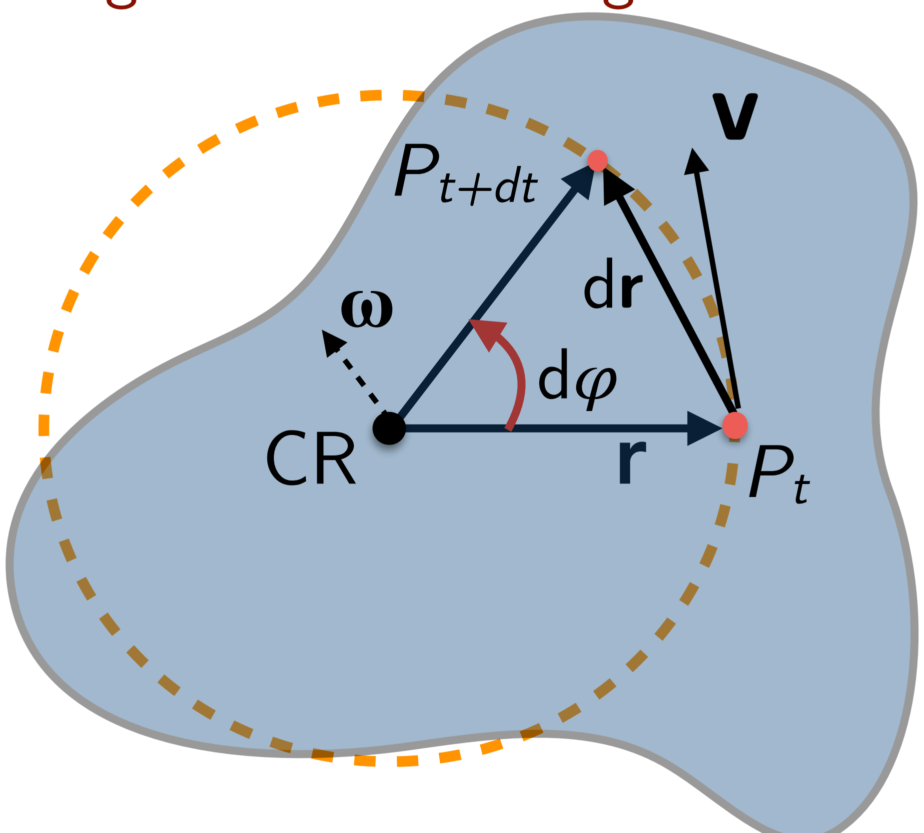
A *few* preliminary notions about kinematics of rigid bodies ...

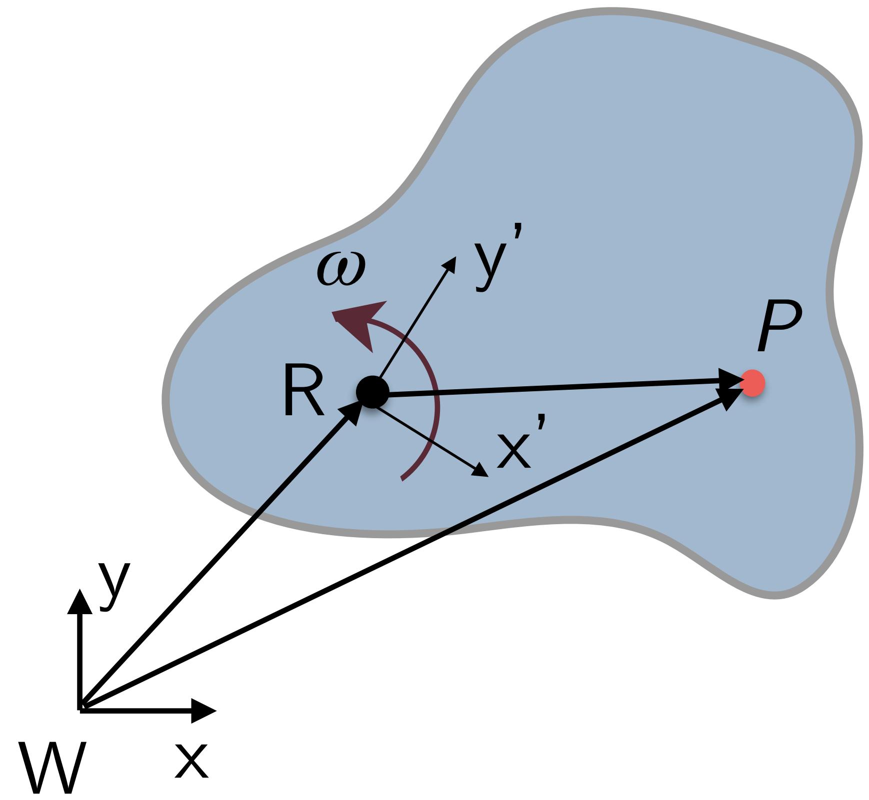


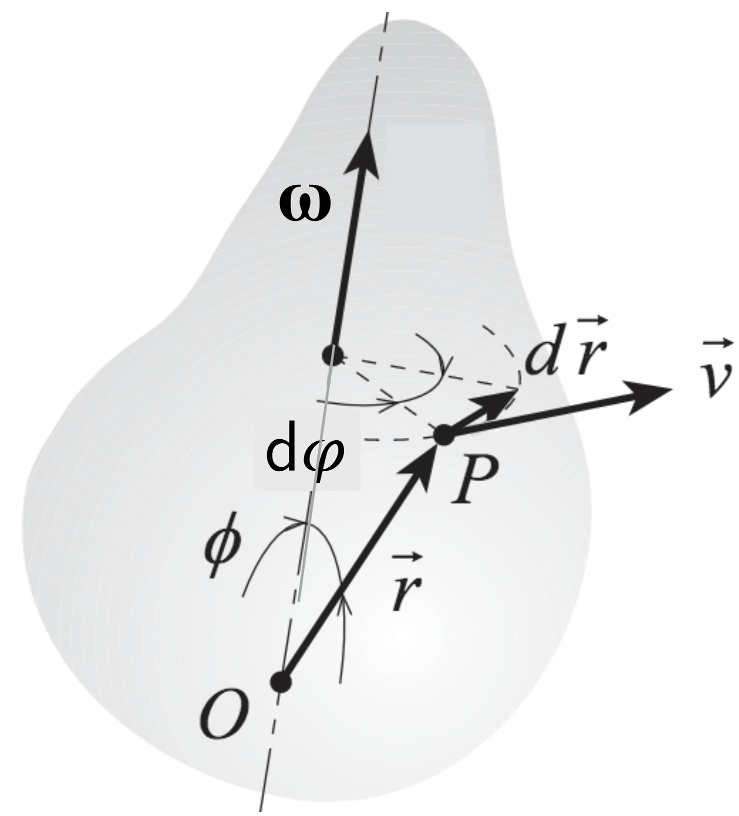




For $dt \to 0$, d**r** becomes ⊥ to **r** ⇒ is aligned with the tangential velocity







Angular velocity (for all points of the rigid body): $\boldsymbol{\omega} = \frac{d\boldsymbol{\varphi}}{dt} = \frac{d\varphi}{dt}\widehat{\boldsymbol{\varphi}}$

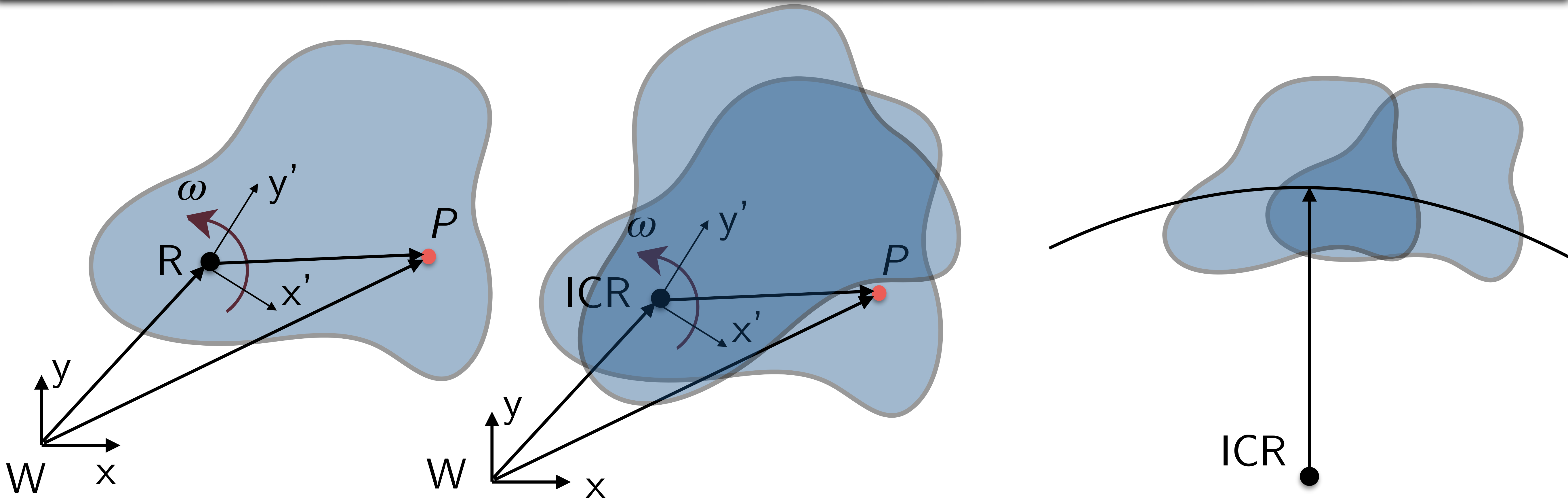
(Angular) Variation of position vector of a point P : $d\boldsymbol{r} = d\boldsymbol{\varphi} \times \boldsymbol{r}$
with respect to center of rotation (CR)

Velocity of point P wrt center of rotation: $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{\varphi} \times d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{\varphi}}{dt} \times d\boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r}$

Position vector of P in {W}: ${}^{W}\boldsymbol{r}_P = {}^{W}\boldsymbol{r}_R + {}^{R}\boldsymbol{r}_P$

Velocity vector of P in {W}: ${}^{W}\boldsymbol{v}_P = {}^{W}\boldsymbol{v}_R + {}^{R}\boldsymbol{v}_P + \boldsymbol{\omega} \times {}^{R}\boldsymbol{r}_P = {}^{W}\boldsymbol{v}_R + \boldsymbol{\omega} \times {}^{R}\boldsymbol{r}_P$

INSTANTANEOUS CENTER OF ROTATION



- For pure rotation, the center of rotation is a point that has zero velocity in {W}
- If the case of general motion, if we can find a point, *ICR*, for which the ***instantaneous velocity in {W} is zero***, then the velocity of the body at that particular instant would be described as a pure rotation about the ICR

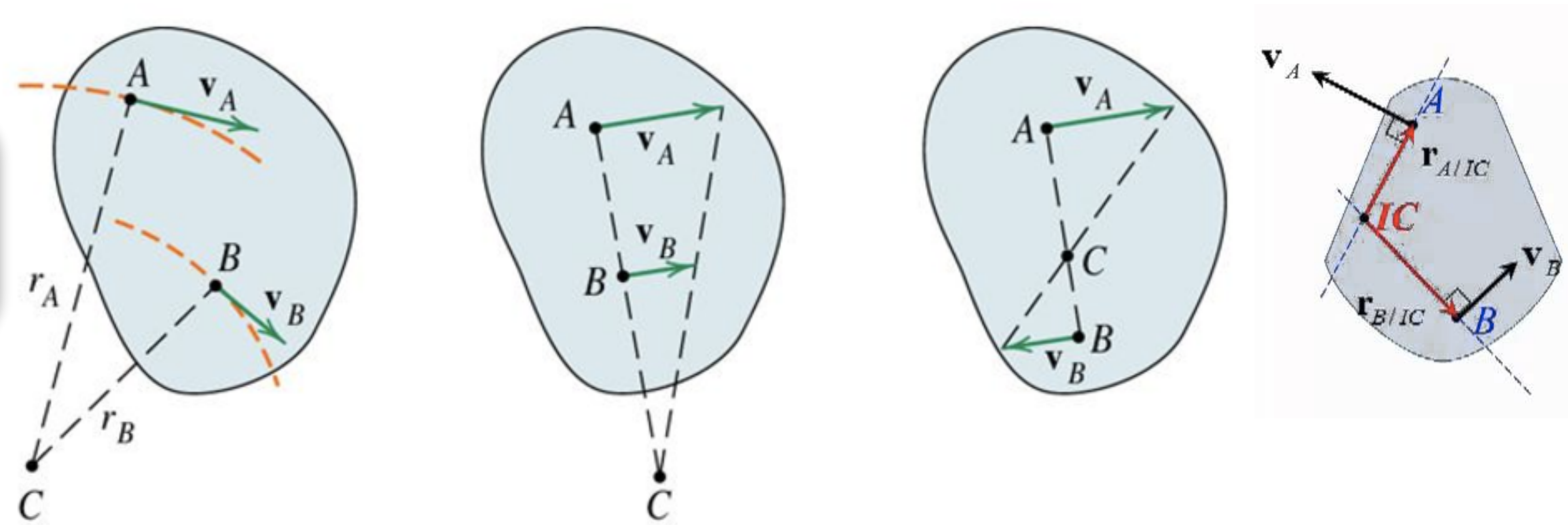
Velocity vector of P in {W}: $\quad {}^W\boldsymbol{v}_P = {}^W\boldsymbol{v}_R + \boldsymbol{\omega} \times {}^R\boldsymbol{r}_P$

Point ICR with 0 velocity in {W}: $\quad {}^W\boldsymbol{v}_{ICR} = 0 = {}^W\boldsymbol{v}_R + \boldsymbol{\omega} \times {}^R\boldsymbol{r}_{ICR}$

multiplying by $-\omega$ and rearranging: $\quad {}^R\boldsymbol{r}_{ICR} = \frac{1}{\omega^2}(\boldsymbol{\omega} \times {}^W\boldsymbol{v}_R)$

INSTANTANEOUS CENTER OF ROTATION

$$ {}^{R}\boldsymbol{r}_{ICR} = \frac{1}{\omega^2}(\boldsymbol{\omega} \times {}^{W}\boldsymbol{v}_R) $$



- If the there is no translation, the ICR is the same as the center of rotation: the velocity of {R} is zero in {W} and accordingly, the ICR coincides with {R}.
- The position vector of the ICR is perpendicular to $\mathbf{v}_R$, the velocity vector in {R}. More in general, selected a point **A** in the body, the position vector (**ICR — A**) is perpendicular to the velocity vector in **A**
- → If we know the velocity at two points of the body, **A** and **B**, then the location of **ICR** can be determined geometrically as the intersection of the lines which go through points **A** and **B** and are perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$
- When the angular velocity, $\boldsymbol{\omega}$, is very small, the center of rotation is very far away; when it is zero (i.e. a pure translation), the center of rotation is at infinity.