On the compact real form of the Lie algebra $\mathfrak{g}_2$

Robert A. Wilson,
School of Mathematical Sciences,
Queen Mary, University of London,
Mile End Road,
London E1 4NS
E-mail: R.A.Wilson@qmul.ac.uk.

October 16, 2007

Abstract

We give an elementary construction of the compact real form of the Lie algebra $\mathfrak{g}_2$.

It is known that every simple complex Lie algebra has a (unique) compact real form, defined by the property that the Killing form is negative definite. For example, the compact real form of $\mathfrak{a}_1$ is just the well-known 3-dimensional vector cross product given by $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}$ and images under the symmetry rotating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

However, I have been unable to find in the literature any nice constructions of the compact real forms of the exceptional Lie algebras. In this note I give an elementary construction of the compact real form of $\mathfrak{g}_2$, exhibiting an irreducible subgroup $2^{3\cdot}L_3(2)$ of the automorphism group.

This group acts on the Lie algebra by permuting seven mutually orthogonal Cartan subalgebras, and the stabilizer of one of these is a group $2^{3\cdot}S_4 \cong 4^2.(2 \times S_3)$. The latter description shows that the induced action on the Cartan subalgebra is the full Weyl group, $W(G_2) \cong 2 \times S_3$, and that the representation of $2^{3\cdot}L_3(2)$ on the Lie algebra is induced from the natural 2-dimensional representation of the Weyl group. (In fact, there are two such representations, but they are interchanged by the outer automorphism of $2^{3\cdot}L_3(2)$, so it does not matter which one we pick.)

To construct this representation, we first take the 7-dimensional representation of $2^{3\cdot}L_3(2)$ generated with respect to a basis $\{i_t \mid t \in \mathbb{F}_7\}$ of a (real) vector space V by the maps $\alpha : t \mapsto t+1$, $\beta : t \mapsto 2t$ and the involution $\gamma = (i_2, -i_2)(i_4, -i_4)(i_3, i_5)(i_6, i_0)$. It is clear that these maps preserve the set of lines $\{t, t+1, t+3\}$ of the projective plane of order 2, and it is easy to check that the kernel of the action is precisely the group 2^3 of sign-changes on the complements of the lines. Thus they generate a group of shape $2^{3\cdot}L_3(2)$.

Now construct the exterior square of this representation, on the basis

$$\{u_t = i_{t+2} \wedge i_{t+6}, v_t = i_{t+4} \wedge i_{t+5}, w_t = i_{t+1} \wedge i_{t+3}\}.$$

The three generators given above act as follows (where x stands for an arbitrary one of u, v, w):

$$\begin{array}{rcl} \alpha & : & x_t \mapsto x_{t+1} \\ \beta & : & u_t \mapsto v_{2t} \mapsto w_{4t} \mapsto u_t \\ \gamma & : & u_1 \leftrightarrow v_1, x_2 \leftrightarrow -x_2, u_4 \leftrightarrow -w_4, \\ & & u_3 \leftrightarrow v_5, v_3 \leftrightarrow w_5, w_3 \leftrightarrow u_5, \\ & & u_6 \leftrightarrow w_0, v_6 \leftrightarrow v_0, w_6 \leftrightarrow u_0 \end{array}$$

There is a submodule spanned by the $u_t+v_t+w_t$, isomorphic to the original 7-dimensional module spanned by the i_t. Factoring this out leaves the 14-dimensional module we require. We use the same notation, now with the understanding that $u_t + v_t + w_t = 0$. Write L_t for the 2-space spanned by u_t, v_t, and w_t, and let $L = \bigoplus_{t\in\mathbb{F}_7} L_t$.

Next consider what products on this 14-space are invariant under the group $2^{3\cdot}L_3(2)$. Invariance under the normal 2^3 implies that the product of a vector in L_r with one in L_s lies in L_t, where $\{r, s, t\}$ is a line in the projective plane. Invariance under α means we only need to consider the products on one line, say the line $\{1, 2, 4\}$. Modulo the sign-changes on the L_t, the stabilizer of this line is a group S_4, generated by β, γ, and $\gamma^{\alpha^{-1}}$.

Invariance under γ implies that $[u_1 + v_1, x_2]$ is a scalar multiple of $u_4 + w_4$, and that $[u_1 - v_1, x_2]$ is a scalar multiple of $u_4 - w_4$. So we may assume that $[u_1, v_2] = \lambda w_4 + \mu u_4$ and $[v_1, v_2] = \mu w_4 + \lambda u_4$, and scale so that $\lambda + \mu = 1$. Applying the symmetry γ^β gives the values of $[u_1, w_2]$ and $[v_1, w_2]$, and the multiplication table can be filled in using the relations $u_t + v_t + w_t = 0$:

	u_2	v_2	w_2
u_1	v_4	$\lambda w_4 + \mu u_4$	$\lambda u_4 + \mu w_4$
v_1	v_4	$\lambda u_4 + \mu w_4$	$\lambda w_4 + \mu u_4$
w_1	$-2v_4$	v_4	v_4

This gives us a 1-parameter family of (non-associative) algebras invariant under $2^{3\cdot}L_3(2)$. The symmetry $\gamma^{\beta^{-1}}$ shows that the multiplication is anti-commutative, and the symmetry γ shows that the multiplication is zero on L_2, and therefore on each L_t.

LEMMA 1. *The multiplication given above satisfies the Jacobi identity if and only if* $\mu = 0$ *(equivalently,* $\lambda = 1$*).*

Proof. The Jacobi identity $[[a,b],c]+[[b,c],a]+[[c,a],b]=0$ holds trivially if the three vectors a,b,c are respectively in L_r, L_s and L_t where $\{r,s,t\}$ is a line. Now the symmetries show that it is necessary and sufficient for the Jacobi identity to hold for the triples (u_1,v_1,v_2), (u_3,u_6,v_5), and (u_3,v_6,u_5). In the first case we have

$$\begin{aligned}[[u_1,v_1],v_2]+[[v_1,v_2],u_1]+[[v_2,u_1],v_1] &= 0+[\lambda u_4+\mu w_4,u_1]-[\lambda w_4+\mu u_4,v_1]\\ &= 0\end{aligned}$$

since $[u_4,u_1]=[w_4,v_1]$ and $[w_4,u_1]=[u_4,v_1]$. So this case of the identity holds whatever the values of λ and μ. In the second case we have

$$\begin{aligned}[[u_3,u_6],v_5]+[[u_6,v_5],u_3]+[[v_5,u_3],u_6] &= -[\lambda w_4+\mu v_4,v_5]-[v_1,u_3]+2[w_2,u_6]\\ &= -\lambda v_0-\lambda\mu u_0-\mu^2 w_0-v_0+2v_0\\ &= \mu(v_0-\lambda u_0-\mu w_0)\end{aligned}$$

But the expression in brackets is never zero, so $\mu=0$, $\lambda=1$, and the multiplication table is determined. We can now verify the third case of the Jacobi identity as follows:

$$\begin{aligned}[[u_3,v_6],u_5]+[[v_6,u_5],u_3]+[[u_5,u_3],v_6] &= -[u_4,u_5]-[w_1,u_3]-[w_2,v_6]\\ &= -v_0-u_0-w_0=0\end{aligned}$$

□

COROLLARY 2. *Up to scalar multiplication, there is a unique* 14*-dimensional Lie algebra invariant under the given action of* $2^{3\cdot}L_3(2)$*.*

As usual, we denote by $\operatorname{ad} x$ the linear map $y\mapsto[x,y]$ on L, and define the Killing form by $(x,y):=\operatorname{Tr}(\operatorname{ad}x.\operatorname{ad}y)$. It is obvious that the Killing form is a symmetric bilinear form.

LEMMA 3. *The Killing form on L is negative definite.*

Proof. First observe that if $x\in L_0$ then $\operatorname{ad}x$ maps L_t into $L_{\pi(t)}$ where π is the permutation $(1,3)(2,6)(4,5)$. Similarly if $y\in L_1$ then $\operatorname{ad}y$ effects the permutation $(2,4)(3,0)(5,6)$. Therefore $\operatorname{ad}x.\operatorname{ad}y$ maps every L_t into a different L_t, so has trace 0. Hence the L_t are mutually orthogonal with respect to the Killing form.

Now we can calculate the Killing form on L_0 using the following two rows of the multiplication table of the algebra:

	v_1	w_1	w_2	u_2	w_3	u_3	u_4	v_4	v_5	w_5	u_6	v_6
u_0	w_3	u_3	w_6	w_6	$-v_1$	$-w_1$	w_5	v_5	$-v_4$	$-u_4$	$-v_2$	$-v_2$
v_0	u_3	w_3	u_6	v_6	$-w_1$	$-v_1$	u_5	u_5	$-w_4$	$-w_4$	$-w_2$	$-u_2$

We find that $\operatorname{Tr}(\operatorname{ad}u_0.\operatorname{ad}u_0)=-16$ and $\operatorname{Tr}(\operatorname{ad}u_0.\operatorname{ad}v_0)=8$, so that the Killing form is negative definite on each L_t. □

Theorem 4. *The Lie algebra L is the compact real form of $\mathfrak{g}_2$.*

Proof. We have shown that the Killing form is non-singular, which implies that the Lie algebra L is semisimple. It is easy to see that each L_t is a Cartan subalgebra, so that L has rank 2. The classification of complex semisimple Lie algebras shows immediately that L is of type G_2. Since the Killing form is negative definite, L is the compact real form.

Alternatively, a proof from first principles can be obtained by explicitly diagonalising $\operatorname{ad} u_0$ and $\operatorname{ad} v_0$ simultaneously (over $\mathbb{C}$). Their simultaneous eigenspaces are the root spaces, and one recovers the standard construction of the split real form. □

Since each L_t is a Cartan subalgebra, it contains a natural copy of the G_2 root system. Our spanning vectors have been chosen so that the short roots in L_t are (up to a suitable scaling factor) $\pm u_t$, $\pm v_t$ and $\pm w_t$. The long roots, similarly, are $\pm(u_t - v_t)$, $\pm(v_t - w_t)$ and $\pm(w_t - u_t)$.

Remark. The multiplication table

	u_2	v_2	w_2
u_1	v_4	w_4	u_4
v_1	v_4	u_4	w_4
w_1	$-2v_4$	v_4	v_4

can be used to define a (non-associative) algebra on a 21-space over any field of characteristic not 2, invariant under the given action of $2^{3\cdot}L_3(2)$. The 7-dimensional subspace spanned by the $u_t + v_t + w_t$ is a subalgebra (indeed, an ideal) isomorphic to the algebra of pure imaginary octonions. This subalgebra satisfies the Jacobi identity if and only if the field has characteristic 3. Moreover, its orthogonal complement is closed under multiplication if and only if the field has characteristic 3. In this case, the multiplication on the orthogonal complement also satisfies the Jacobi identity, and we recover the fact that the pure imaginary octonion algebra is a subalgebra (and an ideal) of the Lie algebra of type $\mathfrak{g}_2$ in characteristic 3.