Exercices on the fourth course.

1. Answer the quizz p. 29.

2. Show that there exist entire functions of arbitrarily large order giving counterexamples to Bieberbach's claim p. 44.

3. Let f be an entire function. Let $A \geq 0$. Assume

$$\limsup_{r\to\infty} \mathrm{e}^{-r}\sqrt{r}|f|_r < \frac{\mathrm{e}^{-A}}{\sqrt{2\pi}}\cdot$$

(a) Prove that there exists $n_0 > 0$ such that, for $n \geq n_0$ and for all $z \in \mathbb{C}$ in the disc $|z| \leq A$, we have

$$|f^{(n)}(z)| < 1.$$

(b) Assume that f is a transcendental function. Deduce that the set

$$\bigl\{(n, z_0) \in \mathbb{N} \times \mathbb{C} \ \mid \ |z_0| \leq A,\ f^{(n)}(z_0) \in \mathbb{Z} \setminus \{0\}\bigr\}$$

is finite.

4. Let $(e_n)_{n\geq 1}$ be a sequence of elements in $\{1, -1\}$. Check that the function

$$f(z) = \sum_{n\geq 0} \frac{e_n}{2^n!} z^{2^n}$$

is a transcendental entire functions which satisfies

$$\limsup_{r\to\infty} \sqrt{r}\mathrm{e}^{-r}|f|_r = \frac{1}{\sqrt{2\pi}}\cdot$$

5. Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume the exponential type $\tau(f)$ satisfies

$$\tau(f) < \min\left\{1, \frac{\pi}{|s_0 - s_1|}\right\}.$$

Prove that f is a polynomial.
Prove that the assumption on $\tau(f)$ is optimal.

6. Let s_0 and s_1 be two complex numbers and f an entire function satisfying $f^{(2n+1)}(s_0) \in \mathbb{Z}$ and $f^{(2n)}(s_1) \in \mathbb{Z}$ for all sufficiently large n. Assume the exponential type $\tau(f)$ satisfies

$$\tau(f) < \min\left\{1, \frac{\pi}{2|s_0 - s_1|}\right\}.$$

Prove that f is a polynomial.
Prove that the assumption on $\tau(f)$ is optimal.

7. Recall Abel's polynomials $P_0(z) = 1$,

$$P_n(z) = \frac{1}{n!} z(z-n)^{n-1} \quad (n \geq 1).$$

Let ω be the positive real number defined by $\omega e^{\omega+1} = 1$. The numerical value is $\omega = 0.278\,464\,542\ldots$

(a) For $t \in \mathbb{C}$, $|t| < \omega$ and $z \in \mathbb{C}$, check

$$e^{tz} = \sum_{n \geq 0} t^n e^{nt} P_n(z),$$

where the series in the right hand side is absolutely and uniformly convergent on any compact of $\mathbb{C}$.

`Hint.` *Let $t \in \mathbb{R}$ satisfy $0 < t < \omega$ and let $z \in \mathbb{R}$. For $n \geq 0$, define*

$$R_n(z) = e^{tz} - \sum_{k=0}^{n-1} t^k e^{kt} P_k(z).$$

Check $R_n(0) = 0$, $R_n'(z) = R_{n-1}(z-1)$, so that

$$R_n(z) = te^t \int_0^z R_{n-1}(w-1)\mathrm{d}w = (te^t)^n \int_0^z \mathrm{d}w_1 \int_1^{w_1} \mathrm{d}w_2 \cdots \int_{n-1}^{w_{n-1}} R_0(w_n - 1)\mathrm{d}w_n.$$

Deduce

$$|R_n(z)| \leq (te^t)^n \frac{(|z|+n)^n}{n!} e^{t|z|}$$

(see [Gontcharoff, 1930, p. 11-12] *and* [Whittaker, 1935, Chap. III, (8.7)]*).*

(b) Let f be an entire function of finite exponential type $< \omega$. Prove

$$f(z) = \sum_{n \geq 0} f^{(n)}(n) P_n(z),$$

where the series in the right hand side is absolutely and uniformly convergent on any compact of $\mathbb{C}$.

(c) Prove that there is no nonzero entire function f of exponential type $< \omega$ satisfying $f^{(n)}(n) = 0$ for all $n \geq 0$.

Give an example of a nonzero entire function f of finite exponential type satisfying $f^{(n)}(n) = 0$ for all $n \geq 0$.

(d) Let $t \in \mathbb{C}$ satisfy $|t| < \omega$. Set $\lambda = te^t$. Let f be an entire function of exponential type $< \omega$ which satisfies

$$f'(z) = \lambda f(z-1).$$

Prove

$$f(z) = f(0)e^{tz}.$$

References

[Gontcharoff, 1930] Gontcharoff, W. Recherches sur les dérivées successives des fonctions analytiques. Généralisation de la série d'Abel. *Ann. Sci. Éc. Norm. Supér. (3)* (1930), **47**:1–78.

[Whittaker, 1935] Whittaker, J. M. *Interpolatory function theory*, volume **33** (1935), Cambridge University Press, Cambridge.