

An Optimization-based Multilevel Algorithm for Total Variation Image Denoising

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Abstract

This paper proposes a fast multilevel method using primal relaxations for the total variation image denoising and analyzes its convergence. The basic primal relaxation is known to get stuck at a non-stationary point (nearly a local minimum) of the minimization, whose solution is known to be ‘non-smooth’ in the space of functions with bounded variation. Our idea is to use coarse level corrections, overcoming the deadlock in a basic primal relaxation scheme and achieving much improvement over relaxation. Moreover, to reach a global minimizer, further refinement of the multilevel method is needed and we propose a non-regular coarse level based on a patch-detection idea (relating to hemivariateness) to correct and improve the standard multilevel method. Both algorithmic and analytical results together with numerical experiments on both 1D and 2D images are presented.

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1 Introduction

The variational formulation has become a well-established technique for modeling a class of image processing problems [2, 7, 24, 57, 1, 71]. This paper is mainly concerned with fast solution issues for the most basic of these formulations – the regularization formulation based on the total variation (TV) minimization for image restoration due to Rudin-Osher-Fatemi [57, 17]. Much of the previous work on developing fast solvers attempts to solve the associated nonlinear partial differential equation (PDE), from the Euler-Lagrange solution. This PDE has a nonlinear and highly non-smooth coefficient which leads to convergence problems of many iterative solvers (even after extra smoothing is added [41, 55]).

As far as solving the nonlinear PDE is concerned, the existing work on the topic falls into three categories: (i) Fixed point iteration [1, 69, 72, 73, 70, 71, 56]. Once the coefficients are fixed, various iterative solver techniques have been considered [72, 73, 19, 20, 18, 47, 12, 37, 48, 61]. Further improvements are still useful. (ii) Explicit time marching scheme [57, 52] that turns the nonlinear PDE into a parabolic equation before using an explicit Euler method to march in time to (slow) convergence. (iii) Primal-dual method [24, 25, 7] that solves for both the primal and dual variables together in order to achieve faster convergence with the Newton method (and a constrained optimisation with the dual variable). One observes that all these methods require a small positive parameter β to avoid singularities in the PDE coefficient associated with flat regions of the solution and consequently are sensitive to the parameter. In a recent work, recognizing that primal relaxation alone does not work, Carter [13] tried the dual formulation that does not involve this parameter and achieved some initial success with the so-called barrier (interior point) dual relaxation method. We also note that some fundamental work [15, 35] has been done recently for the dual formulation that deserves further study. Other recent work that solves the same TV model on a single level includes the active set methods [40, 14, 41], the tube method [36] and the second-order cone programming method [32]. The interior point method is also studied in [31] for related models. This paper will focus on multilevel methods.

We propose an alternative to the PDE approach to solve the image minimization problem *directly*. The problems to overcome in this study include the treatment of the non-differentiable functional in the minimization problem, by using local minimization (and hence dimension reduction), and the stagnation problem with the primal relaxation, by using coarse levels in a multilevel scheme aided by a patch detection idea.

Below, we introduce the image problem and the associated nonlinear equations before discussing some solution methods. In Section 2, we present the primal relaxation method and the algorithmic development details in a multilevel setting. In Section 3, we give a simple convergence result for the proposed algorithms and in Section 4 consider an extension of the theory to a non-TV minimization. In Section 5, we show a complexity analysis. In Section 6, we present some supporting numerical results for the denoising case. Throughout the paper, we denote by MGM either a multigrid method or a multilevel method.

The total variation model and its solution methods. Denote by $u = u(x, y)$ the true image and $z = z(x, y)$ the observed image, both defined in the bounded and rectangular domain $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. In practice, only z is available in a discrete (matrix) form [3, 33, 58]. The observed image z has been contaminated in data collection stage. The purpose of image restoration is to recover u as much as we can using a degradation model

$$Ku - z = \eta, \tag{1}$$

where η is a Gaussian white noise (unknown) and K is a known linear degradation operator; for deblurring problems K is often a convolution operator and for denoising problems $K = I$.

Image restoration is thus an inverse problem that may not have a unique solution. Some regularity condition has to be imposed on the solution space in order to turn the underlying ill-posed problem to a well posed one [71]. We shall use the well known TV regularization to ensure that sharp features of an image are preserved [57]. However we note that there are many other regularization functionals (some beyond the variational framework) that might be used; see [2, 46, 23, 10, 51, 47, 11] and the references therein.

Below we briefly review the common methodology to set the context for our algorithm in next sections. Fol-

Following early work [24], we choose the Tikhonov direct regularization technique to solve the inverse problem (1)

$$\min_u J(u), \quad J(u) = \bar{\alpha}R(u) + \frac{1}{2}\|Ku - z\|_2^2, \quad (2)$$

where the regularization functional $R(u)$ is selected as the TV-norm [57, 24]

$$R(u) = \|u\|_{TV} = \int_{\Omega} |\nabla u| dx dy = \int_{\Omega} \sqrt{u_x^2 + u_y^2} dx dy. \quad (3)$$

Here the parameter $\bar{\alpha}$ represents a tradeoff between the quality of the solution and the fit to the observed data. Thus the overall image restoration problem is the following

$$\min_u \bar{\alpha} \|u\|_{TV} + \frac{1}{2} \|Ku - z\|_2^2. \quad (4)$$

The theoretical solution to problem (4) is given by the Euler-Lagrange equation

$$\bar{\alpha} \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) - K^* K u = -K^* z, \quad (5)$$

with homogeneous Neumann boundary conditions, where K^* is the adjoint operator of K . Notice that the nonlinear coefficient may have a zero denominator so the equation is not defined at such points (corresponding to flat regions of the solution). In 1D, one uses the notation $\nabla u = \frac{du}{dx}$.

A commonly adopted idea to deal with $|\nabla u| = 0$ was to introduce (yet) another parameter β to (4) and (5) so the new Euler-Lagrange equation becomes

$$\bar{\alpha} \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \right) - K^* K u = -K^* z, \quad (6)$$

where corresponds to minimizing, instead of (4),

$$\min_u J_{\beta}(u), \quad J_{\beta}(u) = \int_{\Omega} \left[\bar{\alpha} \sqrt{u_x^2 + u_y^2 + \beta} + \frac{1}{2} (Ku - z)^2 \right] dx dy \quad (7)$$

and in theory $u = u_{\beta}(x, y)$ differs from u in (5). Observe that when $\beta = 0$ equation (6) reduces to (5); moreover $u_{\beta} \rightarrow u$ as $\beta \rightarrow 0$ as shown in [1].

The existing solution methods for solving equation (6) differ in how to deal with nonlinearities:

- **Fixed point iteration** [1, 69, 72, 73, 70, 71]:
Solve a lagged diffusion problem until $u^{k+1} - u^k$ is small

$$\bar{\alpha} \nabla \cdot \left(\frac{\nabla u^{k+1}}{\sqrt{|\nabla u^k|^2 + \beta}} \right) - K^* K u^{k+1} = -K^* z. \quad (8)$$

There exists a large literature on this topic, mainly due to wide interest in developing fast iterative solvers for the above linear equations (once discretized). When K is a convolution operator, the challenge is to solve resulting linear system without forming the discretized matrix of $K^* K$ (mimicking the capability of the fast multipole method) [72, 73, 19, 20, 18, 47]. Further improvements on robustness of these solvers are still needed.

- **Explicit time marching scheme** [57, 52]:

The original idea in [57] was refined in [52] as solving the following parabolic PDE until a steady state has been reached

$$u_t = |\nabla u| \left[\bar{\alpha} \nabla \cdot \left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \beta}} \right) - K^* K u + K^* z \right]. \quad (9)$$

As remarked in [54], for linear problems, this type of ideas represents a kind of relaxation schemes. The drawback is that the artificial time step Δt must be small due to stability requirement.

- **Primal-dual method** [24, 25, 7]:

As discussed in [7], Newton method for equation (6) leads to very slow or no convergence because z is often not a sufficiently close initial guess for u . Introducing the dual variable (vector) $\omega = \nabla u / \sqrt{|\nabla u|^2 + \beta}$ appears to have made the combined system

$$\begin{cases} \bar{\alpha} \nabla \cdot \omega - K^* K u = -K^* z, \\ \omega \sqrt{|\nabla u|^2 + \beta} - \nabla u = 0. \end{cases}$$

in two variables (u, ω) more amenable to Newton iterations as the new system is nearly “linear” in the two variables (not so linear as a single variable after elimination). Note that ω is constrained in each iteration step so the overall algorithm needs some careful implementation. However, up to now, an efficient multilevel implementation of this method remains to be developed. The same is true for the alternative primal-dual formulation [36].

In addition, there exist the powerful dual formulations [15, 35] that replace the primal variable u by its dual variable $\mathbf{p} = (p_1, p_2)$. The work of [17, 42, 55] modifies the TV formulation so that the new equations become more amenable to numerical implementation.

Remark 1 *Of the three types of methods, the recommended approach seems to be the Chan-Golub-Mulet [24] method (CGM), from our experience, although there is scope to realize the implementation more efficiently and to provide a complete theory for it. A simple observation of equation (5) reveals that there is nothing wrong with it in the sense that the equation is still “non-singular” as $\omega = \nabla u / |\nabla u|$ is always bounded if $|\nabla u| \neq 0$, although it tends towards undefined as $|\nabla u|$ gets smaller. This suggests that it may not be an appropriate action to take in introducing the parameter β dependent on the size of $|\nabla u|$ which is a free gradient otherwise. For fixed point iterations, this point is more pronounced since the ‘linearized’ PDE (8) has large coefficients near points where the solution is flat (or less interesting!), creating somewhat ‘unnecessary’ jumps in coefficients.*

In what follows we shall attempt to solve the original primal and optimization formulation (4) by a multilevel algorithm and concentrate on the denoising problem with $K = I$.

2 A multilevel scheme with piecewise constant relaxation

Our objective is to solve (4) using a multilevel method. Such a task is more challenging than solving the regularised equation (6) as previously done in [18, 69, 72, 70, 71, 26, 59, 30] where the merit functional is differentiable.

For a minimization problem, the important issue in designing a multilevel method is how to make use of an approximate solution \tilde{u} to improve it further, or how to measure the “distance” from the true minimizer so that this information can be passed onto the coarse levels somehow. (We note that for differentiable functions, first order conditions can be used to define a residual of \tilde{u} and then it is passed onto the modification of a coarse level minimization functional; see [63, 8, 49].) That is to say, here, there is no obvious way to define a residual correction functional (as done for an operator equation [27]).

We shall first discuss standard multilevel schemes to solve (4) and it turns out that such approaches, although always improving on the coordinate descent method, is not sufficient to reach the global minimizer due to non-differentiability. We then present our new multilevel algorithm before addressing convergence.

For a given image $z \in \mathbb{R}^{n \times n}$, assume $n = 2^L$ and let the standard coarsening be used giving rise to $L + 1$ levels $k = 1$ (finest), $2, \dots, L, L + 1$ (coarsest). Denote the dimension of level k by $\tau_k \times \tau_k$ with $\tau_k = n/2^{k-1}$. (For $z \in \mathbb{R}^n$, the same assumption is used in coarsening.)

2.1 Local minimization by coordinate descent methods

We shall use the coordinate descent method [6, 34, 50, 68, 39] as a smoothing method to find a solution near the true minimizer (not to find the minimizer itself). In the image context, Carter [13] appears to be the first who reported on using this approach.

Below we illustrate the well-known problem with the converged solutions to a non-smooth minimization stuck at a local non-stationary point. Then in explaining how to un-stick the relaxation, we are naturally motivated to consider the multilevel approach and give the early hint of why patch relaxation may be needed. More rigorous explanation is given in the next section.

To this end, consider the discretized form of (4) respectively in 1D and 2D:

$$\min_{u \in \mathbb{R}^n} J(u), \quad J(u) = \alpha \sum_{j=1}^{n-1} |u_j - u_{j+1}| + \frac{1}{2} \sum_{j=1}^n (u_j - z_j)^2 \quad (10)$$

and

$$\min_{u \in \mathbb{R}^{n \times n}} J(u), \quad J(u) = \alpha \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sqrt{(u_{i,j} - u_{i,j+1})^2 + (u_{i,j} - u_{i+1,j})^2} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (u_{i,j} - z_{i,j})^2, \quad (11)$$

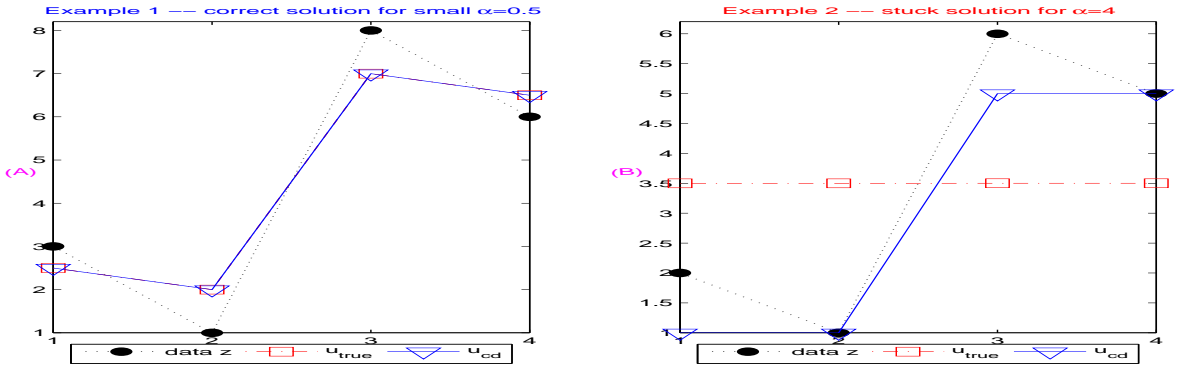
with $\alpha = \bar{\alpha}/h$ and $h = 1/(n-1)$. For the purpose of illustration and motivation of ideas, we shall mainly use the 1D problem (10).

The coordinate descent method may be used to solve (10) as follows:

$$\begin{aligned} \text{Given } & u^{(0)} = [u_1^{(0)}, \dots, u_n^{(0)}] \text{ with } k = 0, \\ \text{Solve } & u_j^{(k)} = \operatorname{argmin}_{u_j \in \mathbb{R}} J^{\text{loc}}(u_j) \text{ for } j = 1, 2, \dots, n \\ \text{Set } & u^{(k+1)} = [u_1^{(k)}, \dots, u_n^{(k)}] \text{ and repeat the above step with } k = k + 1 \\ & \text{until a prescribed stopping step on } k, \end{aligned} \quad (12)$$

where $J^{\text{loc}}(u_j) = \alpha|u_{j-1}^{(k)} - u_j| + \alpha|u_j - u_{j+1}^{(k)}| + \frac{1}{2}(u_j - z_j)^2$ for $1 < j < n$ with $J^{\text{loc}}(u_1) = \alpha|u_1 - u_2^{(k)}| + \frac{1}{2}(u_1 - z_1)^2$ and $J^{\text{loc}}(u_n) = \alpha|u_{n-1}^{(k)} - u_n| + \frac{1}{2}(u_n - z_n)^2$ due to Neumann's condition for the continuous variable u . Here problem (12) can be solved analytically because the local solution u_j is taken from the set $\{z_j - 2\alpha, z_j + 2\alpha, u_{j-1}, u_{j+1}, z_j\}$. Similar results on exact 1D solutions can be found in [9, 60, 74]. Figure 1 shows two examples from using the coordinate descent method where it works fine for the first and not for the second as previously observed in [13] and known to some readers.

Figure 1: Two examples to illustrate the coordinate descent (cd) method: the first ($\alpha = 1/2$) solution is correct while the second ($\alpha = 4$) solution gets stuck at the wrong position. Here the true solution u_{true} of (10) is denoted by \square , the cd solution u_{cd} by ∇ while the image data z is denoted by \bullet .



2.2 Standard multilevel methods

To develop a multilevel method to improve on the coordinate descent method, we may interpret solving (12) as looking for the best scalar constant $u_j^{(k)}$ that minimizes the local merit functional $J^{\text{loc}}(u_j)$ or equally as finding the

best correction constant c that makes $u_j^{(new)} = u_j^{(old)} + c$ the minimizer of $J^{\text{loc}}(u_j)$. These local correction constants may be associated with a piecewise constant function for the finest level. Similarly, we may search for the best piecewise constant corrections on all coarser levels.

Suppose \tilde{u} is our current approximation for the minimizer u of (10) on the finest level 1. Denote a level k constant vector (for $k = 1, 2, \dots, L+1$) by

$$c^{(k)} = [c_1 \ c_2 \ \dots \ c_{\tau_k}]^T. \quad (13)$$

Then define implicitly the interpolation matrix $P_k : \mathbb{R}^{\tau_k} \rightarrow \mathbb{R}^n$ from level k to level 1 from

$$c = P_k c^{(k)} = \underbrace{[c_1 \ \dots \ c_1]}_{\text{block 1}} \underbrace{[c_2 \ \dots \ c_2]}_{\text{block 2}} \dots \dots \underbrace{[c_{\tau_k} \ \dots \ c_{\tau_k}]}_{\text{block } \tau_k}, \quad (14)$$

where each block is of size $b = n/\tau_k = 2^{k-1}$. Thus on each level $k = 1, \dots, L+1$, we propose to solve the correction problem

$$\hat{c}^{(k)} = \underset{c^{(k)}}{\text{argmin}} J(\tilde{u} + P_k c^{(k)}), \quad \hat{c} = P_k \hat{c}^{(k)} \in \mathbb{R}^n, \quad c^{(k)} \in \mathbb{R}^{\tau_k} \quad (15)$$

by a coordinate descent method and then set $\tilde{u} = \tilde{u} + \hat{c}$. This will define a multilevel method which we shall name as the standard multilevel method. Note that there are only k unknowns in problem (15). Here on the coarsest level with $\tau_{L+1} = 1$, \hat{c} is just a single constant while on the finest level with $\tau_1 = n$, \hat{c} is a vector containing n unknowns.

To solve (15) efficiently, it is of interest to simplify its functional. By direct calculation,

$$\begin{aligned} J(\tilde{u} + c) &= \alpha \left[|\tilde{u}_1 + c_1 - (\tilde{u}_2 + c_1)| + \dots + |\tilde{u}_b + c_1 - (\tilde{u}_{b+1} + c_2)| + \right. \\ &\quad |\tilde{u}_{b+1} + c_2 - (\tilde{u}_{b+2} + c_2)| + \dots + |\tilde{u}_{2b} + c_2 - (\tilde{u}_{2b+1} + c_3)| + \\ &\quad \dots + \\ &\quad \left. |\tilde{u}_{n-b+1} + c_{\tau_k} - (\tilde{u}_{n-b+2} + c_{\tau_k})| + \dots + |\tilde{u}_{n-1} + c_{\tau_k} - (\tilde{u}_n + c_{\tau_k})| \right] \\ &+ \frac{1}{2} \sum_{\ell=1}^{\tau_k} \sum_{j=(\ell-1)b+1}^{\ell b} (\tilde{u}_j + c_j - z_j)^2 \\ &= \alpha \sum_{j=1}^{\tau_k-1} |\tilde{u}_{jb} - \tilde{u}_{jb+1} + c_j - c_{j+1}| + \frac{1}{2} \sum_{\ell=1}^{\tau_k} \sum_{j=(\ell-1)b+1}^{\ell b} (\tilde{u}_j + c_\ell - z_j)^2 + J_0(\tilde{u}) \\ &= \alpha \sum_{j=1}^{\tau_k-1} |\bar{c}_j - \bar{c}_{j+1}| + \frac{b}{2} \sum_{\ell=1}^{\tau_k} (c_\ell - \bar{w}_\ell)^2 + J_0(\tilde{u}) + J_1(\tilde{u}) \end{aligned} \quad (16)$$

$$= b \left\{ \underbrace{\frac{\alpha}{b} \sum_{j=1}^{\tau_k-1} |\bar{c}_j - \bar{c}_{j+1}| + \frac{1}{2} \sum_{j=1}^{\tau_k} (\bar{c}_j - \bar{z}_j)^2}_{\text{Coarse level equation with a different "}\alpha\text{"}} + J_0(\tilde{u})/b + J_1(\tilde{u})/b \right\} \quad (17)$$

where we separated the terms associated with c from the other $J_0(\tilde{u})$ and $J_1(\tilde{u})$ terms not involving c . In particular, $\bar{d}_{j+1} = \tilde{u}_{jb} - \tilde{u}_{jb+1}$ (with $\bar{d}_1 = 0$) denotes the j^{th} jump between two adjacent blocks and

$$d_j = \sum_{\ell=1}^j \bar{d}_\ell, \quad \bar{c}_j = c_j - d_j, \quad \bar{z}_j = \bar{w}_j - d_j, \quad \text{for } j = 1, 2, \dots, \tau_k. \quad (18)$$

Here with $\bar{z}_j = z_j - \tilde{u}_j$ for $j = 1, 2, \dots, n$, the mean ‘ z ’ value for each block is

$$\bar{w}_\ell = \sum_{j=(\ell-1)b+1}^{\ell b} \tilde{z}_j/b \quad (19)$$

and the equation (16) is obtained from using the simple equality of the type

$$\sum_{j=1}^b (v - z_j)^2 = bv^2 - 2\left(\sum_{j=1}^b z_j\right)v + \sum_{j=1}^b z_j^2 = b\left(v - \bar{z}\right)^2 + \sum_{j=1}^b z_j^2 - b\bar{z}^2,$$

where $\bar{z} = \sum_{j=1}^b z_j/b$.

Thus from (17), we see that minimizing the finest level correction equation associated with level k is equivalent to solving the following level k correction equation

$$\min_{\bar{c} \in \mathbb{R}^{\tau_k}} \frac{\alpha}{b} \sum_{j=1}^{\tau_k-1} |\bar{c}_j - \bar{c}_{j+1}| + \frac{1}{2} \sum_{j=1}^{\tau_k} (\bar{c}_j - \bar{z}_j)^2 \quad (20)$$

whose solution defines the correction vector $c = \bar{c} + d$. As with most multilevel methods, we shall not solve (20) exactly. Instead we apply the smoothing method of the previous subsection §2.1 and then apply the idea repeatedly with coarser levels until the coarsest level where the solution is simply $\bar{c}_1 = \bar{z}_1$ as $\tau_1 = 1$.

We now summarize the overall multilevel algorithm as follows.

Algorithm 1A Given z and an initial guess $\tilde{u} = z$, with $L + 1$ levels,

- (1) Let $u_0 = \tilde{u}$.
- (2) Smooth the approximation on the finest level 1, i.e. solve (12) for $j = 1, 2, \dots, n$.
- (3) On coarse levels $k = 2, 3, \dots, L + 1$:
 - compute $\bar{z} = z - \tilde{u}$ via (19)
 - compute the local mean $\bar{w}_\ell = \text{mean}\left(\bar{z}((\ell - 1)b + 1 : \ell b)\right)$ via (19)
 - compute $\bar{d}_j, d_j, \bar{c}_j, \bar{z}_j$ via (18)
 - solve (20) by solving local minimization as in (12) if $k \leq L$ or
 - on the coarsest level $k = L + 1$, the correction constant is simply $\bar{c} = \text{mean}(\bar{z}) = c = \text{mean}(z - \tilde{u})$.
 - Add the correction, $\tilde{u} = \tilde{u} + P_k c^{(k)}$ via (15).
- (4) If $\|\tilde{u} - u_0\|_2$ is small enough, stop or return to Step (1).

Algorithm 1A has been applied to solve a number of problems successfully; for example see Figure 2 (right plot), where on the left plot, we show how the coarse level problem is solved correctly by the coordinate descent method.

In the literature, there exists related work to our piecewise constants-based MGM as in Algorithm 1A. For convex minimization, the work of [64, 67, 66, 65, 28] suggests the use of locally-spanned spaces of piecewise polynomials to construct local minimizations. For variational inequalities, the monotone MGM work of [44, 45] suggests to use piecewise linear spaces to construct local minimizations. However the assumption of the differentiability of the merit functional is required. For image restoration, the authors of [14] established that any function from the space of bounded variation (BV) functions can be uniformly approximated by piecewise constant functions.

However, Algorithm 1A as a standard multilevel method can still get stuck just as the coordinate descent method does for some problems as illustrated in Figure 3. In attempting to find a remedy for Algorithm 1A, we observe that the incorrect multilevel solution in Figure 3 is “geometrically” correct i.e. the wrong solution is associated with flat patches and these flat patches are of the correct sizes but the values (patch heights) are incorrect. So if we add an extra coarse grid based on these patches we find that the modified MGM will converge correctly as shown in Figure 4.

From Figures 1-4, we may summarize three points on solving (10) by a coordinate descent method and the multilevel method (Algorithm 1A):

Figure 2: Example to illustrate the success of the standard MGM (Algorithm 1A) : the second ($\alpha = 4$) example from Figure 1 can be solved by the MGM to obtain the correct minimizer Δ . Here again the true solution u_{true} of (10) is denoted by \square , the cd solution u_{cd} by ∇ while the image data z is denoted by \bullet .

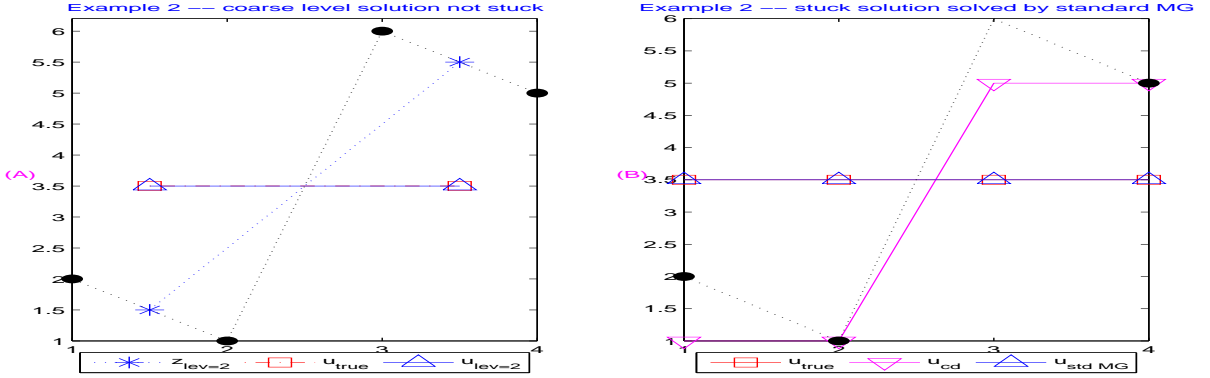
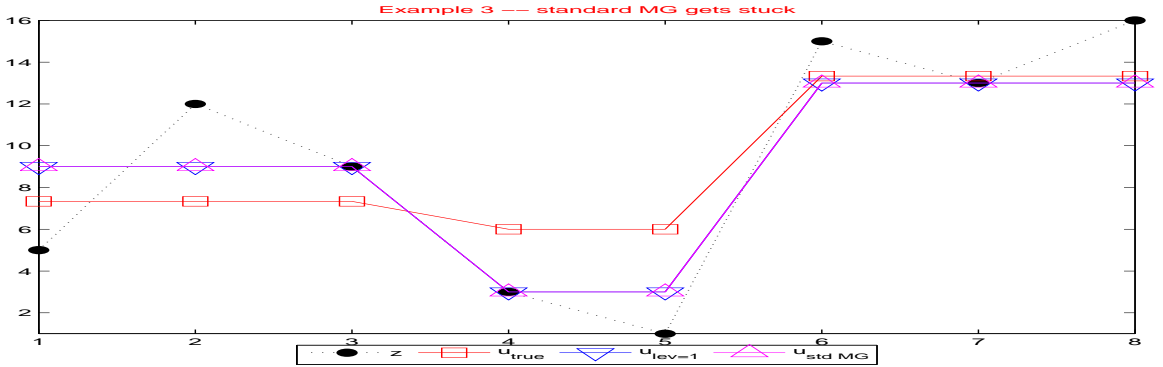


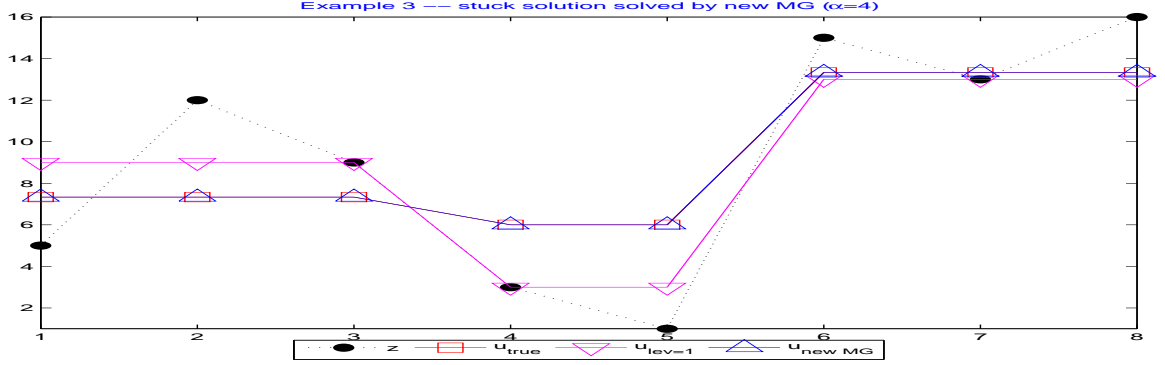
Figure 3: Example to illustrate the failure of the standard MGM (Algorithm 1A) : Here $n = 8$, $\alpha = 4$, z is shown as \bullet and the true solution as \square . Note the solution u_{stdMG} (as Δ) does not improve much from the initial ‘stuck’ solution ∇ .



1. for small α when the true solution has no patches, the coordinate descent (and also the MGM) can be quite good. See Figure 1 (A).
2. for large α , the true solution normally has patches so both the coordinate descent (cd) method and the multilevel method can get stuck easily. See Figure 1 (B). This is quite reasonable for each local minimization at a point, because the local object functional will increase if the wrong solution is set to be the correct solution at this point. To be more specific, consider node 3 in Figure 1 (B): as $z = [2 \ 1 \ 6 \ 5]$ and $\tilde{u} = [1 \ 1 \ 5 \ 5]$, the local problem is $\min_{u_3} 4|u_3 - 1| + 4|u_3 - 5| + (u_3 - 6)^2/2$ which has the local minimizer $u_3 = u_{lev=1}(3) = 5$ at which $J^{loc}(u_3) = 16.5$ while $J^{loc}(u_3^*) = 19.124$ at the true global minimizer $u_3^* = 3.5004$. Note that the TV term takes the same value of 16 at either u_3 or u_3^* . However if nodes 1, 2 and nodes 3, 4 are viewed as 2 separate constant patches (on level 2), the new minimizing problem is

$$\begin{aligned}
 & \min_{u_1, u_2} 4|u_1 - u_2| + (u_1 - 2)^2/2 + (u_1 - 1)^2/2 + (u_2 - 6)^2/2 + (u_2 - 5)^2/2 \\
 & = 2 \min_{u_1, u_2} \left[2|u_1 - u_2| + \frac{1}{2}(u_1 - 1.5)^2 + \frac{1}{2}(u_2 - 5.5)^2 \right] + \frac{1}{2} \quad (21) \\
 & \qquad \qquad \qquad \underbrace{\hspace{10em}}_{\text{coarse problem at level 2 with } \alpha=2}
 \end{aligned}$$

Figure 4: The example showing that the new Algorithm produces u_{newMG} converging to the true solution (as \square) i.e. improving on the initial ‘stuck’ solution ∇ .



which has the local minimizer $u_1 = u_{lev=2}(1) = u_{lev=2}(2) = 3.4996$ and $u_2 = u_{lev=2}(3) = u_{lev=2}(4) = 3.5004$, obtainable by local relaxation, and coincides with the global minimizer!

Thus a natural way to introduce ‘global interaction’ is to let the patches (group of points) achieve minimization together – this is exactly the idea of using coarse grids in a multilevel strategy as done in Figure 2 (A), where 2 groups move together on a coarse level to drive the solution to the correct value.

3. for large α , the patch sizes are not likely to conform with the multigrid coarsening all the time. So a standard multilevel method may not be sufficient, as shown in Figure 3 where the standard multilevel solution is stuck just as a coordinate descent solution is. We may need a special coarse grid to accommodate the varying sizes i.e. 3 patches move on a coarse level to drive the solution to the correct value in Figure 4 — this is the early indication of why our new multilevel idea will be successful.

It turns out that the above observations provided all the insights needed to overcome a stuck solution. They not only motivate our algorithm next but also can be justified theoretically (Section 3).

A new multilevel algorithm. We now consider how to implement the patch idea. The flat patch based coarsening idea is to supplement the use of a standard coarsening. Our refinement is consisted of two parts based on solving (25) i.e.

$$\min_c J(\tilde{u} + c).$$

First assume the current approximation \tilde{u} has a union of m flat patches of respective lengths b_1, b_2, \dots, b_m (with $b_j \geq 1$). Second we take an extra coarse level with m piecewise constants, as in (14) and (15), and seek to optimize these constants. Let

$$c^{<m>} = [c_1 \ c_2 \ \dots \ c_m]^T$$

and

$$c = P_m c^{<m>} = \underbrace{[c_1 \ \dots \ c_1]}_{\text{block } b_1} \ \underbrace{[c_2 \ \dots \ c_2]}_{\text{block } b_2} \ \dots \ \underbrace{[c_m \ \dots \ c_m]}_{\text{block } b_m}. \quad (22)$$

To solve $\min_c J(\tilde{u} + c)$, one can derive an equation for c similarly to before with the only change of computing the local mean using b_j instead of b . More precisely, similar to (17), we obtain

$$J(\tilde{u} + c) = \alpha \sum_{j=1}^{m-1} |\bar{c}_j - \bar{c}_{j+1}| + \frac{1}{2} \sum_{j=1}^m b_j (\bar{c}_j - \bar{z}_j)^2 + M(\tilde{u}), \quad (23)$$

where $M(\bar{u})$ is a generic term not depending on c or \bar{c} . Clearly the solution of (23) proceeds just like with (20); in particular the local minimization problem at a typical node j takes the form:

$$\min_{\bar{c}_j} J^{\text{loc}}(\bar{c}_j), \quad J^{\text{loc}}(\bar{c}_j) = \bar{\alpha}|\bar{c}_j - \bar{c}_{j-1}^{(k)}| + \bar{\alpha}|\bar{c}_j - \bar{c}_{j+1}^{(k)}| + \frac{1}{2}(\bar{c}_j - \bar{z}_j)^2, \quad (24)$$

where $\bar{\alpha} = \alpha/b_j$, which can be solved exactly as with (12).

In summary, our new piecewise constants-based multilevel method is modified from Algorithm 1A to:

Algorithm 1 Given z and an initial guess $\bar{u} = z$, with $L + 1$ levels,

- (1) Let $u_0 = \bar{u}$.
- (2) Smooth the approximation on the finest level 1, i.e. solve (12) for $j = 1, 2, \dots, n$.
- (3) On coarse levels $k = 2, 3, \dots, L + 1$:
 - compute $\bar{z} = z - \bar{u}$ via (19)
 - compute the local mean $\bar{w}_\ell = \text{mean}(\bar{z}((\ell - 1)b + 1 : \ell b))$ via (19)
 - compute $\bar{d}_j, d_j, \bar{c}_j, \bar{z}_j$ via (18)
 - solve (20) by solving local minimization as in (12) if $k \leq L$ or
 - on the coarsest level $k = L + 1$, the correction constant is simply $\bar{c} = \text{mean}(\bar{z}) = c = \text{mean}(z - \bar{u})$.
 - Add the correction, $\bar{u} = \bar{u} + P_k c^{(k)}$ via (15).
- (4) Detect flat patches in the approximation \bar{u} . Solve the patch level minimization problem (23) via (24). Add the patch correction: $\bar{u} = \bar{u} + P_k c^{<m>}$ via (22).
- (5) If $\|\bar{u} - u_0\|_2$ is small enough, stop or return to Step (1).

As shown in Figure 4, Algorithm 1 has much improved on Algorithm 1A in practice. In fact, there is a fundamental reason why the new Algorithm 1 is better than the standard multigrid method; see Lemmas 1-2.

2.3 The 2D algorithm

Assume that $\bar{u} \in \mathbb{R}^{n \times n}$ is our current approximation to (11). We wish to find the best piecewise constant function $C \in \mathbb{R}^{n \times n}$ so that it is the solution of the following

$$\min_c J(\bar{u} + c). \quad (25)$$

Such a problem is equivalent to the original problem (11). Instead, we propose to solve it on coarse levels:

$$\hat{c} = \text{argmin}_{c \in \mathbb{R}^{\tau_k \times \tau_k}} J(\bar{u} + P_k c), \quad C = P_k \hat{c}, \quad (26)$$

where $P_k : \mathbb{R}^{\tau_k \times \tau_k} \rightarrow \mathbb{R}^{n \times n}$ is the interpolation operator so $C \in \mathbb{R}^{n \times n}$. That is to say, we seek a sequence of corrections of the form $C_k = P_k \hat{c}$ (with \hat{c} a constant matrix of c 's):

$$C_k = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & c & \cdots & c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & c & \cdots & c & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \quad \text{to approximate} \quad \begin{bmatrix} c_{1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & c_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i,1} & \cdots & c_{i,i} & \cdots & c_{i,j} & \cdots & c_{i,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{j,1} & \cdots & c_{j,i} & \cdots & c_{j,j} & \cdots & c_{j,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n,1} & \cdots & \cdots & \cdots & \cdots & \cdots & c_{n,n} \end{bmatrix}.$$

Below we discuss how to solve (26). As our methodology does not depend on the dimension, describing our multilevel algorithm in 2D is straightforward. However for efficient implementation, we show some details of simplifying the main formulae.

2D local minimization. Consider the minimization of (11) by the coordinate direction method as in (12):

$$\begin{aligned}
\text{Given } & u^{(0)} = (u_{i,j}^{(0)}) \text{ with } k = 0, \\
\text{Solve } & u_{i,j}^{(k)} = \operatorname{argmin}_{u_{i,j} \in \mathbb{R}} J^{\text{loc}}(u_{i,j}) \text{ for } i, j = 1, 2, \dots, n \\
\text{Set } & u^{(k+1)} = (u_{i,j}^{(k+1)}) \text{ and repeat the above step with } k = k + 1 \\
& \text{until a prescribed stopping step on } k,
\end{aligned} \tag{27}$$

where

$$J^{\text{loc}}(u_{i,j}) = \alpha \left[\sqrt{(u_{i,j} - u_{i+1,j}^{(k)})^2 + (u_{i,j} - u_{i,j+1}^{(k)})^2} + \sqrt{(u_{i,j} - u_{i-1,j}^{(k)})^2 + (u_{i-1,j}^{(k)} - u_{i-1,j+1}^{(k)})^2} + \sqrt{(u_{i,j} - u_{i,j-1}^{(k)})^2 + (u_{i,j-1}^{(k)} - u_{i+1,j-1}^{(k)})^2} \right] + \frac{1}{2}(u_{i,j} - z_{i,j})^2,$$

where, due to Neumann's condition for the continuous variable u , all difference terms involving indices in subscripts larger than n are set to zero (representing first derivatives in the first sum at image boundaries) e.g. when $i = 2, j = n$, the first square root term in $J^{\text{loc}}(u_{2,n})$ becomes

$$\sqrt{(u_{2,n} - u_{3,n}^{(k)})^2 + (u_{2,n} - u_{2,n+1}^{(k)})^2} = |u_{2,n} - u_{3,n}^{(k)}|.$$

Although problem (27) does not have an analytical solution, our new method is not sensitive to the choice of elementary iterative schemes for solving (27); two simple methods are shown in Appendix 1.

It should be remarked that earlier work on formulation and experiments of (12) and (27) can be found in Carter [13], where a coordinate descent method was extensively used. After a few sweeps, the resulting solution u is smooth in some sense but, unfortunately, the method will converge (quickly) to the wrong solution as the number of steps increases. This is known as the local minimizer getting ‘stuck’; see [13] and refer also to [39] for a coordinate descent method applied to a different image restoration model. It is in fact well-known [68] that *for a general non-differentiable function, even if it is convex, the coordinate descent method may get stuck at a non-stationary point*. We also refer to other studies on nonlinear Gauss-Seidel type methods [6, 34, 68, 28, 21, 22, 4].

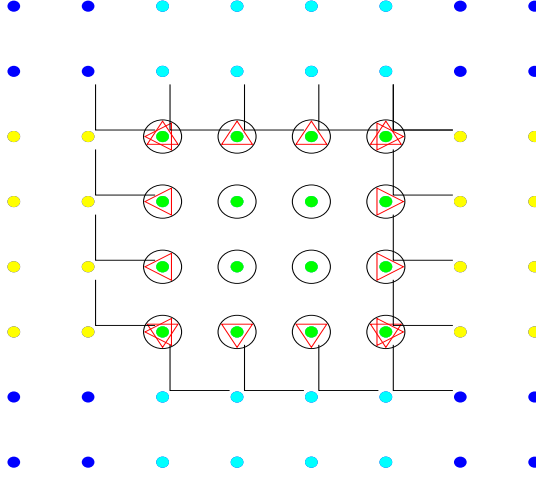
The local minimization on a general level k . On level k , set $b = 2^{k-1}$, $k_1 = (i-1)b + 1$, $k_2 = ib$, $\ell_1 = (j-1)b + 1$, $\ell_2 = jb$. Let $c = (c_{i,j})$. Then the $(i, j)^{\text{th}}$ computational block (stencil) involving $c_{i,j}$ on level k can be depicted in terms of pixel updates on level 1 as follows

$$\begin{array}{c|ccc|c}
\vdots & \vdots & \dots & \vdots & \vdots \\
\frac{\widetilde{u}_{k_1-1, \ell_2+1} + c_{i-1, j+1}}{\widetilde{u}_{k_1-1, \ell_2} + c_{i-1, j}} & \frac{\widetilde{u}_{k_1, \ell_2+1} + c_{i, j+1}}{\widetilde{u}_{k_1, \ell_2} + c_{i, j}} & \dots & \frac{\widetilde{u}_{k_2, \ell_2+1} + c_{i, j+1}}{\widetilde{u}_{k_2, \ell_2} + c_{i, j}} & \frac{\widetilde{u}_{k_2+1, \ell_2+1} + c_{i+1, j+1}}{\widetilde{u}_{k_2+1, \ell_2} + c_{i+1, j}} \\
\dots & \vdots & \dots & \vdots & \dots \\
\frac{\widetilde{u}_{k_1-1, \ell_1} + c_{i-1, j}}{\widetilde{u}_{k_1-1, \ell_1-1} + c_{i-1, j-1}} & \frac{\widetilde{u}_{k_1, \ell_1} + c_{i, j}}{\widetilde{u}_{k_1, \ell_1-1} + c_{i, j-1}} & \dots & \frac{\widetilde{u}_{k_2, \ell_1} + c_{i, j}}{\widetilde{u}_{k_2, \ell_1-1} + c_{i, j-1}} & \frac{\widetilde{u}_{k_2+1, \ell_1} + c_{i+1, j}}{\widetilde{u}_{k_2+1, \ell_1-1} + c_{i+1, j-1}} \\
\vdots & \vdots & \dots & \vdots & \vdots
\end{array} \tag{28}$$

The middle block as a stencil is now illustrated for the case of $k = 3$ (i.e. $b = 2^{k-1} = 4$) in Figure 5, where only the boundary pixels denoted by symbols $\triangleleft, \triangle, \triangleright, \Delta$ involve $c_{i,j}$ in their TV terms. Here the ‘interior’ pixels do not involve $c_{i,j}$ because the finite differences are not changed at interior pixels by the same constant update as in

$$\sqrt{[(c_{i,j} + \widetilde{u}_{k,\ell}) - (c_{i,j} + \widetilde{u}_{k+1,\ell})]^2 + [(c_{i,j} + \widetilde{u}_{k,\ell}) - (c_{i,j} + \widetilde{u}_{k,\ell+1})]^2} = \sqrt{[\widetilde{u}_{k,\ell} - \widetilde{u}_{k+1,\ell}]^2 + [\widetilde{u}_{k,\ell} - \widetilde{u}_{k,\ell+1}]^2}.$$

Figure 5: Illustration of a level $k = 3$, 4×4 , box \odot interacting with neighbouring pixels \bullet . The interior 2×2 pixels do not contribute to a constant update.



This observation can help us simplify the formulation. (In 1D, similar simplification to avoid interior nodes reduces (14) to a coarse problem involving interactions between jumps of 1D blocks.)

Then, as a local minimization for $c_{i,j}$, the problem is equivalent to minimizing the following (resulting from the four sides of the above stencil; refer to (11) and Figure 5)

$$\begin{aligned}
 F(c_{i,j}) = J^{\text{loc}}(\tilde{u}_{i,j} + c_{i,j}) &= \alpha \sum_{\ell=\ell_1}^{\ell_2} \sqrt{[c_{i,j} - (\tilde{u}_{k_1-1,\ell} - \tilde{u}_{k_1,\ell})]^2 + (\tilde{u}_{k_1-1,\ell} - \tilde{u}_{k_1-1,\ell+1})^2} + \\
 &\quad \alpha \sum_{k=k_1}^{k_2-1} \sqrt{[c_{i,j} - (\tilde{u}_{k,\ell_2+1} - \tilde{u}_{k,\ell_2})]^2 + (\tilde{u}_{k,\ell_2} - \tilde{u}_{k+1,\ell_2})^2} + \\
 &\quad \alpha \sqrt{[c_{i,j} - (\tilde{u}_{k_2,\ell_2+1} - \tilde{u}_{k_2,\ell_2})]^2 + [c_{i,j} - (\tilde{u}_{k_2+1,\ell_2} - \tilde{u}_{k_2,\ell_2})]^2} + \\
 &\quad \alpha \sum_{\ell=\ell_1}^{\ell_2-1} \sqrt{[c_{i,j} - (\tilde{u}_{k_2+1,\ell} - \tilde{u}_{k_2,\ell})]^2 + (\tilde{u}_{k_2,\ell} - \tilde{u}_{k_2,\ell+1})^2} + \\
 &\quad \alpha \sum_{k=k_1}^{k_2} \sqrt{[c_{i,j} - (\tilde{u}_{k,\ell_1-1} - \tilde{u}_{k,\ell_1})]^2 + (\tilde{u}_{k,\ell_1-1} - \tilde{u}_{k+1,\ell_1-1})^2} + \frac{1}{2} \sum_{k=k_1}^{k_2} \sum_{\ell=\ell_1}^{\ell_2} [c_{k,\ell} - (z_{k,\ell} - \tilde{u}_{k,\ell})]^2.
 \end{aligned} \tag{29}$$

To simplify the formulation, we define

$$\begin{aligned}
 \tilde{z}_{k,\ell} &= z_{k,\ell} - \tilde{u}_{k,\ell}, & \tilde{w}_{i,j} &= \text{mean}(\tilde{z}(k_1 : k_2, \ell_1 : \ell_2)) = \frac{1}{b^2} \sum_{k=k_1}^{k_2} \sum_{\ell=\ell_1}^{\ell_2} \tilde{z}(k, \ell), \\
 \tilde{v}_{k,\ell} &= \tilde{u}_{k,\ell+1} - \tilde{u}_{k,\ell}, & \tilde{h}_{k,\ell} &= \tilde{u}_{k+1,\ell} - \tilde{u}_{k,\ell}.
 \end{aligned} \tag{30}$$

To also simplify the third term in (29), we may use the simple equality

$$\sqrt{(c-a)^2 + (c-b)^2} = \sqrt{2} \sqrt{\left(c - \frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2}.$$

Therefore we can rewrite (29) as

$$\begin{aligned}
F(c_{i,j}) &= \alpha \sum_{\ell=\ell_1}^{\ell_2} \sqrt{(c_{i,j} - h_{k_1-1,\ell})^2 + v_{k_1-1,\ell}^2} + \alpha \sum_{k=k_1}^{k_2-1} \sqrt{(c_{i,j} - v_{k,\ell_2})^2 + h_{k,\ell_2}^2} + \\
&\quad \alpha \sum_{\ell=\ell_1}^{\ell_2-1} \sqrt{(c_{i,j} - h_{k_2,\ell})^2 + v_{k_2,\ell}^2} + \alpha \sum_{k=k_1}^{k_2} \sqrt{(c_{i,j} - v_{k,\ell_1-1})^2 + v_{k,\ell_1-1}^2} + \\
&\quad \alpha \sqrt{2} \sqrt{(c_{i,j} - \bar{v}_{k_2,\ell_2})^2 + \bar{h}_{k_2,\ell_2}^2} + \frac{1}{2} \sum_{k=k_1}^{k_2} \sum_{\ell=\ell_1}^{\ell_2} (c_{k,\ell} - \tilde{z}_{k,\ell})^2 \\
&= \alpha \sum_{\ell=\ell_1}^{\ell_2} \sqrt{(c_{i,j} - h_{k_1-1,\ell})^2 + v_{k_1-1,\ell}^2} + \alpha \sum_{k=k_1}^{k_2-1} \sqrt{(c_{i,j} - v_{k,\ell_2})^2 + h_{k,\ell_2}^2} + \\
&\quad \alpha \sum_{\ell=\ell_1}^{\ell_2-1} \sqrt{(c_{i,j} - h_{k_2,\ell})^2 + v_{k_2,\ell}^2} + \alpha \sum_{k=k_1}^{k_2} \sqrt{(c_{i,j} - v_{k,\ell_1-1})^2 + v_{k,\ell_1-1}^2} + \\
&\quad \alpha \sqrt{2} \sqrt{(c_{i,j} - \bar{v}_{k_2,\ell_2})^2 + \bar{h}_{k_2,\ell_2}^2} + \frac{b^2}{2} (c_{i,j} - \tilde{w}_{i,j})^2 + F_0(\bar{u}),
\end{aligned} \tag{31}$$

where F_0 is not dependant on $c_{i,j}$ and

$$\bar{v}_{k_2,\ell_2} = \frac{v_{k_2,\ell_2} + h_{k_2,\ell_2}}{2}, \quad \bar{h}_{k_2,\ell_2} = \frac{v_{k_2,\ell_2} - h_{k_2,\ell_2}}{2}. \tag{32}$$

Further we conclude that the local minimization problem for block (i, j) on level k amounts to minimizing the functional

$$\begin{aligned}
\bar{F}(c_{i,j}) &= \alpha \left[\sum_{\ell=\ell_1}^{\ell_2} \sqrt{(c_{i,j} - h_{k_1-1,\ell})^2 + v_{k_1-1,\ell}^2} + \sum_{k=k_1}^{k_2-1} \sqrt{(c_{i,j} - v_{k,\ell_2})^2 + h_{k,\ell_2}^2} + \right. \\
&\quad \left. \sum_{\ell=\ell_1}^{\ell_2-1} \sqrt{(c_{i,j} - h_{k_2,\ell})^2 + v_{k_2,\ell}^2} + \sum_{k=k_1}^{k_2} \sqrt{(c_{i,j} - v_{k,\ell_1-1})^2 + v_{k,\ell_1-1}^2} + \right. \\
&\quad \left. \sqrt{2} \sqrt{(c_{i,j} - \bar{v}_{k_2,\ell_2})^2 + \bar{h}_{k_2,\ell_2}^2} \right] + \frac{b^2}{2} (c_{i,j} - \tilde{w}_{i,j})^2.
\end{aligned} \tag{33}$$

The solution of minimizing (33) can be sought using any iterative methods for univariable optimization; the Richardson and Newton iterations are reviewed in Appendix 1.

Formulation associated with a patch coarse level. As seen in Figure 1, Gauss-Seidel type iterations from all levels of a multilevel method converge to an approximate solution with patches emerging in the iterate. The detection of a local patch (constant) can be done by checking the relative differences of a current approximation at a pixel point with its neighbouring points. Let the general box (with general indices $k_1, \dots, k_2; \ell_1, \dots, \ell_2$) as depicted in (28) with the only difference that $k_1 \neq (i-1)b+1$, $k_2 \neq ib$, $\ell_1 \neq (j-1)b+1$, $\ell_2 \neq jb$ in general for any i, j, k as these general indices are explicitly detected. Let $b_1 = k_2 - k_1 + 1$ and $b_2 = \ell_2 - \ell_1 + 1$.

Then the local (varying) $b_1 \times b_2$ patch minimization proceeds similarly to (31) and (33), with the only essential change in replacing b^2 (the old patch size) by $b_1 b_2$ (the new patch size).

Solution on the coarsest level. On the coarse level $k = L + 1$, we look for a single constant update for the current approximation i.e. $\min_c J(\bar{u} + c)$. Clearly the TV term remains the same for any c so the functional can be written as

$$J(\bar{u} + c) = \alpha TV(\bar{u}) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\bar{u}_{i,j} + c - z_{i,j})^2$$

where $TV(\bar{u})$ denotes the first term in (11). Therefore the exact solution to the minimization with respect to c is

$$c = \text{mean}(z - \bar{u}), \quad u_{\text{new}} = \bar{u} + \text{mean}(z - \bar{u}). \tag{34}$$

The 2D multilevel algorithm. We are ready to state our proposed piecewise constants based multilevel method for solving (11).

Algorithm 2 Given z and an initial guess $\tilde{u} = z$, with $L + 1$ levels,

- (1) Let $u_0 = \tilde{u}$.
- (2) Smooth the approximation on the finest level 1, i.e. solve (27) for $j, \ell = 1, 2, \dots, n$.
- (3) On coarse levels $k = 2, 3, \dots, L + 1$:
 - compute $\tilde{z} = z - \tilde{u}$ via (30)
 - compute the local mean $\tilde{w}_{i,j} = \text{mean}\left(\tilde{z}\left((k-1)b+1 : kb, (\ell-1)b+1 : \ell b\right)\right)$ via (30)
 - compute $\tilde{v}_{k,\ell}, \tilde{h}_{k,\ell}$ via (32)
 - solve (33) by solving local minimization as in (38) if $k \leq L$ or
 - on the coarsest level $k = L + 1$, the correction constant is simply

$$c = \text{mean}(\tilde{w}) = \text{mean}(z - \tilde{u}).$$
 - Add the correction, $\tilde{u} = \tilde{u} + P_k c$ via (26).
- (4) On level $k = 1$, check the possible patch size for each position (i, j) :

This is done by comparing the neighbouring consecutive pixels (i_ℓ, j_ℓ) of pixel (i, j) i.e.

$$\text{patch} = \{(i_\ell, j_\ell) \mid |u_{i_\ell, j_\ell} - u_{i,j}| < \varepsilon\}$$

for some small ε (usually $\varepsilon = 10^{-3}$).

Implement the piecewise constant update as with Step (3).
- (5) If $\|\tilde{u} - u_0\|_2$ is small enough, stop or return to Step (1).

Although Algorithm 2 (and Algorithm 1) uses the V-cycling like pattern in a multilevel setting, the precise manner of information transfers is different as illustrated by Figure 6. It is also feasible to develop other cycling patterns e.g. the W-cycling as described in [27]. In fact, some (though small) improvements have been observed with W-cycling.

We remark that there exists related work on solving a global minimization (or a nonlinear system) by block Gauss-Seidel alternating optimization methods [6, 34, 50, 68]. Such applications are similar to the use of a fixed coarse level to some extent. However, although our methods are not the standard block Gauss-Seidel alternating optimisation methods, they may be viewed as adaptively partitioned alternating optimisation methods.

3 A convergence analysis

To prove the convergence of Algorithm 2 (and its 1D version Algorithm 1), we have to show some monotone decay of the optimisation object functional and address the issue of global minimization. It should be remarked that if one solves the regularized formulation (7) with $\beta > 0$ then the functional is differentiable (and in C^∞) so the analysis done in [44, 45, 64, 65, 66, 67] may be adapted for such a purpose. However we aim to solve the unregularised formulation (4) which has a non-differentiable functional so we shall develop a direct approach of analysis. In fact, as our experiments have shown, the standard MGM does not work for (4) anyway.

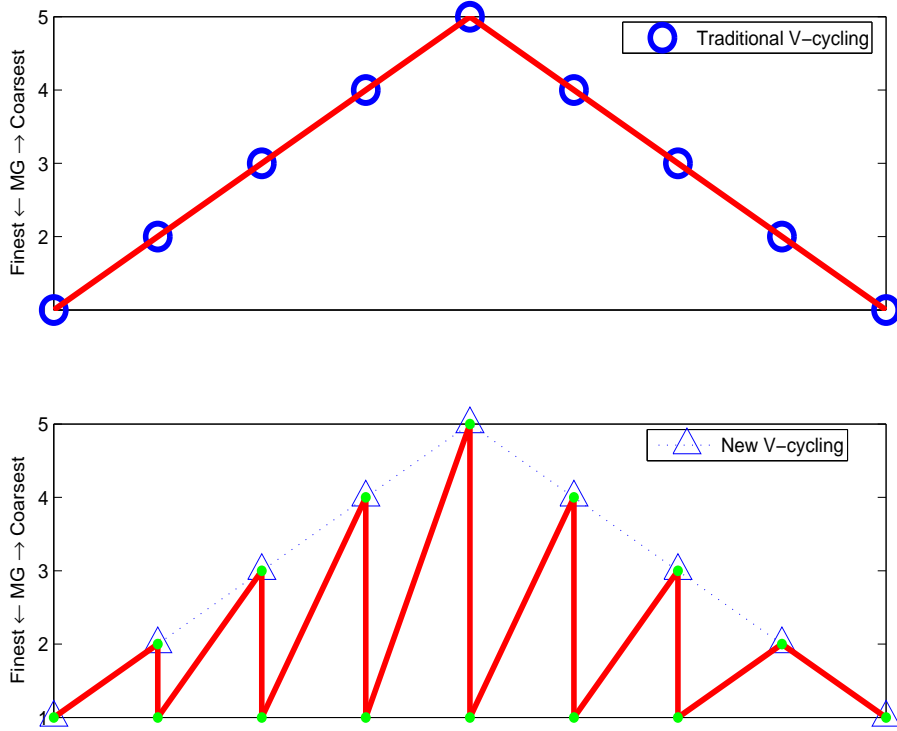
For readers' benefit, we recall some well-known convergence results first.

Theorem 1 (Known Results) Consider the problem $\min_{u \in \mathbb{R}^n} g(u)$, where $g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ solved by the descent method $x^{k+1} = x^k + \alpha_k p^k$ using a sequence of search directions $\{p^j\}$'s or if $p^j = e_j$

$$x^{k+1} = \arg \min_{\xi_j} g(x_1^k, x_2^k, \dots, \xi_j, \dots, x_n^k).$$

Let $D_o \subset D$ be a compact set.

Figure 6: V-cycling comparison. Top: the normal MGM V-cycling and Bottom: the new V-cycling. Clearly one can see two differences: (1). The new V-cycling has a simple interpolation scheme; (2). Each level of the new V-cycling interacts with the finest level directly.



1. If g is continuous and hemivariate on D_o , then every strongly downward sequence $\{x^k\}$ (similar to (35) but stronger) satisfies that $\lim_{k \rightarrow \infty} (x^k - x^{k+1}) = 0$ i.e. no such sequences will oscillate or do not converge. See [50, Ch.14]. (Note the converged point may not be a local or global minimizer.)
2. If D_o is convex and g is continuously differentiable on D and uniformly convex in D_o , then the sequence $\{x^k\}$ converges to the global minimizer of g in either of these cases:
 - (a) if all iterates $\{x^k\}$ lie in D_o , the sequence $\{p^j\}$ is free-steering (i.e. whose span covers \mathbb{R}^n) and satisfy $g(x^{k+1}) \leq g(x^k)$. See [50, Ch.14].
 - (b) if g is hemivariate in D_o . See [5, 39, 68].

It turns out that for our problems, differentiability amounts to checking and overcoming non-hemivariateness associated with our TV term which is in turn linked with our patch detection step of Algorithms 1 and 2. Without differentiability or hemivariateness, a converged solution might be a non-stationary point, which is essentially observed in Figure 1.

Remark 2 Let u_{old} be the current approximation before the k^{th} iteration associated with a local block with index I_i and u_{new} be the new approximation after the k^{th} iteration on I_i . Then we see show the minimizations from Algorithms 1 and 2 satisfy

$$J(u_{new}) \leq J(u_{old}). \quad (35)$$

Note that for continuously differentiable functionals, this decaying property (35) is sufficient to guarantee a global minimization [50] as our search directions (the coordinate unit vectors) form a free-steering sequence of nonzero vectors but our J is non-differentiable.

Now we consider the convergence of Algorithms 1 and 2. We first consider the ideal case in one dimension where no constant patches are present in the solution.

Lemma 1 *Assume the true minimizer u to the problem (10) is hemivariate i.e. $u_i \neq u_{i+1}$ for any i . Then the solution from Algorithm 1 even with $L = 0$ will be the global minimizer.*

Proof. We shall use the elementary decomposition approach. Since $u_i \neq u_{i+1}$ for any i , we may assume that $|u_i - u_{i+1}| = s_i u_i - s_i u_{i+1}$ where the scalar (sign) quantity $s_i = 1$ or -1 (but not 0) and s_i does not depend on values of u . Therefore we can rewrite the functional in (10) as

$$J(u) = \alpha \sum_{j=1}^{n-1} |u_j - u_{j+1}| + \frac{1}{2} \sum_{j=1}^n (u_j - z_j)^2 = \sum_{j=1}^n \phi_j(u_j)$$

where $\phi_j(u_j) = \alpha(s_j - s_{j-1})u_j + \frac{1}{2}(u_j - z_j)^2$ is independent of other ϕ 's with $s_0 = 0$. Furthermore we can connect the local minimization to the global minimization directly:

$$\min_u J(u) = \min_u \sum_{j=1}^n \phi_j(u_j) = \sum_{j=1}^n \min \phi_j(u_j).$$

Here each local minimization is done locally in Algorithm 1; this proves the lemma. ■

If we relax the assumption of non-existence of local patches, we can refine the above result.

Lemma 2 *Let the true minimizer u to problem (10) possess ℓ flat patches ($\ell < n$) with the k th patch I_k of the size τ_k such that $\sum \tau_k = \ell$. Assume that all such patches are correctly identified (but do not necessarily have the correct pixel values). Then the solution from Algorithm 1 will be the global minimizer.*

Proof. The proof is similar to that of the previous lemma. Let the true solution be denoted by

$$u = [u_1 \dots u_n] = [u_{\tau_1}, \dots, u_{\tau_1}; u_{\tau_2}, \dots, u_{\tau_2}; \dots; u_{\tau_\ell}, \dots, u_{\tau_\ell}].$$

Then clearly $u_{\tau_i} \neq u_{\tau_{i+1}}$ for any i . As the functional can be rewritten as

$$J(u) = \alpha \sum_{j=1}^{n-1} |u_j - u_{j+1}| + \frac{1}{2} \sum_{j=1}^n (u_j - z_j)^2 = \alpha \sum_{j=1}^{\ell-1} |u_{\tau_j} - u_{\tau_{j+1}}| + \frac{1}{2} \sum_{j=1}^{\ell} \tau_j (u_{\tau_j} - \bar{z}_j)^2 + \frac{1}{2} \sum_{j=1}^{\ell} [\bar{z}_j^2 - \tau_j \bar{z}_j^2],$$

where $\bar{z}_j = 1/\tau_j \sum_{t \in I_j} z_t$ and $\bar{z}_j^2 = \sum_{t \in I_j} z_t^2$. Then we can follow the decomposition idea used previously to show that

$$\min_u J(u) = \sum_{j=1}^{\ell} \min \phi_j(u_{\tau_j}),$$

with ϕ_j defined similarly, as done in the patch step of Algorithm 1. ■

Next we analyze Algorithm 2. The difficulty here is that we cannot find a decomposition of the underlying functional to derive a simple theory. However we can use the established theory from minimization [50] under suitable conditions.

Lemma 3 *Algorithm 2 yields a global minimizer to (11) under any one of the following conditions*

1. *the solution u to problem (11) is hemivariate i.e. $u_{i,j} \neq u_{i,j+1}, u_{i,j} \neq u_{i+1,j}$ for any (i, j) , and all iterates (in vector form) of Algorithm 2 lie in a compact and convex set $D_o \subset \mathbb{R}^{n^2}$ that contains the solution.*
2. *the solution u to problem (11) has ℓ patches ($\ell < n^2$) that can be identified, and all iterates (in vector form) from Algorithm 2 lie in a compact and convex set $D_o \subset \mathbb{R}^{\ell}$ that contains the solution.*

Proof. Here hemivariateness of u is associated with Algorithm 2 detecting no constant patches anywhere. With this assumption in part 1, the functional $J(u)$ is continuously differentiable in a neighborhood of the true solution. Note the search directions of local minimization are unit coordinate vectors so they form a free-steering sequence. Together with the decaying property (35), convergence to the global minimizer is immediate; refer to Theorem 1.

The proof of part 2 follows from part 1 in the similar way Lemma 2 following Lemma 1; our Algorithm 2 essentially merges variables on a patch into one. ■

It is of interest to point out that detecting the patches in our algorithms is not difficult. As shown by Lemma 2, indeed, Algorithm 1 can find the local minimizers (also global) accurately (even within one sweep) if there are no patches nearby. Therefore fine level local minimizations can reveal the patches quickly as illustrated in Figure 1.

4 An application to a related restoration model

A related and interesting result from Lemma 2 is stated in the following. Consider the following non-rotationally-invariant model [47, 51]:

$$\min_u J(u), \quad J(u) = \int_{\Omega} \left[\bar{\alpha}(|u_x| + |u_y|) + \frac{1}{2}(Ku - z)^2 \right] dx dy \quad (36)$$

whose discrete counterpart is

$$\min_{u \in \mathbb{R}^{n \times n}} J(u) = \min_{u \in \mathbb{R}^n} \alpha \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} |u_{i,j} - u_{i,j+1}| + |u_{i,j} - u_{i+1,j}| + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (u_{i,j} - z_{i,j})^2. \quad (37)$$

This model is known to exhibit interesting characteristics in solution quality quite opposite to the TV model [51], depending on the image.

Lemma 4 *Algorithm 2 applying to (37) converges to the global minimizer. Moreover, if there are no patches in u , Algorithm 2 can find the global minimizer in one sweep over the finest level.*

We also remark that several authors [38, 29, 16, 43] have studied very efficient solution methods of models similar to (36), via developing methods for solving Markov random fields (MRF) problems. It will be of interest to consider multilevel methods based on MRF methods [16]. See also [31] for use of interior point approaches.

Readers who are familiar with optimization techniques may wonder how our low dimensional subproblems (relaxations) are related to line search methods. Below we give such details in Appendix 2.

5 Complexity analysis

The effectiveness of any iterative solver can be measured by the total number of iterative steps (hopefully not many) it takes to yield a solution up to an acceptable accuracy, and the cost associated with a single iterative step. Our analysis below shows that the cost per step is $O(LN)$ where $L + 1$ is the number of levels employed and N is the dimension of our problems i.e. $N = n$ in 1D (Algorithm 1) and $N = n^2$ in 2D (Algorithm 2). We remark that the complexity of the robust algorithm in [32] has the complexity $O(N \sqrt{N})$.

Firstly we analyze Algorithm 1. Let $N = n = 2^{L_m}$ and $L \leq L_m$. Denote as before $b_k = 2^{k-1}$ and $\tau_k = N/b_k$. We can work out the number of floating point operations (flops) as follows:

	Quantities	Flop Counts
Level k	$d_j, \tilde{d}_j, \tilde{c}_j, \tilde{z}_j$ $\tilde{z}, \tilde{\omega}$	$4\tau_k$ $2N$
	s smoothing steps	$4s\tau_k$
All levels	$W_1 = \sum_{k=1}^{L+1} (2N + 4\tau_k + 4s\tau_k)$	

Therefore we can bound the total number of flops by

$$W_1 = \sum_{k=1}^{L+1} \left(2N + \frac{4N}{b_k} + \frac{4sN}{b_k} \right) = 2(L+1)N + 4N(s+1) \sum_{k=0}^L \frac{1}{2^k} < 2(L+1)N + 8N(s+1) \approx O(LN).$$

Secondly for Algorithm 2, the notation is only changed slightly; here $N = n^2$ and set $\tau_k = N/b_k^2$ with $b_k = 2^{k-1}$. Then the flops counts are as follows

	Quantities	Flop Counts
Level k	$h_{i,j}, v_{i,j}$ $\tilde{z}_{i,j}, \tilde{w}_{i,j}$ s smoothing steps	$4b_k\tau_k$ $2N$ $(24b_k + 1)s\tau_k$
All levels	$W_2 = \sum_{k=1}^{L+1} \left(2N + 4b_k\tau_k + (24b_k + 1)s\tau_k \right)$	

Therefore the upper bound is the following

$$W_2 = 2(L+1)N + \sum_{k=1}^{L+1} \left(\frac{4(6s+1)}{b_k} + \frac{s}{b_k^2} \right) = 2(L+1)N + \sum_{k=0}^L \left(\frac{4(6s+1)}{2^k} + \frac{s}{4^k} \right) < 2(L+1)N + (2 + 4s/3)N \approx O(LN).$$

Note that $\max(L) = \log n = O(\log N)$. This 'extra' complexity, although practically insignificant, arises from the cycling pattern as shown in the bottom plot of Figure 6 i.e. each coarse level interacts with the finest level directly and brings $O(\log N)$ work per interaction. One way to get rid of this $O(\log N)$ is to modify the cycling pattern from $k = 1, 2, \dots, L+1$ to $k = 1, L+1, L, \dots, 2$ and to compute the quantity $\tilde{w}_{i,j}$ without forming $\tilde{z}_{i,j}$ explicitly (as $\tilde{w}_{i,j}$ is only changed by a constant on each block at a new level k in the new cycling order).

6 Numerical experiments

In this section, we shall test several aspects of the proposed multilevel method. Firstly we demonstrate the convergence of the method for several test cases and show that the proposed method gives fast and quality solutions in both 1D and 2D. Secondly we engage in the challenging task of comparing it to the well-known CGM [24] to show some advantages of our MGM approaches. The latter is particularly pleasing as our MGM can give sharper solutions in addition to delivering a solution at a faster speed.

Fast convergence. We first consider four 1D denoising problems with the signal-to-noise ratio (SNR, see [24]) of 10 as shown in Figure 7. The processed solutions by our multilevel algorithm are shown in Figure 8, where one observes that the method is extremely fast (as expected of a multilevel method), converging to the tolerance of $tol = 10^{-4}$ in respectively 4, 4, 4, 4 multilevel steps; here α takes the value of 7.68×10^{-5} , 19.2, 160, 25.6 respectively for the 4 examples. We then test further four examples of 2D denoising problems as shown in Figure 9, where SNR=5. The computed solutions by our multilevel algorithm are shown in Figure 10. Again one observes that the method is fast to obtain solutions which are comparable to other methods that we tested. We next consider details of one comparison.

Comparison with CGM. As remarked, the method CGM [24] has been known to produce the most reliable solutions (in terms of speed and reduction of merit functional) when compared to the time-marching and fixed iteration methods. We shall only compare with it for suitable parameter $\beta > 0$; in particular if $\beta = 0$ (or nearly 0), CGM does not converge. We remark that there exist other methods [37, 15, 32] that have not been compared; a future evaluation of all such methods will be valuable.

As an exact solution is not available in general, any comparison must be based on other measures. What we propose to compare is the value of the minimizing function $J(u)$ as in (10) for 1D. From the Table 1, one can see that our method is quite competitive to CGM in terms of reduction of $J(u)$ and CGM depends heavily on the choice

Figure 7: Four 1D test examples

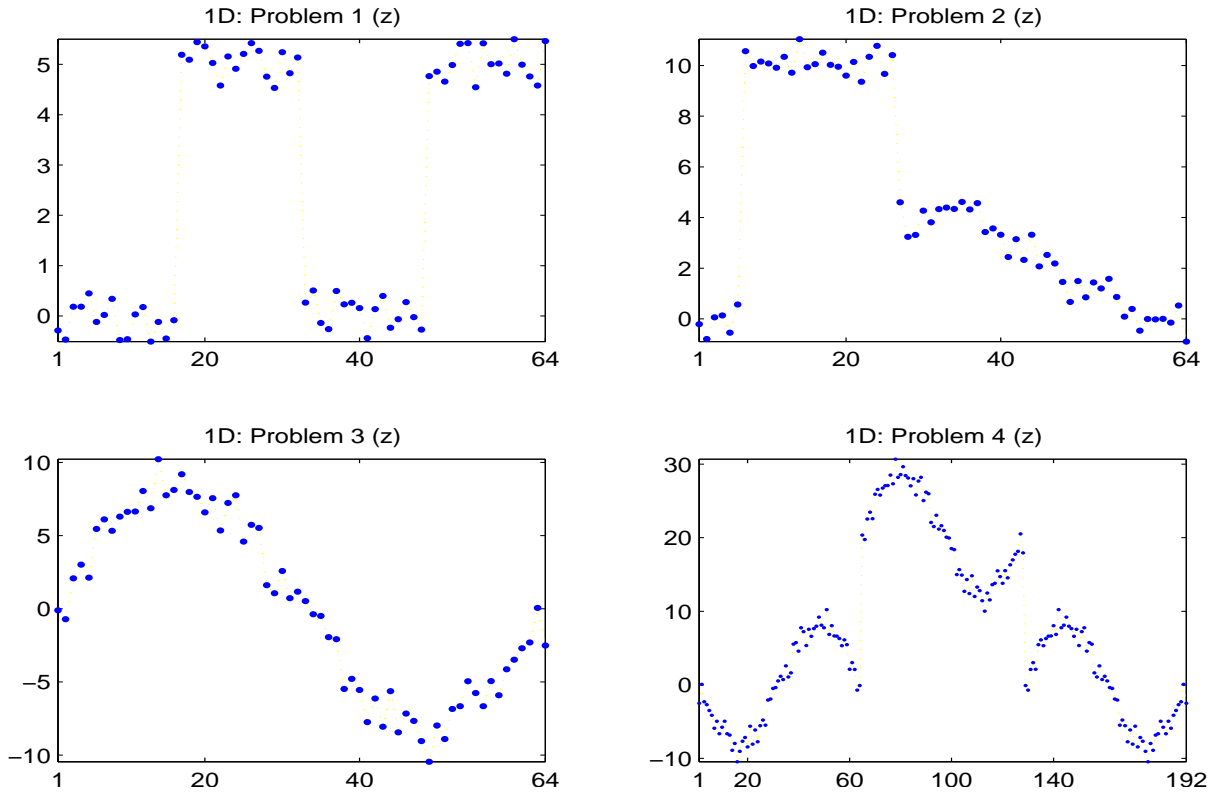


Figure 8: New MGM for the four 1D test examples

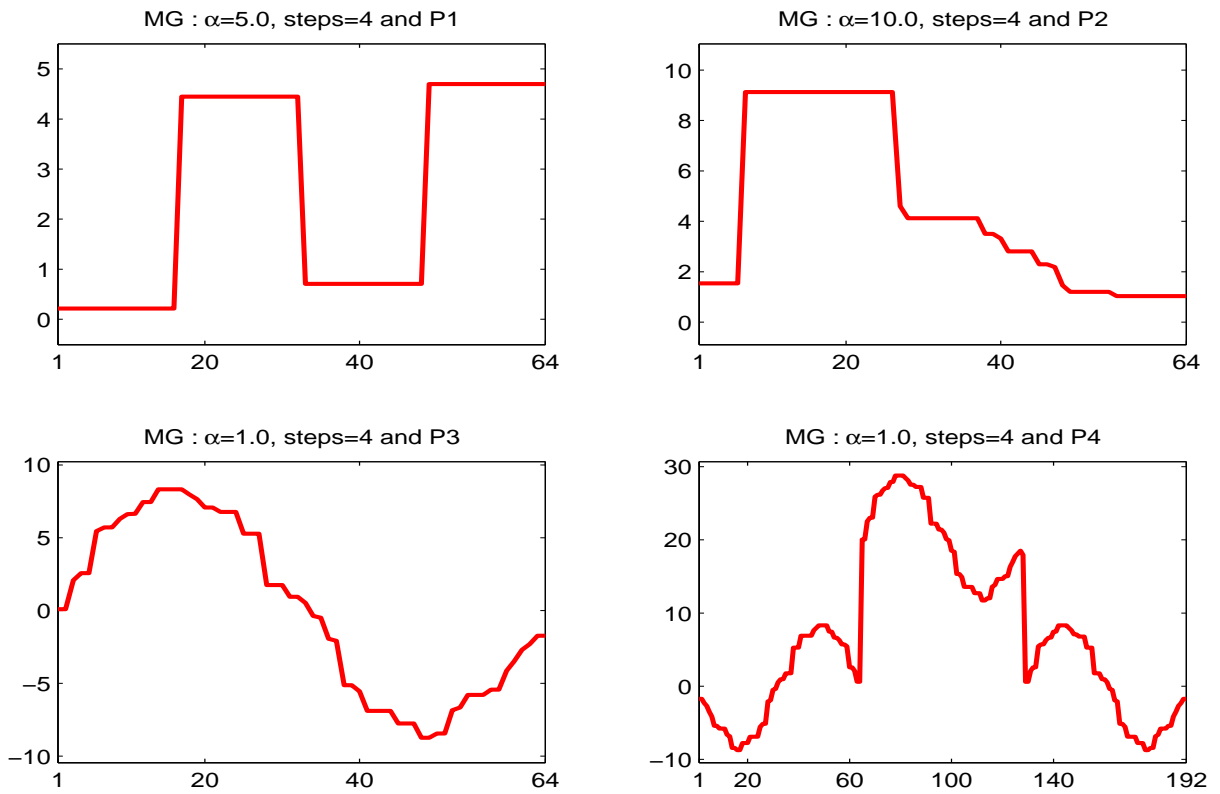


Figure 9: Four 2D test examples

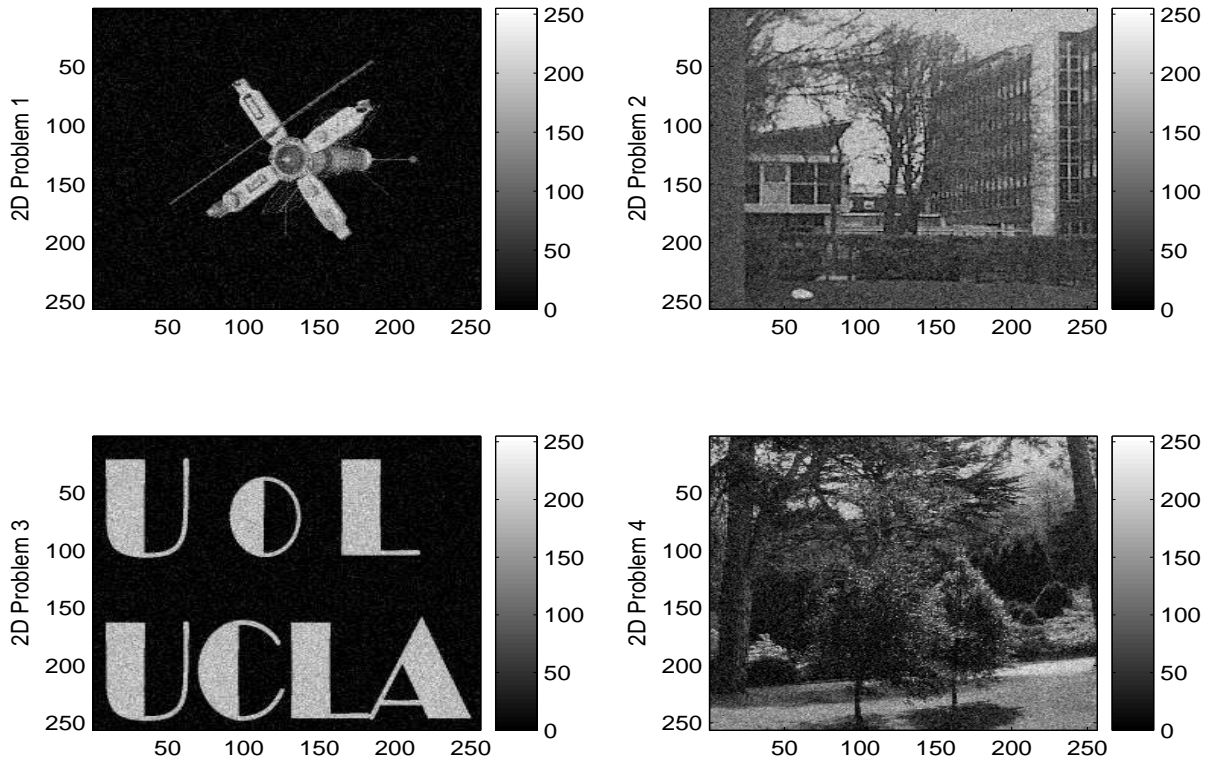


Figure 10: New multilevel solutions of the 2D test examples

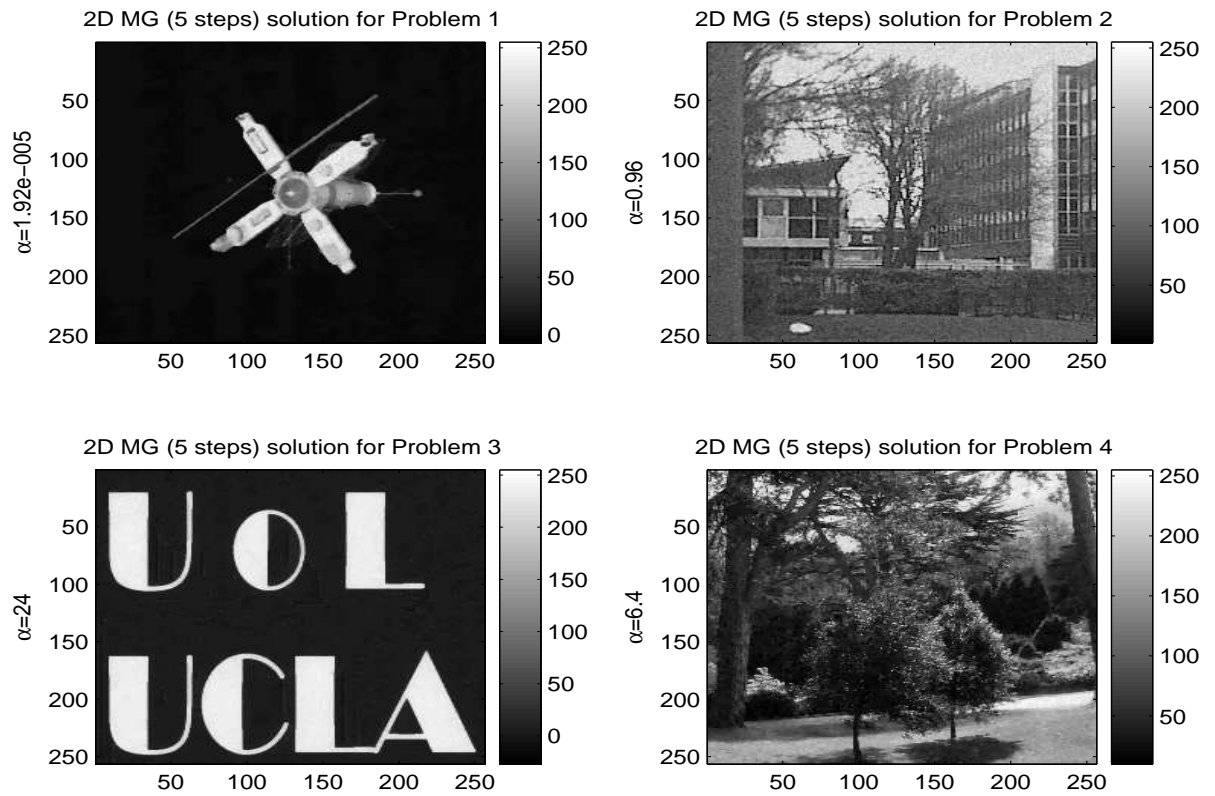


Table 1: Comparison of our new MGM with CGM [24] in 1D

Method	Problem	Convergence Steps	J(u)
CGM $\beta = 10^{-1}$	1	8	82.5160
	2	7	207.648
	3	6	34.1475
	4	6	135.143
CGM $\beta = 10^{-5}$	1	11	70.6723
	2	17	185.745
	3	11	32.8829
	4	11	131.433
CGM $\beta = 10^{-10}$	1	15	70.4687
	2	22	185.413
	3	13	32.8668
	4	11	131.389
CGM $\beta = 10^{-14}$	1	16	70.4680
	2	22	185.409
	3	13	32.8668
	4	11	131.390
MG (new)	1	2	70.4680
	2	4	185.407
	3	4	32.8664
	4	4	131.385

of β . For the 2D examples, we take the parameter $\gamma = \beta$ as we intend to compare our new multilevel solutions with the CGM. For the image size 128×128 (for all the above 4 problems), we observe the following results in Table 2. Clearly the MGM is much less sensitive to the parameter γ than the CGM and of course it may be possible to replace the 2D local minimization by some new methods e.g. [53] and then we expect the multilevel approach improves further.

Next we illustrate what the improvements over CGM solutions look like in terms of image sharpness. We shall take problem 2 (1D) as an example. With $\beta = 10^{-10}$, we have seen that $J(u_{CGM}) = 185.413$ while $J_{MGM} = 185.407$ with the data size $n = 64$ (refer to the second plot of Figure 7). However, if we zoom in two locations of the solutions, we can see an interesting (but expected) difference between CGM and MGM i.e. the multilevel solution is sharper (more like a staircase function) than the CGM. This is visible from Figure 11 near node 28 and Figure 12 near node 55. Clearly the CGM solutions have smoother solutions (though almost piecewise constants).

Finally we show some results in Table 3 comparing the speed of our method with CGM [24] and in Table 4 illustrating the mild dependence of our method on (specially large) α where solution patches dominate. Clearly as the problem size increases, our MGM performs faster and as expected for larger α , our MGM takes more MG cycles to detect the underlying patches (recall that when $\alpha \rightarrow \infty$, all TV solutions are global constants).

7 Conclusions

This paper proposed a new and effective multilevel method based on piecewise constant refinements of the minimization problem modeling the image denoising problem in the TV-norm. We have demonstrated that standard multilevel methods do not work all the time. Our new method is different from those multigrid methods applied to the PDEs and also from others which assume the minimizing functional is differentiable. The apparently non-smooth solutions from primal relaxations of local minimization are corrected using multilevels and varying size coarse level elements.

As our method is less sensitive to parameters, as a by-product (when compared to the well-known CGM [24])

Table 2: Comparison of our new MGM with CGM [24] in 2D

Method	Problem	Convergence Steps	J(u)
CGM $\beta = 10^{-1}$	1	2	1.23384E-5
	2	7	6.59000E5
	3	10	2.55551E6
	4	10	4.31471E6
MGM (new)	1	1	4.60572E-6
	2	3	6.59045E5
	3	5	2.67872E6
	4	5	4.31899E6
CGM $\beta = 10^{-5}$	1	3	1.17532E-5
	2	10	6.58996E5
	3	22	2.55007E6
	4	16	4.31415E6
MGM (new) $\beta = 10^{-5}$	1	1	4.58596E-6
	2	5	6.59045E5
	3	11	2.70140E6
	4	8	4.31898E6
CGM $\beta = 10^{-10}$	1	7	4.50816E-6
	2	15	6.58996E5
	3	40	2.55002E6
	4	21	4.31414E6
MGM (new) $\beta = 10^{-10}$	1	3	4.24560E-6
	2	7	6.59045E5
	3	20	2.68725E6
	4	10	4.31898E6

Table 3: CPU comparison of our new MGM with CGM [24] with $\beta = 10^{-12}$ and $tol = 10^{-6}$.

Problem	Size n (for $n \times n$)	CPU by CGM	CPU by MGM
2	128	6.14	33.19
	256	117.95	133.38
	512	2192.59	548.19
3	128	343.78	33.06
	256	1418.30	133.94
	512	6149.64	567.88

Table 4: Dependence of our new MGM on α for 2D problems 1-2 of size 256×256 . Here $tol = 10^{-4}$.

Problem	α used	MGM cycles	Problem	α used	MGM cycles
1	3.84×10^{-6}	1	2	9.6×10^{-2}	2
	1.92×10^{-5}	1		1.92×10^{-1}	2
	1	2		9.60×10^{-1}	2
	10	4		9.6	4
	15	8		20	12

Figure 11: Detailed comparison of CGM and MGM solutions near node 27 (Problem 2)

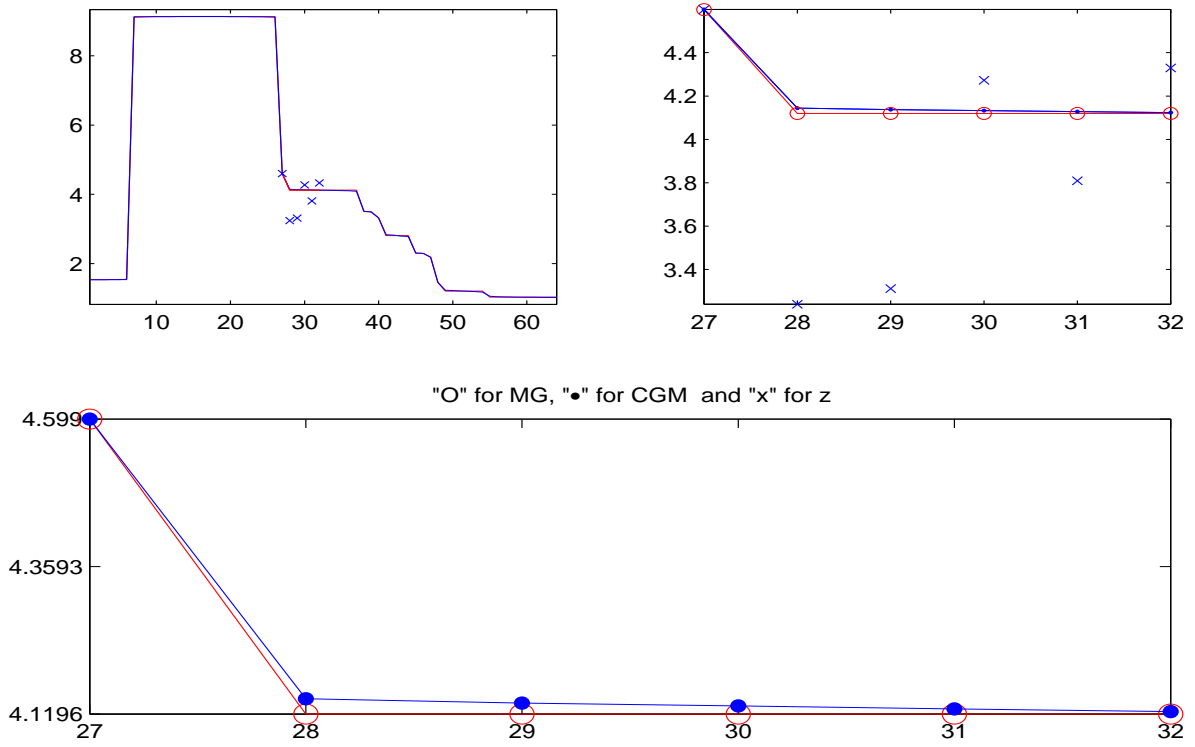
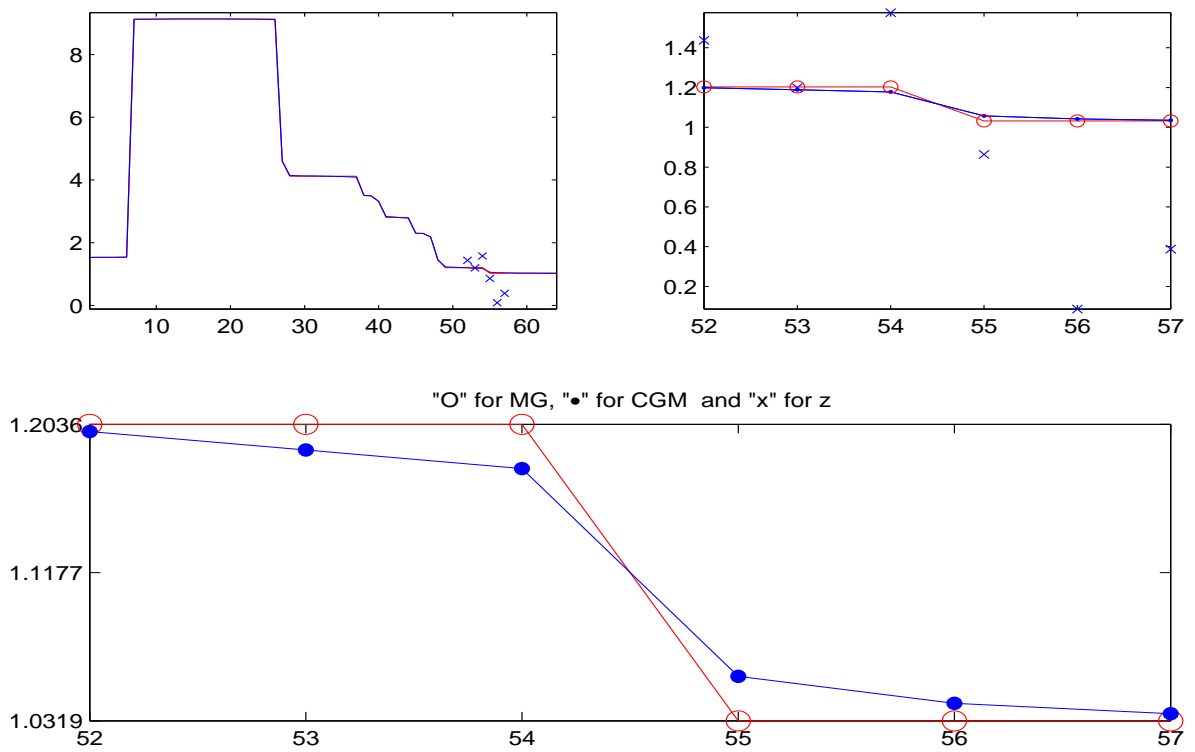


Figure 12: Detailed comparison of CGM and MGM solutions near node 55 (Problem 2)



method), there is strong evidence to suggest that sharpe solutions are obtained in many cases in the true spirit of TV norms. On the other hand, our method also works for variational models with a smooth object functional.

Appendix 1 – Local minimization formulation

We first use the Richardson iteration to solve (27) from repeating $\tilde{u} \rightarrow u \rightarrow \tilde{u}$

$$\alpha \left[\frac{2u_{i,j} - \tilde{u}_{i+1,j} - \tilde{u}_{i,j+1}}{\sqrt{(u_{i,j} - \tilde{u}_{i+1,j})^2 + (u_{i,j} - \tilde{u}_{i,j+1})^2 + \gamma}} + \frac{u_{i,j} - \tilde{u}_{i-1,j}}{\sqrt{(u_{i,j} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma}} + \frac{u_{i,j} - \tilde{u}_{i,j-1}}{\sqrt{(u_{i,j} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma}} \right] + (u_{i,j} - z_{i,j}) = 0,$$

where $\gamma > 0$ is a regularizing parameter which is much taken smaller than β above (e.g. $\gamma = 10^{-30}$ which is far too small for β in (6)). Then our Richardson iterations for solving (27) take the form

$$u_{i,j}^{new} = T^{old} / B^{old} \quad (38)$$

where

$$\begin{aligned} B^{old} &= 2\alpha \left[\sqrt{(u_{i,j}^{old} - \tilde{u}_{i+1,j})^2 + (u_{i,j}^{old} - \tilde{u}_{i,j+1})^2 + \gamma} + \right. \\ &\quad \left. + \alpha \sqrt{(u_{i,j}^{old} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma} + \right. \\ &\quad \left. + \alpha \sqrt{(u_{i,j}^{old} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma} \right] + 1, \\ T^{old} &= \alpha(\tilde{u}_{i+1,j} + \tilde{u}_{i,j+1}) \left[\sqrt{(u_{i,j}^{old} - \tilde{u}_{i+1,j})^2 + (u_{i,j}^{old} - \tilde{u}_{i,j+1})^2 + \gamma} + \right. \\ &\quad \left. + \alpha \tilde{u}_{i-1,j} \sqrt{(u_{i,j}^{old} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma} + \right. \\ &\quad \left. + \alpha \tilde{u}_{i,j-1} \sqrt{(u_{i,j}^{old} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma} \right] + z_{i,j}. \end{aligned}$$

An alternative to the Richardson iterations is the Newton method, which we found to perform similarly, as shown below:

$$u_{i,j}^{new} = u_{i,j}^{old} - T^{old} / B^{old}$$

where

$$\begin{aligned} T^{old} &= \alpha \frac{2u_{i,j}^{old} - \tilde{u}_{i+1,j} - \tilde{u}_{i,j+1}}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i+1,j})^2 + (u_{i,j}^{old} - \tilde{u}_{i,j+1})^2 + \gamma}} + \alpha \frac{u_{i,j}^{old} - \tilde{u}_{i-1,j}}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma}} \\ &\quad + \alpha \frac{u_{i,j}^{old} - \tilde{u}_{i,j-1}}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma}} + u_{i,j}^{old} - z_{i,j}, \\ B^{old} &= \alpha \frac{2}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i+1,j})^2 + (u_{i,j}^{old} - \tilde{u}_{i,j+1})^2 + \gamma}} - \alpha \frac{(2u_{i,j}^{old} - \tilde{u}_{i+1,j} - \tilde{u}_{i,j+1})^2}{\left((u_{i,j}^{old} - \tilde{u}_{i+1,j})^2 + (u_{i,j}^{old} - \tilde{u}_{i,j+1})^2 + \gamma \right)^{3/2}} \\ &\quad + \alpha \frac{1}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma}} - \alpha \frac{(u_{i,j}^{old} - \tilde{u}_{i-1,j})^2}{\left((u_{i,j}^{old} - \tilde{u}_{i-1,j})^2 + (\tilde{u}_{i-1,j} - \tilde{u}_{i-1,j+1})^2 + \gamma \right)^{3/2}} \\ &\quad + \alpha \frac{1}{\sqrt{(u_{i,j}^{old} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma}} - \alpha \frac{(u_{i,j}^{old} - \tilde{u}_{i,j-1})^2}{\left((u_{i,j}^{old} - \tilde{u}_{i,j-1})^2 + (\tilde{u}_{i,j-1} - \tilde{u}_{i+1,j-1})^2 + \gamma \right)^{3/2}} \\ &\quad + 1. \end{aligned}$$

The full Newton method is known to converge slowly [7, 13, 71, 27] when α is relatively large and γ is small. Here the point-wise Newton appears to be adequate but other optimization methods (e.g. [53]) may be considered to solve the above local minimization.

Appendix 2 — Interpretation as a line search method

We give an alternative interpretation of the proposed method which may be of interest to readers in optimization community. Firstly for Algorithm 1, the multigrid idea for solving

$$\min_{\lambda} J(u_k + \lambda \vec{s}), \quad u^{(k)} = u_k + \lambda \vec{s}$$

modifies the simple “alternating variables methods”

$$\vec{s} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{etc}$$

to take a more systematic set of search directions

$$\vec{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\text{level } 0], \quad \vec{s} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\text{level } 1], \quad \vec{s} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} [\text{level } 2], \quad \text{etc.}$$

Likewise, a patch (of first 3 nodes) may be denoted by the search direction

$$\vec{s} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad \text{etc.}$$

Secondly for 2D, we may specify

$$\vec{s} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\text{level } 0], \quad \vec{s} = \frac{1}{2} \begin{bmatrix} \vec{s}_1 \\ \vec{s}_1 \end{bmatrix}, \quad \vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\text{level } 1], \quad \vec{s} = \frac{1}{4} \begin{bmatrix} \vec{s}_2 \\ \vec{s}_2 \\ \vec{s}_2 \\ \vec{s}_2 \end{bmatrix}, \quad \vec{s}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [\text{level } 2], \quad \text{etc.}$$

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