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Compact Polynomial Mollifiers For Poisson's Equation

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Abstract

In this paper we describe a family of polynomial mollifiers of compact support with a parameterized degree of differentiability. We show that the members of this family that have unit integral and satisfy a specified number of differentiability constraints are unique. We give analytic formulas for the potentials and derivatives of the potentials that these mollifiers induce and describe two methods for obtaining high order compact polynomial mollifiers.

1 Introduction

In many scientific computing problems, one requires differentiable functions of unit integral that have compact or nearly compact support. These functions, which are widely used for smoothing or filtering by convolution, we refer to as mollifiers, but in other contexts they are known by other names, e.g. “vortex blobs” in vortex methods [4], or “kernel functions” in smooth particle hydrodynamics [6]. When incorporated into numerical procedures, there are two aspects of such functions that play an important role for obtaining accurate solutions; their differentiability and the number of vanishing moments. In addition, when these functions are used in solution procedures for Poisson’s equation it highly desirable that the potentials induced by these functions can be determined analytically.

The need for differentiability and vanishing moments arises when mollifiers are used as a means of creating continuous approximations of functions based upon discrete data. Specifically, given a discrete set of points \vec{x}_i , and function values at those points, $f_i = f(\vec{x}_i)$, a continuous approximation can be constructed of the form

$$f(\vec{x}) \approx \sum_i B_\delta(x - \vec{x}_i) f_i w_i \quad (1)$$

w_i is a weighting factor. The error in such an approximation can be estimated by adding and subtracting the continuous convolution with the mollifier,

$$\begin{aligned} & \|f(\vec{x}) - \sum_i B_\delta(x - \vec{x}_i) f_i w_i\| \leq \\ & \|f(\vec{x}) - \int f(\vec{s}) B_\delta(\vec{x} - \vec{s}) d\vec{s}\| + \left\| \int f(\vec{s}) B_\delta(\vec{x} - \vec{s}) d\vec{s} - \sum_i B_\delta(x - \vec{x}_i) f_i w_i \right\| \end{aligned} \quad (2)$$

The need for vanishing moments is due to the fact that size of the error made by approximating a function value by its’ convolution with a mollifier, the first term in (2), is related to the width of the mollifier and number of moments of the mollifier that vanish. The differentiability of the mollifier is important in the estimation of the second term of (2). If the weights w_i are chosen to be the weights of an integration formula, then this term is the error of a numerical integration approximation [1], an error whose size and rate of convergence will generally depend upon the differentiability of the integrand [2].

In this note we describe a family of polynomial mollifiers in \mathbb{R}^N that have compact support and, for members of this family with $M+1$ terms, are $M-1$ continuously differentiable in \mathbb{R}^N . The general idea behind constructing this family is not new; for example, it’s implicit in the

mollifiers used in early theoretical work on vortex methods [5] and test problems for vortex methods [7], and certainly occurs to anyone thinking about constructing smooth radially symmetric functions. An interesting result that we present is that if one restricts oneself to polynomials that are just functions of r^2 where $r = |\vec{r}|$ for $\vec{r} \in \mathbb{R}^N$, then the specification of unit integral and differentiability constraints uniquely determines the polynomial. Moreover, this polynomial has a particularly simple form. One consequence of this simple polynomial form is that analytic representations of the potentials induced by this family of polynomial mollifiers can be readily constructed and combinations of members of the family can be combined to create mollifiers whose higher moments vanish.

While the family of mollifiers is stated for \mathbb{R}^N we give normalization constants and other numerical coefficients of interest for mollifiers of one, two, and three dimensions. The coefficients given lead to mollifiers that are up to eight times continuously differentiable. For those requiring just a single member of the family, or an example of one of the family to check implementations of the formulas, we describe in B two specific mollifiers, one that is three times continuously differentiable and one that is four times continuously differentiable. We also give the formulas for the potentials and the derivatives of the potentials that are associated with these mollifiers.

2 Compact Polynomial Mollifiers

The foundation for the family of mollifiers are polynomials of $M + 1$ terms in r^2 given by

$$B(r) = \begin{cases} \frac{\sigma_{M(N)}}{\omega_N} (1 - r^2)^M & r \leq 1 \\ 0 & r > 1 \end{cases} \quad (3)$$

with $r = \sqrt{\sum_{i=1}^N x_i^2}$ and $M \geq 1$. Here ω_N is the surface area of the unit sphere in \mathbb{R}^N ($\omega_1 = 1, \omega_2 = 2\pi, \omega_3 = 4\pi$ etc.) and the scaling factor $\sigma_{M(N)}$ is determined so that functions $B(r)$ have unit integral in \mathbb{R}^N . Since $B(r)$ is a polynomial in r^2 it is a polynomial in the x_i 's, and hence infinitely differentiable for $r < 1$. All derivatives of $B(r)$ up to order $M - 1$ vanish at $r = 1$, so $B(r)$ is $M - 1$ times continuously differentiable in \mathbb{R}^N . Also, for $r < 1$, $\frac{dB}{dr} = -\frac{\sigma_{M(N)}}{\omega_N} 2M(1 - r^2)^{M-1}r < 0$, so $B(r)$ is monotonically decreasing and is strictly positive for $r < 1$.

The following table gives the scaling factors $\sigma_{M(N)}$, appropriate for functions of one, two and three dimensions for $M = 1 \dots 9$,

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\sigma_{M(1)}$	$\frac{3}{4}$	$\frac{15}{16}$	$\frac{35}{32}$	$\frac{315}{256}$	$\frac{693}{512}$	$\frac{3003}{2048}$	$\frac{6435}{4096}$	$\frac{109395}{65536}$	$\frac{230945}{131072}$
$\sigma_{M(2)}$	4	6	8	10	12	14	16	18	20
$\sigma_{M(3)}$	$\frac{15}{2}$	$\frac{105}{8}$	$\frac{315}{16}$	$\frac{3465}{128}$	$\frac{9009}{256}$	$\frac{45045}{1024}$	$\frac{109395}{2048}$	$\frac{2078505}{32768}$	$\frac{4849845}{65536}$

Table 1: Mollifier normalization factors

To obtain a mollifier with support in the disk of radius δ , one uses

$$B_\delta(r) = \begin{cases} \frac{1}{\delta^N} \frac{\sigma_{M(N)}}{\omega_N} \left(1 - \left(\frac{r}{\delta}\right)^2\right)^M & r \leq \delta \\ 0 & r > \delta \end{cases} \quad (4)$$

When using mollifiers and their derivatives to create a smooth function that matches moments of a given function up to a specified order, it is useful to have analytic expressions for the moments and derivatives of the mollifiers. The derivatives are readily evaluated by differentiating (3). The polynomial form of the mollifiers also makes the computation of these moments relatively easy with symbolic integration software. For example, for the mollifiers of the form (3), all first order moments vanish due to radial symmetry, but not all second order moments vanish. The moments with respect to x_i^2 are non-zero and one finds that in two dimensions for a mollifier with exponent M and radius δ ,

$$\int_{\mathbb{R}^2} B_\delta(\vec{r}) x_i^2 = \frac{\delta^2}{2(M+2)}$$

and for three dimensions, one finds

$$\int_{\mathbb{R}^3} B_\delta(\vec{r}) x_i^2 = \frac{\delta^2}{2(M+2)+1}$$

3 High Order Mollifiers

A mollifier is of high order if for smooth functions, $f(\vec{x})$, the convolution of $f(x)$ with $B_\delta(\vec{x})$ satisfies

$$\|f - f * B_\delta\|_2 = \left\| f(\vec{x}) - \int_{\mathbb{R}^N} f(\vec{s}) B_\delta(\vec{x} - \vec{s}) d\vec{s} \right\|_2 = O(\delta^p) \quad (5)$$

with $p > 1$. The order of a mollifier is directly related to the number of its non-zero moments that vanish. The mollifier will be of order p if the mollifier has unit integral, and if all moments q for $1 \leq q < p$ vanish [3], e.g.

$$\int_{\mathbb{R}^N} B(\vec{x}) \vec{x}^i d\vec{x} \quad (6)$$

where i is a multi-index with $|i| = q$.

The family of mollifiers (3) that we are considering are radially symmetric, so all odd order moments vanish, and thus are of second order. Moreover, using Fourier transforms, one can show that for smooth functions $f(\vec{x})$, there is an asymptotic expansion of the mollifier error of the form

$$\|f - f * B_\delta\| = c_2 \delta^2 + c_4 \delta^4 + c_6 \delta^6 + \dots \quad (7)$$

As suggested by [3], one can use repeated Richardson extrapolation [2] or Aitken extrapolation to create mollifiers that are of order $p > 2$ by combining K mollifiers of different radii,

$$\tilde{B}(r) = \sum_{j=0}^{K-1} \alpha_j B_{\gamma_j}(r) \quad (8)$$

where the γ_j 's and α_j 's are chosen so that $\tilde{B}(r)$ has unit integral and the first K terms in an error expansion similar to (7) vanish. The radii of the mollifiers used, the γ_j 's, are normalized so $\gamma_j \leq 1$ and are usually selected so that the derivatives of the resulting mollifier are not large and the associated coefficients α_j 's are not large as well. Combinations that we have found useful that balance these concerns are

$$\tilde{B}(r) = -\frac{1}{3}B(r) + \frac{4}{3}B\left(\frac{1}{2}r\right) \quad (9)$$

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for fourth order, and

$$\tilde{B}(r) = \frac{1}{10}B(r) - \frac{3}{5}B\left(\frac{2}{3}r\right) + \frac{3}{2}B\left(\frac{1}{3}r\right) \quad (10)$$

for sixth order.

One advantage of using linear combinations of radially symmetric mollifiers to create high order mollifiers is that the coefficients in the combination do not depend on the specific mollifier being used. Two disadvantages of using such combinations is that the computational cost of evaluating mollifiers of the form (8) will generally be K times the work of evaluating a single mollifier and the mollifier combination will typically have larger derivatives than the single mollifiers because $\gamma_j \leq 1$.

When using mollifiers of the form (3) an alternative method for constructing higher order mollifiers consists of combining members of the family with different exponents, specifically

$$\tilde{B}(r) = \sum_{j=0}^{K-1} \beta_j (1-r^2)^{M+j} \quad (11)$$

where the β_j are selected so the $\tilde{B}(r)$ has unit mass and the first K terms of the asymptotic expansion of the mollification error vanish. For each M , these mollifiers can be written as

$$\tilde{B}(r) = \frac{\gamma_{M(N)}}{\omega_N} (1-r^2)^M \left(1 + \sum_{j=1}^{K-1} \alpha_{j(N)} (1-r^2)^j \right) \quad (12)$$

so that there is only a small additional cost in their evaluation over the cost of evaluating a single mollifier with exponent M . In Tables 3-8 in A we list the coefficients for fourth and sixth order formulas for values of M from 1 to 9.

4 Potentials of polynomial mollifiers

The polynomial form of the mollifiers enables one to construct analytic formulas for the potentials they induce. We describe here the procedure for determining ϕ_δ 's so that $\Delta\phi_\delta = B_\delta(r)$ where $B_\delta(r)$ is a mollifier of the form (3). The construction of the potentials induced by higher order mollifiers is similar. Since

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} r^{2k} = (2k)^2 r^{2k-2} \quad \text{and} \quad \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} r^{2k} = (2k)(2k+1)r^{2k-2}$$

it follows that

$$v_2(r) = c_{0(2)} + \sum_{j=1}^{M+1} \frac{(-1)^{(j-1)}}{(2j)^2} \binom{M}{j-1} r^{2j} \quad (13)$$

$$v_3(r) = c_{0(3)} + \sum_{j=1}^{M+1} \frac{(-1)^{(j-1)}}{(2j)(2j+1)} \binom{M}{j-1} r^{2j} \quad (14)$$

are solutions of

$$\Delta v_N = (1-r^2)^M = \sum_{k=0}^M (-1)^k \binom{M}{k} r^{2k}$$

for $r \leq 1$ for two and three dimensions respectively. We find it convenient to choose $c_{0(N)}$ so that $v_N(r)$ vanish at $r = 1$. Values of $c_{0(N)}$ are given in Table 2. The complete potential is a sum of $\frac{\sigma_{M(N)}}{\omega_N} v_N(r)$ and harmonic functions inside and outside the unit disk chosen to insure the continuity of the resulting potential and its' derivatives at $r = 1$. For the solution ϕ to $\Delta\phi = B(r)$, in two dimensions we have

$$\phi(r) = \begin{cases} \frac{\sigma_{M(2)}}{2\pi} v_2(r) & r \leq 1 \\ \frac{\log(r)}{2\pi} & r > 1 \end{cases}$$

In three dimensions we have

$$\phi(r) = \begin{cases} \frac{\sigma_{M(3)}}{4\pi} v_3(r) - \frac{1}{4\pi} & r \leq 1 \\ -\frac{1}{4\pi r} & r > 1 \end{cases}$$

The solutions associated with mollifiers of non-unit radius, e.g. the solutions of $\Delta\phi_\delta = B_\delta(r)$, where $B_\delta(r)$ is given by (4), are obtained by the appropriate scaling;

$$\phi_\delta(r) = \begin{cases} \frac{\sigma_{M(2)}}{2\pi} v_2\left(\frac{r}{\delta}\right) + \frac{\log(\delta)}{2\pi} & r \leq \delta \\ \frac{\log(r)}{2\pi} & r > \delta \end{cases}$$

in two dimensions, and in three dimensions by

$$\phi_\delta(r) = \begin{cases} \frac{1}{\delta} \left(\frac{\sigma_{M(3)}}{4\pi} v_3\left(\frac{r}{\delta}\right) - \frac{1}{4\pi} \right) & r \leq \delta \\ -\frac{1}{4\pi r} & r > \delta \end{cases}$$

In all of the above formulas $\sigma_{M(2)}$ and $\sigma_{M(3)}$ are the normalization factor for $B(r)$ whose values are given in Table 1.

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$c_{0(2)}$	$-\frac{3}{16}$	$-\frac{11}{72}$	$-\frac{25}{192}$	$-\frac{137}{1200}$	$-\frac{49}{480}$	$-\frac{363}{3920}$	$-\frac{761}{8960}$	$-\frac{7129}{90720}$	$-\frac{7381}{100800}$
$c_{0(3)}$	$-\frac{7}{60}$	$-\frac{19}{210}$	$-\frac{187}{2520}$	$-\frac{437}{6930}$	$-\frac{1979}{36036}$	$-\frac{4387}{90090}$	$-\frac{76627}{1750320}$	$-\frac{165409}{4157010}$	$-\frac{141565}{3879876}$

Table 2: Potential constant factors $c_{0(N)}$ so that $v_N(1) = 0$, $N = 2, 3$.

Formulas for the derivatives of the potential are readily constructed by noting that for $r \leq 1$, $\phi(r) = w(r^2)$ where $w(s)$ is a polynomial of degree $M + 1$. Thus, for $r \leq 1$,

$$\frac{\partial \phi}{\partial x_i} = 2 \left(\frac{dw}{ds} \Big|_{s=r^2} \right) x_i$$

and for the second derivatives

$$\frac{\partial^2 \phi}{\partial x_j \partial x_i} = 4 \left(\frac{d^2 w}{ds^2} \Big|_{s=r^2} \right) x_i x_j + \epsilon_{i,j} 2 \left(\frac{dw}{ds} \Big|_{s=r^2} \right)$$

where $\epsilon_{i,j} = 1$ for $i = j$ and $\epsilon_{i,j} = 0$ otherwise. Utilizing this observation and the explicit form of the potential induced by the mollifiers $B_\delta(x)$ of the form (4) one obtains the following formulas for the derivatives of the potential for $r \leq \delta$ in two dimensions,

$$\frac{\partial \phi_\delta}{\partial x_i} = \frac{\sigma_{M(2)}}{2\pi\delta} \sum_{k=1}^{M+1} \frac{(-1)^{(k-1)}}{(2k)} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-2} x_i$$

and in three dimensions

$$\frac{\partial \phi_\delta}{\partial x_i} = \frac{\sigma_{M(3)}}{4\pi\delta^2} \sum_{k=1}^{M+1} \frac{(-1)^{(k-1)}}{(2k+1)} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-2} x_i$$

For the second derivatives of the two dimensional potential for $r \leq \delta$,

$$\frac{\partial^2 \phi_\delta}{\partial x_j \partial x_i} = \frac{\sigma_{M(2)}}{2\pi\delta^2} \left[\sum_{k=2}^{M+1} \frac{(k-1)(-1)^{(k-1)}}{k} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-4} x_i x_j + \epsilon_{i,j} \sum_{k=1}^{M+1} \frac{(-1)^{(k-1)}}{(2\sigma k)} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-2} \right]$$

and for three dimensions

$$\frac{\partial^2 \phi_\delta}{\partial x_j \partial x_i} = \frac{\sigma_{M(3)}}{4\pi\delta^3} \left[\sum_{k=2}^{M+1} \frac{2(k-1)(-1)^{(k-1)}}{(2k+1)} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-4} x_i x_j + \epsilon_{i,j} \sum_{k=1}^{M+1} \frac{(-1)^{(k-1)}}{(2k+1)} \binom{M}{k-1} \left(\frac{r}{\delta}\right)^{2k-2} \right]$$

In the above formulas the values of $\sigma_{M(2)}$ and $\sigma_{M(3)}$ are given in Table 1.

Plots of the three dimensional mollifiers of order 2, 4 and 6 with $\delta = 1$ and $M = 9$ are given in Figure 1(a). The potential functions associated with each of these mollifiers is shown in Figure 1(b) along with the exact potential.

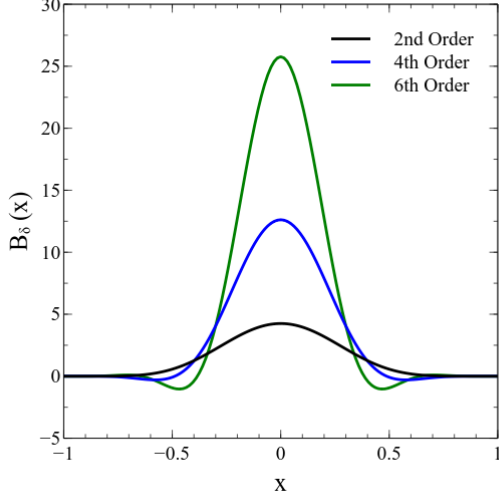


Figure 1(a)

Three dimensional $B_\delta(x)$ for $\delta = 1$

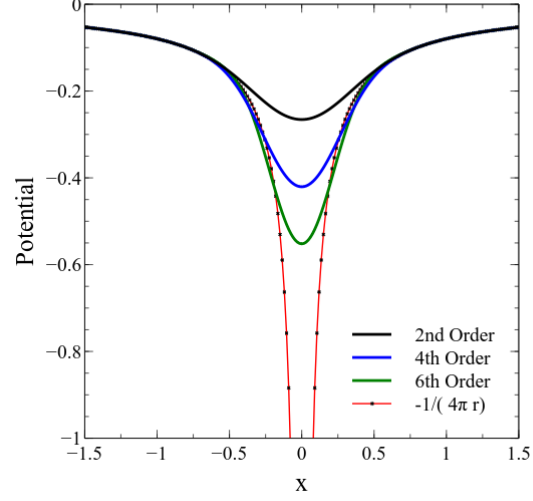


Figure 1(b)

Three dimensional $\phi_\delta(x)$ for $\delta = 1$

5 Mollifier Uniqueness

The unit integral and differentiability properties at $r = 1$ of mollifiers of the form (3) determine the mollifiers uniquely, as is proven by the following theorem:

Theorem 1 Consider polynomials in r^2 with $M + 1$ terms of the form

$$p(r) = \sum_{k=0}^M c_k r^{2k} \quad (15)$$

If, for a given dimension N , a polynomial of this form satisfies the $M + 1$ conditions,

$$\int_{D^N} p(\vec{r}) d\vec{r} = 1 \quad (16)$$

$$\frac{d^q p}{dr^q} \Big|_{r=1} = 0 \quad q = 0 \dots M - 1 \quad (17)$$

where $r = |\vec{r}|$, $\vec{r} \in \mathbb{R}^N$, and D^N is the unit ball in \mathbb{R}^N , then that polynomial is unique, and is given by

$$p(r) = \frac{\sigma_{M(N)}}{\omega_N} (1 - r^2)^M \quad (18)$$

where ω_N the surface area of the N -sphere, and $\frac{1}{\sigma_{M(N)}} = \int_0^1 r^{N-1} (1 - r^2)^M dr$.

The proof of this fact relies on the following lemma which shows that if one replaces the integral constraint (16) by an interpolation constraint, there is a unique polynomial of the form (15) that satisfies these constraints.

Lemma 1 *For a given β , the polynomial*

$$p(r) = \beta(1 - r^2)^M \quad (19)$$

is the unique polynomial of the form (15) that satisfies the conditions

$$p(0) = \beta \quad (20)$$

$$\left. \frac{d^q p}{dr^q} \right|_{r=1} = 0 \quad q = 0 \dots M - 1 \quad (21)$$

Proof. Proof of Lemma 1

If $p(r) = \sum_{k=0}^M c_k r^{2k}$ then (20) uniquely determines that $c_0 = \beta$. Conditions (21) imply that the remaining coefficients satisfy the M equations

$$\sum_{k=1}^M c_k = -\beta \quad (22)$$

$$\sum_{k=1}^M \left(\prod_{s=1}^m (2k - (s - 1)) \right) c_k = 0 \quad m = 1 \dots M - 1 \quad (23)$$

In equations (23), terms that have zero value are retained so that the sum indexing starts at $k = 1$.

Any solution of (22)-(23) also satisfies the following equations,

$$\sum_{k=1}^M c_k = -\beta \quad (24)$$

$$\sum_{k=1}^M k^m c_k = 0 \quad m = 1 \dots M - 1 \quad (25)$$

The equivalence of equations (25) to (23) follows by induction. Specifically, for $m = 1$, equation (25) is satisfied, as it is identical, up to a multiplicative scaling factor of (23). If we assume that the equations (25) are satisfied for $m' < m$ then, multiplying out the coefficients in the m th equation of (23) there exist coefficients v_j , such that

$$\sum_{k=1}^M \left(\prod_{s=1}^m (2k - (s-1)) \right) c_k = \sum_{k=1}^M \left(2^m k^m + \sum_{j=1}^{m-1} v_j k^j \right) c_k = 0$$

If one interchanges the order of summation and invokes the induction hypothesis, one finds

$$\sum_{k=1}^M k^m c_k = -\frac{1}{2^m} \sum_{j=1}^{m-1} v_j \left(\sum_{k=1}^M k^j c_k \right) = 0$$

so that (25) holds for all $m = 0 \dots M-1$.

If $\vec{c} = (c_1, c_2, \dots, c_M)$ is a solution of (22)-(23), then \vec{c} therefore satisfies a linear system of equations of the form $A\vec{c} = \vec{b}$ where A is the $M \times M$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & M \\ 1 & 2^2 & 3^2 & \dots & M^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{M-1} & 3^{M-1} & \dots & M^{M-1} \end{pmatrix} \quad (26)$$

This matrix is non-singular because it's transpose is the matrix associated with the problem of constructing a polynomial interpolant of degree $M-1$ with data given at the M distinct points $x_k = k$ for $k = 1 \dots M$; a linear problem with a unique solution. Therefore, conditions (20) and (21) uniquely determine the coefficients of a polynomial of the form $p(r) = \sum_{k=0}^M c_k r^{2k}$, since, if they didn't, this would lead to a contradiction to the fact that the system of equations $A\vec{c} = \vec{b}$ is non-singular. One can readily verify that $p(r) = \beta(1-r^2)^M$ is a polynomial in r^2 with $M+1$ terms that satisfies (20) and (21) and is thus the unique polynomial that satisfies these conditions. ■

Proof. Proof of Theorem 1

Assume $v(r)$ and $w(r)$ are two polynomials of the form (15) that both satisfy condition (16) for some N and conditions (17). $v(0)$ and $w(0)$ must be non-zero, because, otherwise, by Lemma 1 with $\beta = 0$, they would have to coincide with the zero polynomial, and this would contradict the fact that they have a non-vanishing integral. If $v(0) = w(0)$, then, by Lemma 1 with $\beta = v(0) = w(0)$, we must have $v(r) = w(r)$. If $v(0) \neq w(0)$, then let $\tilde{v}(r) = \frac{w(0)}{v(0)}v(r)$, then $\tilde{v}(r)$ is a polynomial of the form (15) and $\tilde{v}(0) = w(0)$ so that by Lemma 1, $\tilde{v}(r) = w(r)$. Since both $v(r)$ and $w(r)$ satisfy (16),

$$\int_{\mathbb{D}^N} w(\vec{r}) d\vec{r} = \int_{\mathbb{D}^N} \tilde{v}(\vec{r}) d\vec{r} = \frac{w(0)}{v(0)} \int_{\mathbb{D}^N} v(\vec{r}) d\vec{r} = \frac{w(0)}{v(0)} \neq 1$$

Thus $v(0) \neq w(0)$ leads to a contradiction and we must have $v(r) = w(r)$. One can directly verify that for a given N , $p(r) = \frac{\sigma_{M(N)}}{\omega_N} (1-r^2)^M$ satisfies (16) and conditions (17), and is of the form (15) and so is the unique polynomial that satisfies these conditions. ■

6 Conclusion

In this paper we have presented a family of mollifiers that have polynomial form, that vanish outside of a specified radius, and have a parameterized degree of differentiability. Moreover, if one assumes a polynomial representation involving only even powers of the radial coordinate, then these mollifiers are unique. The relatively simple representation of these mollifiers enables one to construct analytic representations of the potentials induced by these mollifiers. While we have focused on the construction of formulas for the potential, analytic representations of other functionals can often be obtained. In particular, analytic representations for highly differentiable approximations to the Heaviside function can be obtained by forming the indefinite integral of the mollifiers in one dimension. Lastly, we've presented two methods for constructing highly differentiable mollifiers whose high order moments vanish. These mollifiers are suitable for the creation of methods in which the errors due to mollification are proportional to the smoothing radius to a power greater than two.

A Coefficients for High Order Mollifiers

$$\tilde{B}(r) = \frac{\gamma_{M(N)}}{\omega_N} (1 - r^2)^M \left(1 + \sum_{j=1}^{S-1} \alpha_{j(N)} (1 - r^2)^j \right) \quad (27)$$

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(1)}$	$-\frac{15}{8}$	$-\frac{105}{32}$	$-\frac{315}{64}$	$-\frac{3465}{512}$	$-\frac{9009}{1024}$	$-\frac{45045}{4096}$	$-\frac{109395}{8192}$	$-\frac{2078505}{131072}$	$-\frac{4849845}{262144}$
$\alpha_{0(1)}$	$-\frac{7}{4}$	$-\frac{3}{2}$	$-\frac{11}{8}$	$-\frac{13}{10}$	$-\frac{5}{4}$	$-\frac{17}{14}$	$-\frac{19}{16}$	$-\frac{7}{6}$	$\frac{23}{20}$

Table 3: Fourth Order Mollifier Coefficients - 1D

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(1)}$	$\frac{105}{32}$	$\frac{945}{128}$	$\frac{3465}{256}$	$\frac{45045}{2048}$	$\frac{135135}{4096}$	$\frac{765765}{16384}$	$\frac{2078505}{32768}$	$\frac{43648605}{524288}$	$\frac{111546435}{1048576}$
$\alpha_{0(1)}$	$-\frac{9}{2}$	$-\frac{11}{3}$	$-\frac{13}{4}$	-3	$-\frac{17}{6}$	$-\frac{19}{7}$	$-\frac{21}{8}$	$-\frac{23}{9}$	$-\frac{5}{2}$
$\alpha_{1(1)}$	$\frac{33}{8}$	$\frac{143}{48}$	$\frac{39}{16}$	$\frac{17}{8}$	$\frac{323}{168}$	$\frac{57}{32}$	$\frac{161}{96}$	$\frac{115}{72}$	$\frac{135}{88}$

Table 4: Sixth Order Mollifier Coefficients - 1D

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(2)}$	-12	-24	-40	-60	-84	-112	-144	-180	-220
$\alpha_{0(2)}$	2	$-\frac{5}{3}$	$-\frac{3}{2}$	$-\frac{7}{5}$	$-\frac{4}{3}$	$-\frac{9}{7}$	$-\frac{5}{4}$	$-\frac{11}{9}$	$-\frac{6}{5}$

Table 5: Fourth Order Mollifier Coefficients - 2D

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(2)}$	24	60	120	210	336	504	720	990	1320
$\alpha_{0(2)}$	-5	-4	$-\frac{7}{2}$	$-\frac{16}{5}$	-3	$-\frac{20}{7}$	$-\frac{11}{4}$	$-\frac{8}{3}$	$-\frac{13}{5}$
$\alpha_{1(2)}$	5	$\frac{7}{2}$	$\frac{4}{5}$	$\frac{12}{5}$	$\frac{15}{7}$	$\frac{55}{28}$	$\frac{11}{6}$	$\frac{26}{15}$	$\frac{91}{55}$

Table 6: Sixth Order Mollifier Coefficients - 2D

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(3)}$	$-\frac{105}{4}$	$-\frac{945}{16}$	$-\frac{3465}{32}$	$-\frac{45045}{256}$	$-\frac{135135}{512}$	$-\frac{765765}{2048}$	$-\frac{2078505}{4096}$	$-\frac{43648605}{65536}$	$-\frac{111546435}{131072}$
$\alpha_{0(3)}$	$-\frac{9}{4}$	$-\frac{11}{6}$	$-\frac{13}{8}$	$-\frac{3}{2}$	$-\frac{17}{12}$	$-\frac{19}{14}$	$\frac{21}{16}$	$-\frac{23}{18}$	$-\frac{5}{4}$

Table 7: Fourth Order Mollifier Coefficients - 3D

	M = 1	M = 2	M = 3	M = 4	M = 5	M = 6	M = 7	M = 8	M = 9
$\gamma_{M(3)}$	$\frac{945}{16}$	$\frac{10395}{64}$	$\frac{45045}{128}$	$\frac{675675}{1024}$	$\frac{2297295}{2048}$	$\frac{14549535}{8192}$	$\frac{43648605}{16384}$	$\frac{1003917915}{262144}$	$\frac{2788660875}{524288}$
$\alpha_{0(3)}$	$-\frac{11}{2}$	$-\frac{13}{3}$	$-\frac{15}{4}$	$-\frac{17}{5}$	$-\frac{19}{6}$	-3	$-\frac{23}{8}$	$-\frac{25}{9}$	$-\frac{27}{10}$
$\alpha_{1(3)}$	$\frac{143}{24}$	$\frac{65}{16}$	$\frac{51}{16}$	$\frac{323}{120}$	$\frac{19}{8}$	$\frac{69}{32}$	$\frac{575}{288}$	$\frac{15}{8}$	$\frac{783}{440}$

Table 8: Sixth Order Mollifier Coefficients - 3D

B

In this appendix we give explicit formulas for a three times continuously differentiable mollifier and its associated potential and potential derivatives. We also give a three times continuously differentiable fourth order mollifier and its associated potential and potential derivatives.

When $M = 4$, and denoting $r = |\vec{x}|$ we have in two dimensions for $r \leq \delta$

$$B_\delta(\vec{x}) = \frac{10}{2\pi\delta^2} \left(1 - \left(\frac{r}{\delta}\right)^2\right)^4$$

$$\phi_\delta(\vec{x}) = \frac{10}{2\pi} \left[-\frac{137}{1200} + \frac{1}{4} \left(\frac{r}{\delta}\right)^2 - \frac{1}{4} \left(\frac{r}{\delta}\right)^4 + \frac{1}{6} \left(\frac{r}{\delta}\right)^6 - \frac{1}{16} \left(\frac{r}{\delta}\right)^8 + \frac{1}{100} \left(\frac{r}{\delta}\right)^{10} \right] + \frac{\log(\delta)}{2\pi}$$

$$\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{10}{2\pi\delta} \left[\frac{1}{2} - \left(\frac{r}{\delta}\right)^2 + \left(\frac{r}{\delta}\right)^4 - \frac{1}{2} \left(\frac{r}{\delta}\right)^6 + \frac{1}{10} \left(\frac{r}{\delta}\right)^8 \right] x_i$$

while for $r > \delta$, $B_\delta(r) = 0$, $\phi_\delta(r) = \frac{\log(r)}{2\pi}$ and $\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{x_i}{2\pi r^2}$.

In three dimensions we have, for $r \leq \delta$

$$B_\delta(r) = \frac{3465}{128} \frac{1}{4\pi\delta^3} \left(1 - \left(\frac{r}{\delta}\right)^2\right)^4$$

$$\phi_\delta(r) = \frac{3465}{128} \frac{1}{4\pi\delta} \left[-\frac{437}{6930} + \frac{1}{6} \left(\frac{r}{\delta}\right)^2 - \frac{1}{5} \left(\frac{r}{\delta}\right)^4 + \frac{1}{7} \left(\frac{r}{\delta}\right)^6 - \frac{1}{18} \left(\frac{r}{\delta}\right)^8 + \frac{1}{110} \left(\frac{r}{\delta}\right)^{10} \right] + \frac{1}{4\pi\delta}$$

$$\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{3465}{128} \frac{1}{4\pi\delta^2} \left[\frac{1}{3} - \frac{4}{5} \left(\frac{r}{\delta}\right)^2 + \frac{6}{7} \left(\frac{r}{\delta}\right)^4 - \frac{4}{9} \left(\frac{r}{\delta}\right)^6 + \frac{1}{11} \left(\frac{r}{\delta}\right)^8 \right] x_i$$

while for $r > \delta$, $B_\delta(r) = 0$, $\phi_\delta(r) = -\frac{1}{4\pi r}$ and $\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{x_i}{4\pi r^3}$.

For fourth order mollifiers based upon (12), with $M = 4$, for $r < \delta$ in two dimensions we have

$$B_\delta(r) = -\frac{60}{2\pi\delta^2} \left(1 - \left(\frac{r}{\delta}\right)^2\right)^4 \left[1 - \frac{7}{5} \left(1 - \left(\frac{r}{\delta}\right)^2\right)\right]$$

$$\phi_\delta(r) = -\frac{60}{2\pi} \left[\frac{23}{800} - \frac{1}{10} \left(\frac{r}{\delta}\right)^2 + \frac{3}{16} \left(\frac{r}{\delta}\right)^4 - \frac{2}{9} \left(\frac{r}{\delta}\right)^6 + \frac{5}{32} \left(\frac{r}{\delta}\right)^8 - \frac{3}{50} \left(\frac{r}{\delta}\right)^{10} + \frac{7}{720} \left(\frac{r}{\delta}\right)^{12} \right] + \frac{\log(\delta)}{2\pi}$$

$$\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = -\frac{60}{2\pi\delta} \left[-\frac{1}{5} + \frac{3}{4} \left(\frac{r}{\delta}\right)^2 - \frac{4}{3} \left(\frac{r}{\delta}\right)^4 + \frac{5}{8} \left(\frac{r}{\delta}\right)^6 - \frac{3}{5} \left(\frac{r}{\delta}\right)^8 + \frac{7}{60} \left(\frac{r}{\delta}\right)^{10} \right] x_i$$

For $r > \delta$, $B_\delta(r) = 0$, $\phi_\delta(r) = \frac{\log(r)}{2\pi}$ and $\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{x_i}{2\pi r^2}$.

In three dimensions we have for $r < \delta$,

$$B_\delta(r) = -\frac{45045}{256} \frac{1}{4\pi\delta^3} \left(1 - \left(\frac{r}{\delta}\right)^2\right)^4 \left[1 - \frac{3}{2} \left(1 - \left(\frac{r}{\delta}\right)^2\right)\right]$$

$$\phi_\delta(r) =$$

$$-\frac{45045}{256} \frac{1}{4\pi\delta} \left[\frac{6961}{360360} - \frac{1}{12} \left(\frac{r}{\delta}\right)^2 + \frac{7}{40} \left(\frac{r}{\delta}\right)^4 - \frac{3}{14} \left(\frac{r}{\delta}\right)^6 + \frac{11}{72} \left(\frac{r}{\delta}\right)^8 - \frac{13}{220} \left(\frac{r}{\delta}\right)^{10} + \frac{1}{104} \left(\frac{r}{\delta}\right)^{12} \right] + \frac{1}{4\pi\delta}$$

$$\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = -\frac{45045}{256} \frac{1}{4\pi\delta^2} \left[-\frac{1}{6} + \frac{7}{10} \left(\frac{r}{\delta}\right)^2 - \frac{9}{7} \left(\frac{r}{\delta}\right)^4 + \frac{11}{9} \left(\frac{r}{\delta}\right)^6 - \frac{13}{22} \left(\frac{r}{\delta}\right)^8 + \frac{3}{26} \left(\frac{r}{\delta}\right)^{10} \right] x_i$$

and for $r > \delta$, $B_\delta(r) = 0$, $\phi_\delta(r) = -\frac{1}{4\pi r}$ and $\frac{\partial\phi_\delta(\vec{x})}{\partial x_i} = \frac{x_i}{4\pi r^3}$.

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