Hausdorff étale groupoids and their C^*-algebras

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Contents

Foreword

These notes were written as supplementary material for a five-hour lecture series presented at the Centre de Recerca Mathemàtica at the Universitat Autònoma de Barcelona from the 13th to the 17th of March 2017. This was part of the Intensive Research Program *Operator algebras: dynamics and interactions* which

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ran from March to July 2017. I thank the CRM for the excellent support, facility, and atmosphere it provided. The whole IRP provided an exceptional environment for research and research interactions, and it was a real pleasure to be a part of the enthusiastic and productive research activity that was going on. The place was really buzzing.

The intention of these notes is to give a brief overview of some key topics in the area of C^*-algebras associated to étale groupoids. The scope has been deliberately contained to the case of étale groupoids with the intention that much of the representation-theoretic technology and measure-theoretic analysis required to handle general groupoids can be suppressed in this simpler setting. Because these notes are based on a short course, they feature only a small selection of topics, chosen for their relevance and interest to participants in the masterclass. My choice to include or omit any particular topic is not a comment on the interest of that topic in general—it was a question of developing something consistent and coherent that could sensibly be presented in a week's worth of lectures, and would hopefully seem sensible in the context of the masterclass and the other two lecture series being presented. So, for example, I have not included any discussion here of inverse semigroups and their connections with étale groupoids even though, arguably, inverse semigroups and étale groupoids are more or less inseparable. I apologise wholeheartedly to all those people whose very nice work on groupoids and groupoid C^*-algebras has every right to be discussed in a set of notes like this but has not been mentioned.

Finally, my thanks to those participants in the masterclass, and to those subsequent readers, who pointed out errors and possible improvements on the draft version of the notes that was circulated at the time of the lecture series. Special thanks to Kevin Brix from the University of Copenhagen for a number of helpful corrections and suggestions, to Valentin Deaconu for spotting a number of typos, and to Becky Armstrong for many helpful suggestions and corrections; and also to Mikael Rørdam for suggesting the neat proof of Lemma 3.2.2.

Errata

There are doubtless errors in these notes, despite my best efforts to weed them out and the generous help I've had from others. I will maintain an up-to-date list of errata for these notes as part of a common errata file for these notes and the other two sets of notes in this volume at http://www.uow.edu.au/~asims/2017crm/errata.pdf. Please let me know by email to asims@uow.edu.au if you find any typos or other errors—I'll be very grateful.

Chapter 1

Introduction

Groupoids are algebraic objects that behave like a group except that the multiplication operation is only partially defined. Topological groupoids provide a useful unifying model for groups and group actions, and equivalence relations induced by continuous maps between topological spaces. They also provide a good algebraic model for the quotient of a topological space by a group or semigroup action in instances where the quotient space itself is, topologically, poorly behaved—for example, the quotient of a shift-space determined by the shift map, or the quotient of the circle by an irrational rotation.

The collection $\mathcal{G}^{(0)}$ of idempotent elements in a groupoid $\mathcal{G}$ is called its unit space, since these are precisely the elements x that satisfy $x\gamma = \gamma$ and $\eta x = \eta$ whenever these products are defined. This leads to a natural fibred structure of $\mathcal{G}$ over $\mathcal{G}^{(0)}$: the fibre over a unit x is the collection $\mathcal{G}_x$ of elements γ for which the product γx is defined. If $\mathcal{G}$ is a topological groupoid, then $\mathcal{G}^{(0)}$, as well as each $\mathcal{G}_x$, is a topological space in the subspace topology, and it is often helpful to think of the subspaces $\mathcal{G}_x$ as transverse to $\mathcal{G}^{(0)}$. In the special case that $\mathcal{G}$ is a group, its unit space has just one element e, and then $\mathcal{G} = \mathcal{G}_e$. So the analogues, in the setting of groupoids, of topological properties of groups, typically involve corresponding topological conditions on the sets $\mathcal{G}_x$. In particular, the analogue of a discrete group is a groupoid in which the sets $\mathcal{G}_x$ are all discrete in a coherent way. More specifically, we ask that the map that sends each γ to the unique element $s(\gamma)$ for which $\gamma s(\gamma)$ makes sense should be a local homeomorphism. Renault [37] called such groupoids "étale" (which means something like "loose" or "spread out"), and the terminology has stuck.

The study of C^*-algebras associated to groupoids was initiated by Renault in his PhD thesis [37], and was motivated by earlier work, particularly that of Feldman and Moore [19, 20, 21] for von Neumann algebras. Groupoid C^*-algebras have been studied intensively ever since, and provide useful and concrete models of many classes of C^*-algebras. The construction and study of groupoid C^*-algebras in general is fairly involved and requires a significant amount of representation-theoretic

background. But just as the study of C^*-algebras of discrete groups and their crossed products requires less background than the study of group C^*-algebras in general, so the restriction to the situation of étale groupoids significantly reduces the overheads involved in studying groupoid C^*-algebras. And, as Renault realised from the outset, étale-groupoid C^*-algebras are sufficient to capture a lot of examples. For example, all crossed products of commutative C^*-algebras by discrete-group actions; all AF algebras; all Cuntz–Krieger algebras and graph algebras; all Kirchberg algebras in the UCT class; and many others.

My intention in preparing these notes was to develop a concise account of the elementary theory of étale groupoids and their C^*-algebras that minimises the representation-theoretic and analytic background needed. For this reason, I have chosen, for example, not to include any discussion of non-Hausdorff groupoids, even though there are many good reasons for studying these (for example, Exel and Pardo's C^*-algebras associated to self-similar actions of groups on graphs admit étale-groupoid models, but these groupoids are frequently non-Hausdorff). I have also skimmed over some of the more technical aspects of groupoid theory (for example the question of amenability). I have tried to include enough examples along the way, together with two sections that outline a couple of important applications of the theory, to illustrate what is going on with key concepts.

We start in Chapter 2 with a discussion of groupoids themselves: the axioms and set up, a number of illustrative examples, the definition of a topological groupoid, and a discussion of the étale condition.

In Chapter 3, we describe the construction of the convolution algebra of an étale groupoid, then of its two C^*-algebras—the full C^*-algebra and the reduced C^*-algebra—and finally of the notion of equivalence of étale groupoids and Renault's equivalence theorem. We already see advantages to sticking to étale groupoids here, since we are able to get through all of this material using fairly elementary techniques, and in particular without having to include a treatment of Renault's Disintegration Theorem, which is one of the mainstays of groupoid C^*-algebra theory in general, but quite a complicated piece of work.

In Chapter 4, we discuss some of the elementary structure theory of groupoid C^*-algebras. As mentioned earlier, I have chosen to skim over the details of amenability for groupoids, though I have tried to give enough references to help the interested reader find out more. There is a whole book on the subject of amenability for groupoids [2], and unlike the situation for groups, it's far from a done deal. My focus in discussing amenability has been to describe its C^*-algebraic consequences, and some standard techniques for showing that a given groupoid is amenable. We then go on to discuss effective groupoids (these are like topologically free group actions). Again, things simplify significantly in the étale setting, and we are able to present a short and self-contained proof that every nonzero ideal of the reduced C^*-algebra of an effective étale (Hausdorff) groupoid must have nonzero intersection with the abelian subalgebra of C_0-functions on the unit space. We follow this with a discussion of invariant sets of units and the ideal structure of

the C^*-algebra of an amenable étale groupoid. We use the results we have put together on ideal structure to characterise the amenable étale (Hausdorff) groupoids whose C^*-algebras are simple. To finish Chapter 4, we describe Anantharaman-Delaroche's notion of locally contracting groupoids, and prove that the reduced C^*-algebra of a locally contracting groupoid is purely infinite.

Interestingly, modulo treating amenability as a black box, all of the structure theory developed in these first few chapters is done without recourse to any heavy machinery like Renault's Disintegration theorem. Though probably known to, or at least expected by, experts, I am not aware of such an approach having appeared in print previously.

In Chapter 5 we discuss beautiful results of Renault, extending earlier work of Kumjian, that provide a C^*-algebraic version of Feldman–Moore theory, and then go on to discuss an application of this machinery to the classification of Fell algebras.

In Section 5.1, we discuss the notion of a twist $\mathcal{E}$ over an étale groupoid $\mathcal{G}$, and of the associated full and reduced C^*-algebras. These can be thought of as the analogue, for groupoids, of the reduced and full twisted C^*-algebras of a discrete group with respect to a $\mathbb{T}$-valued 2-cocycle. The details here begin to get significantly more complicated than in the previous three sections, and so I have given an overview with sketches of proofs rather than a detailed treatment of all the results. We discuss Renault's definition of a Cartan pair of C^*-algebras, indicate how a twist over an effective étale groupoid gives rise to a Cartan pair of C^*-algebras, and outline Renault's proof that the twist can be recovered from its Cartan pair, so that Cartan pairs are in bijection with twists over effective étale groupoids. This implies, for example, due to work of Barlak and Li, that any separable nuclear C^*-algebra that admits a Cartan subalgebra belongs to the UCT class.

We then wrap up in Section 5.2 by outlining an application of the groupoid technology we have developed to the classification of Fell algebras. Fell algebras are Type I C^*-algebras that generalise the continuous trace C^*-algebras, which in turn are the subject of the famous Dixmier–Douady classification theorem. The classical approach to the Dixmier–Douady theorem does not work well for Fell algebras, but another approach is available: we sketch how to construct, from each Fell algebra A, a Cartan pair (C, D) in which C is Morita equivalent to A. We then outline how to make the collection of isomorphism classes of twists over a given groupoid $\mathcal{G}$ into a group $\mathrm{Tw}_{\mathcal{G}}$. If $\mathcal{G}$ is the equivalence relation determined by a local homeomorphism of a locally compact Hausdorff space Y onto a locally locally compact, locally Hausdorff space X, then its twist group is isomorphic to a second sheaf-cohomology group of X. In particular, the pair (C, D) discussed above determines an element of $H^2(\widehat{A}, \mathcal{S})$; moreover this class is independent of any of the choices involved in our constructions, so we can regard it as an invariant $\delta(A)$ of A. The main result discussed in the section says that $\delta(A)$ is a complete invariant of A, and also that the range is exhausted in the sense that every element of $H^2(X, \mathcal{S})$ can be obtained as $\delta(A)$ for some Fell algebra A with spectrum X.

In this section, I have been very brief. I provide no detailed proofs, and very few proof sketches, and instead try to present the big picture. The details can be found in [25].

These notes are just a brief introduction to a small part of the theory of groupoid C^*-algebras. There are many useful references for the more general theory. The theory began with Renault [37], and this remains the definitive text. Exel [18] and Paterson [31] have both given excellent discussions of étale groupoids—particularly as they relate to inverse semigroups—but in the non-Hausdorff setting, where the details are a little trickier. Renault's equivalence theorem for (full) groupoid C^*-algebras first appeared in print in the work of Muhly–Renault–Williams [29], but the approach used here, via linking groupoids, is based on [42] and also owes a lot to many valuable conversations I have had with Alex Kumjian. It is also closely related to [30, 45, 49]. A trove of information amount about amenability of groupoids is contained in Anantharaman-Delaroche and Renault's book [2] on the topic. There are countless other very useful references that I have forgotten, or that are hard to come by (for example Paul Muhly's excellent but lamentably unfinished book on the subject). I apologise to the surely long list of people whose work I have overlooked (despite its being eminently worthy of mention and attention) in this brief and far-from-comprehensive discussion.

What I believe *is* missing from the literature is an elementary and self-contained introduction to the C^*-algebras of étale Hausdorff groupoids (though Exel's paper [18] does contain an excellent treatment of the construction of the groupoid C^*-algebra for non-étale Hausdorff groupoids); these notes should go a little way to filling this gap. I hope that they give a flavour for the subject and a useful reference for those who find themselves in the enviable position of having all their groupoids turn out to be Hausdorff and étale. I think that most of the arguments in the first three sections here (with the exceptions of anything about amenability, and of Anantharaman–Delaroche's pure-infiniteness result) were developed from scratch; but of course the results are not new, and the treatment reflects the many ideas and techniques that I have accumulated both from the literature, and from discussion and collaboration with many people including Jon Brown, Lisa Clark, Valentin Deaconu, Ruy Exel, Astrid an Huef, Alex Kumjian, Paul Muhly and Dana Williams. Some parts are more identifiably attributable to ideas I learned from others: in particular, the elementary approach to the construction of the universal C^*-algebra $C^*(\mathcal{G})$ was showed to me by Robin Deeley during a series of beautiful graduate-level lectures he gave at the research event *Refining C^*-algebraic invariants for dynamics using KK-theory* at the MATRIX facility of the University of Melbourne in July 2016. Robin tells me that the idea came to him in turn from lecture notes of Ian Putnam.

Chapter 2

Étale groupoids

2.1 What is a groupoid?

The following definition of a groupoid comes from [24] (see [31, page 7]); Hahn himself attributes it to a conversation with G. Mackey. This is a fairly minimal set of axioms, so optimal for the purposes of checking whether a given object is a groupoid, but I refer the reader forward to Remark 2.1.5 for an equivalent, but less efficient, list of axioms that might provide a little more intuition.

Definition 2.1.1. A *groupoid* is a set $\mathcal{G}$ together with a distinguished subset $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, a multiplication map $(\alpha, \beta) \mapsto \alpha\beta$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$ and an inverse map $\gamma \mapsto \gamma^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ such that

1. $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$;

2. if (α, β) and (β, γ) belong to $\mathcal{G}^{(2)}$, then $(\alpha\beta, \gamma)$ and $(\alpha, \beta\gamma)$ belong to $\mathcal{G}^{(2)}$, and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; and

3. $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$, and for all $(\gamma, \eta) \in \mathcal{G}^{(2)}$, we have $\gamma^{-1}(\gamma\eta) = \eta$ and $(\gamma\eta)\eta^{-1} = \gamma$.

Axiom (2) shows that for products of three groupoid elements, there is no ambiguity in dropping the parentheses (as we do in groups), and simply writing $\alpha\beta\gamma := (\alpha\beta)\gamma$.

To get a feeling for groupoids, we begin by exploring some of the consequences of the above axioms.

Given a groupoid $\mathcal{G}$ we shall write $\mathcal{G}^{(0)} := \{\gamma^{-1}\gamma \mid \gamma \in \mathcal{G}\}$ and refer to elements of $\mathcal{G}^{(0)}$ as *units* and to $\mathcal{G}^{(0)}$ itself as the *unit space*. Since $(\gamma^{-1})^{-1} = \gamma$ for all γ, we also have $\mathcal{G}^{(0)} = \{\gamma\gamma^{-1} \mid \gamma \in \mathcal{G}\}$. We define $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ by

$$r(\gamma) := \gamma\gamma^{-1} \qquad \text{and} \qquad s(\gamma) := \gamma^{-1}\gamma$$

for all $\gamma \in \mathcal{G}$.

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Lemma 2.1.2. *If $\mathcal{G}$ is a groupoid and $\gamma \in \mathcal{G}$, then $(r(\gamma), \gamma)$ and $(\gamma, s(\gamma))$ belong to $\mathcal{G}^{(2)}$, and*

$$r(\gamma)\gamma = \gamma = \gamma s(\gamma).$$

We have $r(\gamma^{-1}) = s(\gamma)$ and $s(\gamma^{-1}) = r(\gamma)$. Moreover, γ^{-1} is the unique element such that $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ and $\gamma\gamma^{-1} = r(\gamma)$, and also the unique element such that $(\gamma^{-1}, \gamma) \in \mathcal{G}^{(2)}$ and $\gamma^{-1}\gamma = s(\gamma)$.

Proof. The first statement, is just axiom (3) with $\eta = \gamma^{-1}$.

We have $r(\gamma^{-1}) = \gamma^{-1}(\gamma^{-1})^{-1} = \gamma^{-1}\gamma = s(\gamma)$ by axiom (1).

If $(\gamma, \alpha) \in \mathcal{G}^{(2)}$ and $\gamma\alpha = r(\gamma) = \gamma\gamma^{-1}$, then axiom (2) shows that $(\gamma^{-1}\gamma, \alpha) \in \mathcal{G}^{(2)}$ and $\alpha = \gamma^{-1}\gamma\alpha = \gamma^{-1}r(\gamma) = \gamma^{-1}s(\gamma^{-1}) = \gamma^{-1}$. A similar argument shows that $\alpha\gamma = s(\gamma)$ only for $\alpha = \gamma^{-1}$. $\qquad\square$

We also quickly see that groupoids have cancellation.

Lemma 2.1.3. *Let $\mathcal{G}$ be a groupoid. Suppose that $(\alpha, \gamma), (\beta, \gamma) \in \mathcal{G}^{(2)}$ and that $\alpha\gamma = \beta\gamma$. Then $\alpha = \beta$. Similarly if $(\gamma, \alpha), (\gamma, \beta) \in \mathcal{G}^{(2)}$ and $\gamma\alpha = \gamma\beta$ then $\alpha = \beta$.*

Proof. If $\alpha\gamma = \beta\gamma$, then axioms (2) and (3) show that $\alpha = \alpha\gamma\gamma^{-1} = \beta\gamma\gamma^{-1} = \beta$. $\quad\square$

Lemma 2.1.4. *Let $\mathcal{G}$ be a groupoid. Then $(\alpha, \beta) \in \mathcal{G}^{(2)}$ if and only if $s(\alpha) = r(\beta)$. We have*

1. *$r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$;*

2. *$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$; and*

3. *$r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$.*

Proof. First suppose that $s(\alpha) = r(\beta)$; that is, $\alpha^{-1}\alpha = \beta\beta^{-1}$. In particular, we have $(\alpha, \beta\beta^{-1}) = (\alpha, \alpha^{-1}\alpha) \in \mathcal{G}^{(2)}$. Since $(\beta\beta^{-1}, \beta) \in \mathcal{G}^{(2)}$ as well, we deduce from axiom (2) that $(\alpha, \beta\beta^{-1}\beta)$, which is just (α, β), belongs to $\mathcal{G}^{(2)}$. On the other hand, if $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $(\alpha^{-1}, \alpha\beta) \in \mathcal{G}^{(2)}$ with $\alpha^{-1}\alpha\beta = \beta = r(\beta)\beta$, and so Lemma 2.1.3 shows that $s(\alpha) = \alpha^{-1}\alpha = r(\beta)$.

For (1) observe that axiom (2) shows that $(r(\alpha), \alpha\beta)$ belongs to $\mathcal{G}^{(2)}$ and $r(\alpha)(\alpha\beta) = (r(\alpha)\alpha)\beta = \alpha\beta = r(\alpha\beta)(\alpha\beta)$. So Lemma 2.1.3 shows that $r(\alpha) = r(\alpha\beta)$. A similar argument gives $s(\beta) = s(\alpha\beta)$.

For (2) we use (1) to see that $(\beta^{-1}, \alpha^{-1})$ and $(\alpha\beta, \beta^{-1}\alpha^{-1})$ belong to $\mathcal{G}^{(2)}$. Since $r(\beta^{-1}) = s(\beta)$ and since (1) gives $s(\alpha\beta\beta^{-1}) = s(\beta^{-1}) = r(\alpha^{-1})$, we can use (1) twice more to see that the products $(\alpha\beta\beta^{-1})\alpha^{-1}$ and $(\alpha\beta)(\beta^{-1}\alpha^{-1})$ make sense and are equal. We have $(\alpha\beta\beta^{-1})\alpha^{-1} = \alpha\alpha^{-1} = r(\alpha) = r(\alpha\beta)$, and so uniqueness in the final statement of Lemma 2.1.2 implies that $\beta^{-1}\alpha^{-1} = (\alpha\beta)^{-1}$.

For (3), fix $x \in \mathcal{G}^{(0)}$, say $x = \gamma^{-1}\gamma$. Then (1) shows that $r(x) = r(\gamma^{-1}) = s(\gamma) = \gamma^{-1}\gamma = x$. Similarly, $s(x) = s(\gamma^{-1}\gamma) = s(\gamma) = \gamma^{-1}\gamma = x$. $\qquad\square$

In many places you will find the definition of a groupoid summarised with the pithy "a groupoid is a small category with inverses." The above results should convince you that this is equivalent to Definition 2.1.1. It's a slick definition, but if it means much to you, then you probably already knew what a groupoid was anyway...

Lemma 2.1.4 shows that if $\mathcal{G}$ is a groupoid, then $\mathcal{G}^{(2)} = \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$. When describing a groupoid (in an example) it is often convenient, and more helpful, to make use of this: we typically specify the set $\mathcal{G}$, the distinguished subset $\mathcal{G}^0$ and the maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ first; and then specify an associative multiplication map from $\{(\alpha, \beta) \mid s(\alpha) = r(\beta) \in \mathcal{G}^{(0)}\}$ to $\mathcal{G}$ satisfying $r(\alpha\beta) = r(\alpha)$, $s(\alpha\beta) = s(\beta)$ and $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$, and specify an inverse map satisfying $s(\alpha^{-1}) = r(\alpha)$, $r(\alpha^{-1}) = s(\alpha)$ and $\alpha^{-1}\alpha = s(\alpha)$ and $\alpha\alpha^{-1} = r(\alpha)$. Using the results above, you should be able to convince yourself that this is equivalent to Definition 2.1.1. We will specify groupoids this way throughout these notes.

Remark 2.1.5. An earlier version of these notes contained the following definition of a groupoid: A groupoid is a set $\mathcal{G}$ with a distinguished subset $\mathcal{G}^{(0)}$, maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$, a map $(\alpha, \beta) \mapsto \alpha\beta$ from $\{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$ to $\mathcal{G}$ and a map $\gamma \mapsto \gamma^{-1}$ from $\mathcal{G}$ to $\mathcal{G}$ with the following properties:

(G1) $r(x) = x = s(x)$ for all $x \in \mathcal{G}^{(0)}$;

(G2) $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ for all $\gamma \in \mathcal{G}$;

(G3) $r(\gamma^{-1}) = s(\gamma)$ and $s(\gamma^{-1}) = r(\gamma)$ for all $\gamma \in \mathcal{G}$;

(G4) $\gamma^{-1}\gamma = s(\gamma)$ and $\gamma\gamma^{-1} = r(\gamma)$ for all $\gamma \in \Gamma$;

(G5) $r(\alpha\beta) = r(\alpha)$ and $s(\alpha\beta) = s(\beta)$ whenever $s(\alpha) = r(\beta)$; and

(G6) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ whenever $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$.

Lemmas 2.1.2 and 2.1.4 show that every groupoid has these properties, and it is not hard to check that given the structure above, putting $\mathcal{G}^{(2)} := \{(\alpha, \beta) \in \mathcal{G} \times \mathcal{G} \mid s(\alpha) = r(\beta)\}$ yields a groupoid according to Definition 2.1.1. (The only thing to check is that (G1)–(G6) force $(\gamma^{-1})^{-1} = \gamma$; and since (G2), (G4) and (G6) ensure cancellativity just as in Lemma 2.1.3, this follows from the observation that (G3) and (G4) give $(\gamma^{-1})^{-1}\gamma^{-1} = s(\gamma^{-1}) = r(\gamma) = \gamma\gamma^{-1}$.) In the end, I decided to stick with Definition 2.1.1 because it is the definition that appears in the literature. But I still feel that (G1)–(G6), while not maximally efficient, are a good way to present groupoids to an audience unfamiliar with them and trying to come to grips with them: this presentation emphasises the point of view of a groupoid as a collection of reversible arrows, carrying an associative composition operation, between points in $\mathcal{G}^{(0)}$ (see Example 2.1.15 below).

Corollary 2.1.6. *If $\mathcal{G}$ is a groupoid then $\mathcal{G}^{(0)} = \{\gamma \in \mathcal{G} \mid (\gamma, \gamma) \in \mathcal{G}^{(2)} \text{ and } \gamma^2 = \gamma\}$.*

Proof. Lemma 2.1.2 combined with Lemma 2.1.4(3) gives $\subseteq$. For $\supseteq$, suppose that $(\gamma, \gamma) \in \mathcal{G}^{(2)}$ and $\gamma^2 = \gamma$. Then $\gamma^2 = \gamma = \gamma s(\gamma)$ by Lemma 2.1.2, and then Lemma 2.1.3 shows that $\gamma = s(\gamma) \in \mathcal{G}^{(0)}$. $\hfill\square$

Example 2.1.7. Every group Γ can be viewed as a groupoid, with $\Gamma^{(0)} = \{e\}$, multiplication given by the group operation, and inversion the usual group inverse. A groupoid is a group if and only if its unit space is a singleton.

Example 2.1.8 (Group bundles). Let X be a set, and for each $x \in X$, let Γ_x be a group. Let $\mathcal{G} := \bigcup_{x \in X} \{x\} \times \Gamma_x$. This is a groupoid with $\mathcal{G}^{(0)} = \{(x, e_{\Gamma_x}) \mid x \in X\}$ identified with X, $r(x, g) = x = s(x, g)$, $(x, g)(x, h) = (x, gh)$ and $(x, g)^{-1} = (x, g^{-1})$.

Example 2.1.9 (Matrix groupoids). Fix $N \geq 1$. Define

$$R_N := \{1, \ldots, N\} \times \{1, \ldots, N\}.$$

Put $R_N^{(0)} = \{(i, i) \mid i \leq N\}$, $r(i, j) = (i, i)$, $s(i, j) = (j, j)$ and $(i, j)(j, k) = (i, k)$. Then R_N is a groupoid, and $(i, j)^{-1} = (j, i)$ for all i, j. We usually identify $R_N^{(0)}$ with $\{1, \ldots, N\}$ in the obvious way.

Example 2.1.10. There was nothing special about $\{1, \ldots, N\}$. For any set X, the set $R_X := X \times X$ is a groupoid with operations analogous to those above. Again, we identify $R_X^{(0)}$ with X.

Example 2.1.11 (Equivalence relations). More generally again, if R is an equivalence relation on a set X, then $R^{(0)} := \{(x, x) \mid x \in X\}$ is contained in R by reflexivity; we identify $R^{(0)}$ with X again. The maps $r(x, y) = x$, $s(x, y) = y$, $(x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$ make R into a groupoid.

If R is an equivalence relation on X, then the map $\gamma \mapsto (r(\gamma), s(\gamma))$ from R to $R^{(0)} = X$ is the identity map from R to R.

Given groupoids $\mathcal{G}$ and $\mathcal{H}$, we call a map $\phi \colon \mathcal{G} \to \mathcal{H}$ a *groupoid homomorphism* if $(\phi \times \phi)(\mathcal{G}^{(2)}) \subseteq \mathcal{H}^{(2)}$ and $\phi(\alpha)\phi(\beta) = \phi(\alpha\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$.

Lemma 2.1.12. *If $\mathcal{G}$ and $\mathcal{H}$ are groupoids and $\phi \colon \mathcal{G} \to \mathcal{H}$ is a groupoid homomorphism, then $\phi(\mathcal{G}^{(0)}) \subseteq \mathcal{H}^{(0)}$. We have $\phi(r(\gamma)) = r(\phi(\gamma))$, $\phi(s(\gamma)) = s(\phi(\gamma))$ and $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ for all $\gamma \in \mathcal{G}$.*

Proof. For $u \in \mathcal{G}^{(0)}$ we have $\phi(u)^2 = \phi(u^2) = \phi(u)$, so $\phi(u)$ is idempotent and therefore a unit by Corollary 2.1.6. For $\gamma \in \mathcal{G}$ we have

$$\phi(\gamma)\phi(s(\gamma)) = \phi(\gamma s(\gamma)) = \phi(\gamma) = \phi(\gamma)s(\phi(\gamma))$$

and similarly $\phi(r(\gamma))\phi(\gamma) = r(\phi(\gamma))\phi(\gamma)$. So Lemma 2.1.3 shows that $\phi(s(\gamma)) = s(\phi(\gamma))$ and $\phi(r(\gamma)) = r(\phi(\gamma))$. We then have $\phi(\gamma)\phi(\gamma^{-1}) = \phi(\gamma\gamma^{-1}) = \phi(s(\gamma)) = s(\phi(\gamma))$, and so the uniqueness assertion in Lemma 2.1.2 shows that $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$. $\hfill\square$

Lemma 2.1.13. *If $\mathcal{G}$ is a groupoid, then $R = R(\mathcal{G}) \subseteq \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ defined by $R = \{(r(\gamma), s(\gamma)) \mid \gamma \in \mathcal{G}\}$ is an equivalence relation on $\mathcal{G}^{(0)}$, and $\gamma \mapsto (r(\gamma), s(\gamma))$ is a surjective groupoid homomorphism from $\mathcal{G}$ to R. We call R the* equivalence *relation of $\mathcal{G}$.*

Proof. We have $(x, x) = (r(x), s(x)) \in R$ for all $x \in \mathcal{G}^{(0)}$, so R is reflexive. We have $(s(\gamma), r(\gamma)) = (r(\gamma^{-1}), s(\gamma^{-1}))$ for each γ, so R is symmetric. And if $(x, y), (y, z) \in R$, say $x = r(\alpha)$, $y = s(\alpha)$, and $y = r(\beta)$, $z = s(\beta)$, then $(x, z) = (r(\alpha\beta), s(\alpha\beta)) \in R$, so R is transitive. That is, R is an equivalence relation. The map $\gamma \mapsto (r(\gamma), s(\gamma))$ is surjective by definition of R, and is a homomorphism by Lemma 2.1.4(1). $\quad\square$

We say that a groupoid $\mathcal{G}$ is *principal* if $\gamma \mapsto (r(\gamma), s(\gamma))$ is injective.

Lemma 2.1.14. *A groupoid $\mathcal{G}$ is algebraically isomorphic to an equivalence relation if and only if it is principal, in which case it is algebraically isomorphic to $R(\mathcal{G})$.*

Proof. We just saw that equivalence relations are always principal, so the "only if" implication is clear. For the "if" implication, suppose that $\mathcal{G}$ is principal. Lemma 2.1.13 shows that $\gamma \mapsto (r(\gamma), s(\gamma))$ is a surjective groupoid homomorphism onto $R(\mathcal{G})$, and it is injective because $\mathcal{G}$ is principal. $\quad\square$

From the preceding lemma, it may seem a little strange, in the first instance, to make the distinction between an "equivalence relation" and a "principal groupoid." Indeed, algebraically there is no difference. But when we start introducing topology into the mix the distinction makes sense. The term "equivalence relation" is reserved for principal groupoids that have the relative topology inherited from the product topology on $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, whereas "principal groupoids" may have finer topologies.

Example 2.1.15 (Transformation groupoids)**.** Let X be a set, and let Γ be a group acting on X by bijections. Let $\mathcal{G} := \Gamma \times X$, put $\mathcal{G}^{(0)} = \{e\} \times X$ (identified with X in the obvious way), define $r(g, x) := g \cdot x$ and $s(g, x) := x$, and define $(g, h \cdot x)(h, x) := (gh, x)$ and $(g, x)^{-1} := (g^{-1}, g \cdot x)$. Then $\mathcal{G}$ is a groupoid, called the *transformation groupoid.*

Example 2.1.16 (Deaconu–Renault groupoids)**.** Let X be a set, Γ an abelian group, and $S \subseteq \Gamma$ a subsemigroup of Γ that contains 0. Suppose that S acts on X in the sense that we have maps $x \mapsto s \cdot x$ from $X \to X$ satisfying $s \cdot (t \cdot x) = (s + t) \cdot x$, and $0 \cdot x = x$ for all x. Let

$$\mathcal{G} := \{(x, s - t, y) \mid s \cdot x = t \cdot y\}.$$

Put $\mathcal{G}^{(0)} = \{(x, e, x) \mid x \in X\}$ and identify it with X. Define $r(x, g, y) = x$ and $s(x, g, y) = y$, and put $(x, g, y)^{-1} = (y, -g, x)$. Suppose that $s_1 \cdot x = t_1 \cdot y$ and $s_2 \cdot y = t_2 \cdot z$. Then

$$(s_1 + s_2) \cdot x = s_2 \cdot (s_1 \cdot x) = s_2 \cdot (t_1 \cdot y) = (t_1 + s_2) \cdot y = t_1 \cdot (s_2 \cdot y) = t_1 \cdot (t_2 \cdot z) = (t_1 + t_2) \cdot z.$$

So we can define $(x, g, y)(y, h, z) = (x, gh, z)$. Under these operations, $\mathcal{G}$ is a groupoid.

2.2   Isotropy

Before discussing the isotropy of a groupoid, we introduce some standard notation.

Notation. For $x \in \mathcal{G}^{(0)}$ we write $\mathcal{G}_x \coloneqq \{\gamma \in \mathcal{G} \mid s(\gamma) = x\}$, and $\mathcal{G}^x \coloneqq \{\gamma \in \mathcal{G} \mid r(\gamma) = x\}$. In some articles you might see these sets denoted $\mathcal{G}x$ and $x\mathcal{G}$; this is helpful and sensible notation, but the literature is well-established and the superscript-subscript notation is quite standard, so we'll stick with it here. We write $\mathcal{G}^y_x \coloneqq \mathcal{G}_x \cap \mathcal{G}^y$.

We will, however, write UV for $\{\alpha\beta \mid \alpha \in U, \beta \in V, s(\alpha) = r(\beta)\}$ for any pair of subsets U, V of a groupoid $\mathcal{G}$.

If $\mathcal{G}$ is a groupoid, then the *isotropy subgroupoid* of $\mathcal{G}$, or just the isotropy of $\mathcal{G}$, is the subset $\mathrm{Iso}(\mathcal{G}) = \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}^x_x = \{\gamma \in \mathcal{G} \mid r(\gamma) = s(\gamma)\}$. It is straightforward to see that the isotropy subgroupoid really is a subgroupoid; indeed, it is a group bundle as in Example 2.1.8.

Clearly $\mathcal{G}^{(0)} \subseteq \mathrm{Iso}(\mathcal{G})$.

Lemma 2.2.1. *A groupoid $\mathcal{G}$ is principal if and only if $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$.*

Proof. If $\mathcal{G}$ is principal and $\gamma \in \mathrm{Iso}(\mathcal{G})$, then $x = r(\gamma)$ satisfies $(r(\gamma), s(\gamma)) = (r(x), s(x))$, and since $\mathcal{G}$ is principal it follows that $\gamma = x \in \mathcal{G}^{(0)}$. Now suppose that $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$, and that $(r(\gamma), s(\gamma)) = (r(\alpha), s(\alpha))$. Then $\alpha\gamma^{-1} \in \mathrm{Iso}(\mathcal{G})$ and therefore $\alpha\gamma^{-1} = r(\alpha)$. So Lemma 2.1.2 forces $\alpha = \gamma$. $\qquad\square$

Example 2.2.2. If X is a set, and Γ is a group acting on X, then, as usual, the isotropy subgroup of Γ at $x \in X$ is $\Gamma_x \coloneqq \{g \in \Gamma \mid g{\cdot}x = x\}$. The isotropy subgroupoid of the transformation groupoid $\mathcal{G}$ is then $\bigcup_{x \in X} \{x\} \times \Gamma_x$, the union of the isotropy subgroups associated to points $x \in X$.

Lemma 2.2.3. *If $\mathcal{G}$ is a groupoid and $\gamma \in \mathcal{G}$, then the map $\mathrm{Ad}_\gamma : \alpha \mapsto \gamma\alpha\gamma^{-1}$ is a group isomorphism from $\mathcal{G}^{s(\gamma)}_{s(\gamma)}$ to $\mathcal{G}^{r(\gamma)}_{r(\gamma)}$.*

Proof. The map $\mathrm{Ad}_{\gamma^{-1}}$ is plainly an inverse for Ad_γ; and

$$\mathrm{Ad}_\gamma(\alpha)\,\mathrm{Ad}_\gamma(\beta) = \gamma\alpha\gamma^{-1}\gamma\beta\gamma^{-1} = \mathrm{Ad}_\gamma(\alpha\beta). \qquad\qquad\square$$

2.3   Topological groupoids

Since we are going to be interested here in C^*-algebras, we will want to topologise our groupoids.

Definition 2.3.1. A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a locally compact topology under which $\mathcal{G}^{(0)} \subseteq \mathcal{G}$ is Hausdorff in the relative topology, the maps r, s and $\gamma \mapsto \gamma^{-1}$ are continuous, and the map $(g, h) \to gh$ is continuous with respect to the relative topology on $\mathcal{G}^{(2)}$ as a subset of $\mathcal{G} \times \mathcal{G}$.

We naturally expect that the unit space should be closed in a topological groupoid; but in fact, this is true only when $\mathcal{G}$ is Hausdorff.

Lemma 2.3.2. *If $\mathcal{G}$ is a topological groupoid, then $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$ if and only if $\mathcal{G}$ is Hausdorff.*

Proof. First suppose that $\mathcal{G}$ is Hausdorff, and suppose that $(x_i)_{i \in I}$ is a net in $\mathcal{G}^{(0)}$ such that $x_i \to \gamma \in \mathcal{G}$. Since r is continuous, we have $x_i = r(x_i) \to r(\gamma) \in \mathcal{G}^{(0)}$. Since $\mathcal{G}$ is Hausdorff, this limit point is unique, and we deduce that $\gamma = r(\gamma)$.

Now suppose that $\mathcal{G}^{(0)}$ is closed. To see that $\mathcal{G}$ is Hausdorff, it suffices to show that convergent nets have unique limit points. For this, suppose that $(\gamma_i)_{i \in I}$ is a net and that $\gamma_i \to \alpha$ and $\gamma_i \to \beta$. By continuity, we then have $\gamma_i^{-1}\gamma_i \to \alpha^{-1}\beta$. Since each $\gamma_i^{-1}\gamma_i = s(\gamma_i) \in \mathcal{G}^{(0)}$ and $\mathcal{G}^{(0)}$ is closed, we deduce that $\alpha^{-1}\beta \in \mathcal{G}^{(0)}$. Hence $\alpha = \beta$. □

Although there are many interesting and important examples without these properties, in these notes, all the topological groupoids that I discuss will be second-countable and Hausdorff as topological spaces. So in these notes, $\mathcal{G}^{(0)}$ is always a closed subset of $\mathcal{G}$.

Example 2.3.3 (Discrete groupoids). Every groupoid is a topological groupoid in the discrete topology.

Example 2.3.4 (Topological equivalence relations). If X is a second-countable Hausdorff space, and R is an equivalence relation on X, then R is a topological groupoid in the relative topology inherited from $X \times X$.

The previous example, combined with Lemma 2.1.14, shows that if $\mathcal{G}$ is a principal groupoid, then any second-countable Hausdorff topology on $\mathcal{G}^{(0)}$ induces a topological-groupoid structure on $\mathcal{G}$. Moreover, if $\mathcal{G}$ is a principal topological groupoid, then $\gamma \mapsto (r(\gamma), s(\gamma))$ is a bijective continuous map from $\mathcal{G}$ to the topological equivalence relation $R(\mathcal{G})$. So the topology on $\mathcal{G}$ must be finer than the one inherited from the product topology on $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. It can be strictly finer:

Example 2.3.5. Let $X := \prod_{i=1}^{\infty}\{0,1\}$, viewed as right-infinite strings of 0's and 1's, and given the product topology. Define an equivalence relation R on X by $(x,y) \in R$ if and only if there exists $n \in \mathbb{N}$ such that $x_j = y_j$ for all $j \geq n$. Let $\mathcal{R}_{2^\infty} := \{(x,y) \mid x \sim y\}$ be an algebraic copy of R. For finite words $v, w \in \{0,1\}^n$, define $Z(v,w) = \{(vx, wx) \mid x \in X\} \subseteq \mathcal{R}_{2^\infty}$. Observe that

$$Z(v,w) \cap Z(v',w') = \begin{cases} Z(v,w) & \text{if } v = v'u \text{ and } w = w'u \text{ for some } u \\ Z(v',w') & \text{if } v' = vu \text{ and } w' = wu \text{ for some } u \\ \varnothing & \text{otherwise.} \end{cases}$$

So the $Z(v,w)$ form a base for a topology.

We claim that $\mathcal{R}_{2^\infty}$ is a topological groupoid in this topology. It is Hausdorff because if $(x,y) \neq (x',y')$ then there is an n such that $\big(x(0,n), y(0,n)\big) \neq \big(x'(0,n), y'(0,n)\big)$, and then $Z\big(x(0,n), y(0,n)\big)$ and $Z\big(x'(0,n), y'(0,n)\big)$ separate these points. The sets $Z(v,w)$ are also compact: the map $x \mapsto (vx, wx)$ is a bijective continuous map from the compact space X to the Hausdorff space $Z(v,w)$, and therefore a homeomorphism. The maps r, s restrict to homeomorphisms of $Z(v,w)$ onto $Z(v)$ and $Z(w)$, so they are continuous. Inversion is clearly continuous. Multiplication is continuous because the pre-image of $Z(v,w)$ is $\bigcup_{|y|=|v|} \bigcup_u \big(Z(vu, yu) \times Z(yu, wu)\big) \cap \mathcal{R}_{2^\infty}$. This proves the claim.

We now claim that this topology is finer than the one inherited from the product topology. To see this, consider the sequence $\gamma_n = \big((0^n 10^\infty, 0^\infty)\big)_{n=1}^\infty$. In the product topology, we have $\gamma_n \to (0^\infty, 0^\infty)$. But the basic neighbourhoods of $(0^\infty, 0^\infty)$ are of the form $Z(0^m, 0^m)$, and we have $\gamma_n \notin Z(0^m, 0^m)$ for $n > m$. So $\gamma_n \not\to (0^\infty, 0^\infty)$ in $\mathcal{R}_{2^\infty}$.

As we will see later, this example is important: its C^*-algebra is $M_{2^\infty}(\mathbb{C})$.

Example 2.3.6. Suppose that X is a second-countable locally compact Hausdorff space and that Γ is a locally compact group acting on X by homeomorphisms as in [52, Section 1.1]. Then the transformation groupoid $\mathcal{G}$ is a topological groupoid in the product topology.

In these notes, a *local homeomorphism* from X to Y is a continuous map $h : X \to Y$ such that every $x \in X$ has an open neighbourhood U such that $h(U) \subseteq Y$ is open and $h : U \to h(U)$ is a homeomorphism.

Example 2.3.7. If X is a second-countable locally compact Hausdorff space, Γ is a discrete abelian group, S is a subsemigroup of Γ containing 0, and S acts on X by local homeomorphisms, then the Deaconu–Renault groupoid becomes a locally compact Hausdorff groupoid in the topology with basic open sets

$$Z(U, p, q, V) = \{(x, p - q, y) \mid x \in U, y \in V, p \cdot x = q \cdot y\}$$

indexed by pairs U, V of open subsets of X and pairs $p, q \in S$.

2.4   Étale groupoids

In these notes, we will focus on étale groupoids. These are the analogue, in the groupoid world, of discrete groups.

Definition 2.4.1. A topological groupoid $\mathcal{G}$ is *étale* if the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

The first important consequence of this condition is the following.

Lemma 2.4.2. *If $\mathcal{G}$ is an étale groupoid, then $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.*

Proof. We prove the contrapositive: that is, we suppose that $\mathcal{G}^{(0)}$ is not open in $\mathcal{G}$ and show that r is not a local homeomorphism. Since $\mathcal{G}^{(0)}$ is not open, there is a sequence $(\gamma_n)_{n=1}^{\infty}$ in $\mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $\gamma_n \to x \in \mathcal{G}^{(0)}$. Since r is continuous, $r(\gamma_n) \to u$ as well. Fix an open neighbourhood U of x in $\mathcal{G}$. Then there exists n such that γ_n and $r(\gamma_n)$ both belong to U. Since γ_n is not a unit, we have $\gamma_n \neq r(\gamma_n)$, but $r(\gamma_n) = r(r(\gamma_n))$. Hence $r|_U$ is not injective, and we deduce that r is not locally injective and therefore not a local homeomorphism. $\qquad\square$

Example 2.4.3. Every discrete groupoid is étale.

Example 2.4.4. The principal groupoid of Example 2.3.5 is étale: we proved that r is a homeomorphism of $Z(u,v)$ onto $Z(u) \subseteq \mathcal{G}^{(0)}$ for each (u,v).

Example 2.4.5. A transformation groupoid is étale if and only if the acting group Γ is discrete.

Example 2.4.6. Deaconu–Renault groupoids associated to actions by local homeomorphisms are always étale: For each $(x,p,q,y) \in \mathcal{G}$ we can choose open neighbourhoods U of x and V of y such that $u \mapsto p \cdot u$ is a homeomorphism from U to $p \cdot U$, and $v \mapsto q \cdot v$ is a homeomorphism of V onto $p \cdot V$. Let $W = p \cdot U \cap q \cdot V$ and let $U' := \{u \in U \mid p \cdot u \in W\}$ and $V' := \{v \in V \mid q \cdot v \in W\}$. Then $Z(U',p,q,V')$ is an open neighbourhood of (x,p,q,y), and the range map restricts to a homeomorphism of this neighbourhood onto U'.

Example 2.4.7 (Graph groupoids)**.** Let E be a directed graph with vertex set E^0, edge set E^1 and direction of edges described by range and source maps $r,s : E^1 \to E^0$. Assume that E is row-finite with no sources (so $r^{-1}(v)$ is finite and nonempty for every vertex v). See [33] for background on graphs in the context of graph C^*-algebras; we will use Raeburn's conventions and notation for graphs throughout. Let E^{∞} denote the space of right-infinite paths in E, so $E^{\infty} = \{x_1 x_2 x_3 \cdots \mid x_i \in E^1, r(x_{i+1}) = s(x_i)\}$. Give E^{∞} the topology inherited from the product space $\prod_{i=1}^{\infty} E^1$. Then E^{∞} is a totally-disconnected locally compact Hausdorff space, and the sets $Z(\mu) = \{\mu x \mid x \in E^{\infty}, r(x_1) = s(\mu)\}$ form a base of compact open sets for the topology. The map $\sigma : E^{\infty} \to E^{\infty}$ given by $\sigma(x)_i = x_{i+1}$ is a local homeomorphism (it restricts to a homeomorphism on $Z(\mu)$ whenever $|\mu| \geq 1$), so induces an action of $\mathbb{N}$ by local homeomorphisms. The associated Deaconu–Renault groupoid $\mathcal{G}_E = \{(x, m-n, y) \mid \sigma^m(x) = \sigma^n(y)\}$ is called the *graph groupoid* of E.

Since $\gamma \mapsto \gamma^{-1}$ is continuous and self-inverse, if $\mathcal{G}$ is étale, then $s : \mathcal{G} \to \mathcal{G}^{(0)}$ is also a local homeomorphism. So there are plenty of open sets on which r, s are both homeomorphisms.

Definition 2.4.8. A subset B of an étale groupoid $\mathcal{G}$ is a *bisection* if there is an open set U containing B such that $r : U \to r(U)$ and $s : U \to s(U)$ are both homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$.

Lemma 2.4.9. *Let $\mathcal{G}$ be a second-countable Hausdorff étale groupoid. Then $\mathcal{G}$ has a countable base of open bisections.*

Proof. Choose a countable dense subset $\{\gamma_n\}$ of $\mathcal{G}$. For each γ_n, choose countable neighbourhood bases $\{U_{n,i}\}_i$ and $\{V_{n,i}\}_i$ at γ_n such that r is a homeomorphism of each $U_{n,i}$ onto an open set, and s is a homeomorphism of each $V_{n,i}$ onto an open set. Then $\{U_{n,i} \cap V_{n,i} \mid n, i \in \mathbb{N}\}$ is a countable base of open bisections.   □

Corollary 2.4.10. *If $\mathcal{G}$ is an étale groupoid, then each $\mathcal{G}_x$ and each $\mathcal{G}^x$ is discrete in the relative topology.*

Proof. For each $\gamma \in \mathcal{G}_x$, choose an open bisection U_γ containing γ. Then $U_\gamma \cap \mathcal{G}_x = \{\gamma\}$, and so $\{\gamma\}$ is open in $\mathcal{G}_x$.   □

We finish this section with the important observation that multiplication is an open map in an étale groupoid. The following quick proof was shown to me by Dana Williams.

Lemma 2.4.11. *If $\mathcal{G}$ is a topological groupoid and r is an open map, then the multiplication map on $\mathcal{G}$ is open. In particular, if $\mathcal{G}$ is étale, then multiplication is an open map.*

Proof. Fix open sets $U, V \subseteq \mathcal{G}$, and an element $(\alpha, \beta) \in U \times V \cap \mathcal{G}^{(2)}$. Fix a sequence γ_i converging to $\alpha\beta$; it suffices to show that the γ_i eventually belong to UV. Fix a descending neighbourhood base $\{U_j\}_{j \in \mathbb{N}}$ for α contained in U. Since r is an open map, each $r(U_j)$ is an open neighbourhood of $r(\alpha)$. Since $\gamma_i \to \alpha\beta$, we have $r(\gamma_i) \to r(\alpha\beta) = r(\alpha)$, so for each j we eventually have $r(\gamma_i) \in r(U_j)$. Choose α_i in U with $r(\alpha_i) = r(\gamma_i)$ and $\alpha_i \in U_j$ whenever $r(\gamma_i) \in r(U_j)$. Then $\alpha_i \to \alpha$. Hence $\alpha_i^{-1}\gamma \to \beta$, and therefore $\alpha_i^{-1}\gamma \in V$ for large i. But then $\gamma_i = \alpha_i(\alpha_i^{-1}\gamma) \in UV$ for large i.   □

Chapter 3

C^*-algebras and equivalence

In this section, we will associate two C^*-algebras to each étale groupoid. As with groups and dynamical systems, each groupoid has both a reduced C^*-algebra and a full C^*-algebra. We first discuss the convolution product on $C_c(\mathcal{G})$ and then its two key C^*-completions, $C^*(\mathcal{G})$ and $C_r^*(\mathcal{G})$. At the end of the chapter, we discuss equivalence of groupoids and Renault's equivalence theorem for their C^*-algebras.

3.1 The convolution algebra

Proposition 3.1.1. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For $f, g \in C_c(\mathcal{G})$ and $\gamma \in \mathcal{G}$, the set*

$$\{(\alpha, \beta) \in \mathcal{G}^{(2)} \mid \alpha\beta = \gamma \ \text{and} \ f(\alpha)g(\beta) \neq 0\}$$

is finite. The complex vector space $C_c(\mathcal{G})$ is a *-algebra with multiplication given by $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$ and involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$. For $f, g \in C_c(\mathcal{G})$ we have $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f)\operatorname{supp}(g)$.

Proof. If $\alpha\beta = \gamma$, then $\alpha \in G^{r(\gamma)}$ and $\beta \in G_{s(\gamma)}$. We saw in Corollary 2.4.10 that these are discrete sets, so their intersections with the compact sets $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are finite. The rest is routine. $\quad\square$

Remark 3.1.2. The convolution formula can be (and often is) equivalently reformulated as

$$(f * g)(\gamma) = \sum_{\alpha \in \mathcal{G}^{r(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma).$$

We will see later that the multiplication operation, and also the C^*-norms, are simpler for elements of $C_c(\mathcal{G})$ whose supports are contained in a bisection than for general elements. So it's helpful to know that when $\mathcal{G}$ is étale, such functions span the convolution algebra.

Lemma 3.1.3. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. Then*

$$C_c(\mathcal{G}) = \operatorname{span}\{f \in C_c(\mathcal{G}) \mid \operatorname{supp}(f) \text{ is a bisection}\}.$$

Proof. Fix $f \in C_c(\mathcal{G})$. By Lemma 2.4.9, we can cover $\operatorname{supp}(f)$ with open bisections, and then use compactness to pass to a finite subcover $U_1, \ldots, U_n$. Choose a partition of unity $\{h_i\}$ on $\bigcup U_i$ subordinate to the U_i. The pointwise products $f_i := f \cdot h_i$ belong to $C_c(\mathcal{G})$ with $\operatorname{supp}(f_i) \subseteq U_i$, and we have $f = \sum_i f_i$.   □

One reason why the preceding lemma is so useful is because convolution is very easy to compute for functions supported on bisections.

Lemma 3.1.4. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. If $U, V \subseteq \mathcal{G}$ are open bisections and $f, g \in C_c(\mathcal{G})$ satisfy $\operatorname{supp}(f) \subseteq U$ and $\operatorname{supp}(g) \subseteq V$, then $\operatorname{supp}(f * g) \subseteq UV$ and for $\gamma = \alpha\beta \in UV$, we have*

$$(f * g)(\gamma) = f(\alpha)g(\beta). \tag{3.1}$$

We have $C_c(\mathcal{G}^{(0)}) \subseteq C_c(\mathcal{G})$. If $f \in C_c(\mathcal{G})$ is supported on a bisection, then $f^* * f \in C_c(\mathcal{G}^{(0)})$ is supported on $s(\operatorname{supp}(f))$ and $(f^* * f)(s(\gamma)) = |f(\gamma)|^2$ for all $\gamma \in \operatorname{supp}(f)$. Similarly $f * f^*$ is supported on $r(\operatorname{supp}(f))$ and $(f * f^*)(r(\gamma)) = |f(\gamma)|^2$ for $\gamma \in \operatorname{supp}(f)$. For $f \in C_c(\mathcal{G})$ and $h \in C_c(\mathcal{G}^{(0)})$, we have

$$(h * f)(\gamma) = h(r(\gamma))f(\gamma) \quad \text{and} \quad (f * h)(\gamma) = f(\gamma)h(s(\gamma)).$$

Proof. We have $f(\gamma) = \sum_{\eta\zeta=\gamma} f(\eta)g(\zeta)$. For any η, ζ appearing in the sum, we have $r(\eta) = r(\gamma)$ and $s(\zeta) = s(\gamma)$. Since f and g are supported on bisections, and since $\alpha \in \mathcal{G}^{r(\gamma)}$ and $\beta \in \mathcal{G}^{s(\gamma)}$ are contained in $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ respectively, it follows that the only term in the sum that can be nonzero is $f(\alpha)g(\beta)$.

Since $\mathcal{G}^{(0)}$ is open by Lemma 2.4.2, we can regard $C_c(\mathcal{G}^{(0)})$ as a subalgebra of $C_c(\mathcal{G})$ in the usual way: for $f \in C_c(\mathcal{G}^{(0)})$ the corresponding element of $C_c(\mathcal{G})$ agrees with f on $\mathcal{G}^{(0)}$ and vanishes on its complement. The remaining statements follow from the convolution formula (3.1).   □

Example 3.1.5. Consider the matrix groupoid R_N. Since this is a finite discrete groupoid, we have $C_c(R_N) = \operatorname{span}\{1_{(i,j)} \mid i,j \leq N\}$. Lemma 3.1.4 shows that $1_{(i,j)}1_{(k,l)} = \delta_{j,k}1_{(i,l)}$. So the $1_{(i,j)}$ are matrix units, and $C_c(R_N) \cong M_N(\mathbb{C})$.

Example 3.1.6. If Γ is a discrete group, regarded as a groupoid, then its convolution algebra as described above is identical to the usual group algebra $C_c(\Gamma)$.

Example 3.1.7. Let Γ be a discrete group acting on a compact space X, and let $\mathcal{G}$ be the associated transformation groupoid. Let α be the action of Γ on $C(X)$ induced by the Γ action on X. Let $C_c(\Gamma, C(X))$ be the convolution algebra of the C^*-dynamical system $(C(X), \Gamma, \alpha)$ described in [52, Section 1.3.2]. Then there is an isomorphism $\omega : C_c(\mathcal{G}) \to C_c(\Gamma, C(X))$ given by $\omega(f)(g)(x) = f(g, x)$ for all $f \in C_c(\mathcal{G})$, all $g \in \Gamma$ and all $x \in X$.

3.2 The full C^*-algebra

There are two ways to describe the full C^*-algebra of a discrete group. The first is as the universal C^*-algebra generated by a unitary representation of Γ. The second is as the universal C^*-algebra generated by a *-representation of $C_c(\Gamma)$.

There is a version of the first description for groupoids, which we will discuss briefly later; but it's a little technical. The second description, on the other hand, generalises nicely, and is the one we'll make use of in these notes—a luxury that we can afford because we are sticking to étale groupoids throughout. The following elementary construction of the full C^*-norm on the convolution algebra of an étale groupoid was shown to me by Robin Deeley. It appears, for non-Hausdorff groupoids, in Ruy Exel's paper [18, Definition 3.17]; thanks to Eduardo Scarparo for reminding me.

Proposition 3.2.1. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For each $f \in C_c(\mathcal{G})$, there is a constant $K_f \geq 0$ such that $\|\pi(f)\| \leq K_f$ for every *-representation $\pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of $C_c(\mathcal{G})$ on Hilbert space. If f is supported on a bisection, we can take $K_f = \|f\|_\infty$.*

To prove Proposition 3.2.1, we need the following very standard lemma. Thanks to Mikael Rørdam for suggesting this clever little proof.

Lemma 3.2.2. *Let X be a locally compact Hausdorff space, let A be a C^*-algebra, and let $\pi : C_c(X) \to A$ be a *-homomorphism. Them $\|\pi(f)\| \leq \|f\|_\infty$ for all $f \in C_c(X)$.*

Proof. Fix $f \in C_c(X)$. Let $U := \{x \in X : f(x) \neq 0\}$. Then π restricts to a homomorphism from the C^*-algebra $C_0(U) \subseteq C_c(X)$ to A. Since C^*-homomorphisms are norm-decreasing, $\pi|_{C_0(U)}$ is norm-decreasing, and since $f \in C_0(U)$, we deduce that $\|\pi(f)\| \leq \|f\|_\infty$. □

Proof. Fix $f \in C_c(\mathcal{G})$ and use Lemma 3.1.3 to write $f = \sum_{i=1}^n f_i$ with each f_i supported on a bisection. Define $K_f := \sum_{i=1}^n \|f_i\|_\infty$.

Suppose that π is a *-representation. Then $\pi|_{C_c(\mathcal{G}^{(0)})}$ is a *-representation, so Lemma 3.2.2 implies that $\|\pi(h)\| \leq \|h\|_\infty$ for every $h \in C_c(\mathcal{G}^{(0)})$. Lemma 3.1.4 implies first that each $f_i^* * f_i$ is supported on $\mathcal{G}^{(0)}$ and second that $\|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2$. So

$$\|\pi(f_i)\|^2 = \|\pi(f_i^* * f_i)\| \leq \|f_i^* * f_i\|_\infty = \|f_i\|_\infty^2,$$

and so each $\|\pi(f_i)\| \leq \|f_i\|_\infty$. Now the triangle inequality gives $\|\pi(f)\| \leq K_f$. If f is supported on a bisection, then there is just one term in the sum, so $K_f = \|f\|_\infty$. □

This allows us to define the universal C^*-algebra of an étale groupoid.

Theorem 3.2.3. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. There exist a C^*-algebra $C^*(\mathcal{G})$ and a *-homomorphism $\pi_{\max} : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ such that $\pi_{\max}(C_c(\mathcal{G}))$ is dense in $C^*(\mathcal{G})$, and such that for*

every *-representation $\pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ there is a representation ψ of $C^*(\mathcal{G})$ such that $\psi \circ \pi_{\max} = \pi$. The norm on $C^*(\mathcal{G})$ satisfies

$$\|\pi_{\max}(f)\| = \sup \big\{\|\pi(f)\| \,\big|\, \pi \text{ is a } ^*\text{-representation of } C_c(\mathcal{G})\big\}$$

for all $f \in C_c(\mathcal{G})$.

Proof. For each $a \in C_c(\mathcal{G})$, Proposition 3.2.1 shows that the set

$$\{\pi(a) \,|\, \pi \text{ is a } ^*\text{-representation of } C_c(\mathcal{G})\}$$

is bounded above, and it is nonempty because of the zero representation. So we can define $\rho : C_c(\mathcal{G}) \to [0, \infty)$ by

$$\rho(f) := \sup \big\{\|\pi(f)\| \,\big|\, \pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ is a } ^*\text{-representation}\big\}.$$

It is routine to check that ρ is a *-seminorm satisfying the C^*-identity using that each $f \mapsto \|\pi(f)\|$ has the same properties. So we can define $C^*(\mathcal{G})$ to be the completion of the quotient of $C_c(\mathcal{G})$ by $N := \big\{f \in C_c(\mathcal{G}) \,\big|\, \|f\| = 0\big\}$ in the pre-C^*-norm $\|\cdot\|$ induced by ρ. We define $\pi_{\max}(f) := f + N \in C^*(\mathcal{G})$.

By construction of ρ, if π is a *-representation of $C^*(\mathcal{G})$, then $\|\pi(f)\| \leq \rho(f) = \|\pi_{\max}(f)\|$ for all $f \in C_c(\mathcal{G})$. This implies that there is a well-defined norm-decreasing linear map $\psi : C^*(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ satisfying $\psi \circ \pi_{\max} = \pi$. Continuity then ensures that this ψ is a C^*-homomorphism. $\qquad\square$

Of course, we expect that $\pi_{\max}$ is injective; we will prove this in the next section.

It is not immediately obvious that the norm defined in Theorem 3.2.3 agrees with Renault's definition. This is because, to deal with non-étale groupoids, Renault defines the universal norm, not as the supremum over all *-representations of $C_c(\mathcal{G})$, but as the supremum only over *-representations of $C_c(\mathcal{G})$ that are bounded with respect to the "I-norm" on $C_c(\mathcal{G})$. When $\mathcal{G}$ is étale, the I-norm is given by

$$\|f\|_I = \sup_{x \in \mathcal{G}^{(0)}} \max \Big\{ \sum_{\gamma \in \mathcal{G}_x} |f(\gamma)|, \sum_{\gamma \in \mathcal{G}^x} |f(\gamma)| \Big\}.$$

(Think of it like a fibrewise 1-norm.) Renault shows that boundedness in the I-norm is equivalent to continuity in the inductive-limit topology on $C_c(\mathcal{G})$: the topology obtained by regarding $C_c(\mathcal{G})$ as the inductive limit of the subspaces $X_K := \big(\{f \in C(\mathcal{G}) \mid \mathrm{supp}(f) \subseteq K\}, \|\cdot\|_\infty\big)$ indexed by compact subsets K of $\mathcal{G}$ (see, for example, [13, Definition 5.4 and Example 5.10]). In the general non-étale setting, this equivalence between I-norm boundedness and continuity in the inductive-limit topology is nontrivial, and requires an appeal to Renault's Disintegration Theorem, which we will discuss later.

So to make sure we are talking about the same universal C^*-algebra as Renault, we must verify that every *-representation of $C_c(\mathcal{G})$ is continuous in the

inductive-limit topology when $\mathcal{G}$ is étale; and for completeness we should also prove that continuity in this topology implies boundedness with respect to the I-norm. The nice proof of this latter fact given below was developed by Ben Maldon in his honours thesis; it does not seem to have appeared in the literature previously.

Lemma 3.2.4. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. Then every *-representation π of $C_c(\mathcal{G})$ is both continuous in the inductive-limit topology, and bounded in the I-norm.*

Proof. Fix a nondegenerate *-representation of $C_c(\mathcal{G})$. By, for example, [13, Proposition 5.7], to see that π is continuous in the inductive-limit topology, we just have to check that $\pi|_{X_K}$ is continuous for each compact $K \subseteq \mathcal{G}$. To see this, fix a compact $K \subseteq \mathcal{G}$. We can cover K by open bisections, and then use compactness to obtain a finite subcover $K \subseteq \bigcup_{i=1}^n U_i$. Fix a partition of unity $\{h_i\}$ for K subordinate to the U_i. Then for $f \in X_k$, Lemma 3.2.1 gives

$$\|\pi(f)\| = \Big\| \sum_i \pi(h_i \cdot f) \Big\| \le \sum_{i=1}^n \|\pi(h_i \cdot f)\| \le \sum_{i=1}^n \|h_i \cdot f\|_\infty \le n\|f\|_\infty.$$

So π is Lipschitz on X_K with Lipschitz constant at most n. This shows that π is continuous in the inductive-limit topology.

To see that it is I-norm bounded, observe that if $f \in C_c(\mathcal{G})$, then $\|f\|_\infty \le \|f\|_I$. So the inductive-limit topology is weaker than the I-norm topology, and we deduce that π is continuous for the I-norm. Since continuity is equivalent to boundedness for linear maps on normed spaces, we deduce that π is I-norm bounded. Routine calculations show that the I-norm is a *-algebra norm, so π extends to a *-homomorphism from the Banach *-algebra completion $\overline{C_c(\mathcal{G})}^I$ of $C_c(\mathcal{G})$ in the I-norm into $\mathcal{B}(\mathcal{H})$. Now we can apply spectral theory: Write $\rho_A : A \to [0,\infty)$ for the spectral-radius function on a Banach algebra A. For each $f \in C_c(\mathcal{G})$, we have $\|\pi(f)\|^2 = \|\pi(f^*f)\| = \rho_{\mathcal{B}(\mathcal{H})}(\pi(f^*f)) \le \rho_{\overline{C_c(\mathcal{G})}^I}(f^*f) \le \|f^*f\|_I \le \|f\|_I^2.$ □

We will not really need the preceding result from here on in, but we can take comfort that we are discussing the same family of representations as Renault is; so we can appeal to his theorems at need.

Example 3.2.5. If $\mathcal{G}$ is a group, then $C^*(\mathcal{G})$ is the usual full group C^*-algebra.

Example 3.2.6. Let X be a compact Hausdorff space, and Γ a discrete group acting on X. Then the elements $U_g := 1_{\{g\} \times X}$ indexed by $g \in \Gamma$ belong to $C_c(\mathcal{G})$, and there is an inclusion $\pi : C(\mathcal{G}^{(0)}) \to C_c(\mathcal{G})$ such that $\pi(f)(g,x) = \delta_{e,g} f(x)$. For $f \in C(X)$ and $g \in \Gamma$ we have $U_g \pi(f) U_g^*(h,x) = \sum_{\alpha\beta\gamma=(h,x)} 1_{\{g\} \times X}(\alpha)\pi(f)(\beta) 1_{\{g\} \times X}^*(\gamma)$. This can only be nonzero if $h = 0$, and then the only nonzero term occurs when $\alpha = (g, g^{-1} \cdot x)$ and $\gamma = (g^{-1}, x)$. For this α, β we have $1_{\{g\} \times X}(\alpha)\pi(f)(\beta) 1_{\{g\} \times X}^*(\gamma) = \pi(f)(e, \gamma^{-1} \cdot x) = f(\gamma^{-1} \cdot x)$. So $U_g \pi(f) U_g^* \in \pi(C(X))$, and agrees with $\pi(x \mapsto$

$f(g^{-1} \cdot x))$. So the universal property of the crossed product $C(X) \rtimes \Gamma$ (see [52, Theorem 1.3.3] and [51, Theorem 2.6.1]) gives a homomorphism $C(X) \rtimes \Gamma \to C^*(\mathcal{G})$ that takes each $i_\Gamma(g)$ to U_g and each $i_{C(X)}(f)$ to $\pi(f)$. Conversely, for $f \in C_c(\mathcal{G})$ and $g \in \mathcal{G}$, define $f_g \in C(X)$ by $f_g(x) = f(g, x)$. Easy calculations show that the map $\psi(f) := \sum_{g \in \Gamma} i_{C(X)}(f_g) i_\Gamma(g)$ gives a *-homomorphism $\psi : C_c(\mathcal{G}) \to C(X) \rtimes \Gamma$, and that ψ is inverse to $\pi \times U$. So ψ is an isomorphism $C^*(\mathcal{G}) \cong C(X) \rtimes \Gamma$.

Example 3.2.7. Consider the groupoid $\mathcal{R}_{2^\infty}$ of Example 2.3.5. For each $n \geq 0$ and each pair $u, v \in \{0, 1\}^n$, let $\theta_n(u, v) := 1_{Z(u,v)} \in C_c(\mathcal{R}_{2^\infty})$. Easy calculations using Lemma 3.1.4 show that $\theta_n(u, v)^* = \theta_n(v, u)$ and $\theta_n(u, v)\theta_n(w, y) = \delta_{v,w}\theta_n(u, y)$. So $A_n := \text{span}\{\theta_n(u, v) \mid u, v \in \{0, 1\}^n\}$ is isomorphic to $M_{\{0,1\}^n}(\mathbb{C})$ via the map $\sum_{u,v} a_{u,v}\theta_n(u, v) \mapsto (a_{u,v})_{u,v \in \{0,1\}^n}$. Since $Z(u, v) = Z(u0, v0) \sqcup Z(u1, v1)$, we have $\theta_n(u, v) = \theta_{n+1}(u0, v0) + \theta_{n+1}(u1, v1)$, and so $A_n \subseteq A_{n+1}$. If we identify $M_{\{0,1\}^{n+1}}(\mathbb{C})$ with $M_2(M_{\{0,1\}^n}(\mathbb{C}))$ via

$$(a_{u,v})_{u,v \in \{0,1\}^{n+1}} \mapsto \big((a_{wi,yj})_{w,y \in \{0,1\}^n}\big)_{i,j \in \{0,1\}},$$

then the inclusion $A_n \to A_{n+1}$ is compatible with the canonical block-diagonal inclusion $M_{\{0,1\}^n}(\mathbb{C}) \to M_{\{0,1\}^{n+1}}(\mathbb{C})$ with multiplicity 2. So the uniqueness of M_{2^∞} shows that $\overline{\bigcup_n A_n} \cong M_{2^\infty}$. A straightforward application of the Stone-Weierstrass theorem shows that $C(Z(u, v)) \subseteq \overline{\bigcup_n A_n}$ for all $|u| = |v|$, and we deduce that that $C_c(R_{2^\infty}) \subseteq \overline{\bigcup_n A_n}$. Hence $C^*(R_{2^\infty}) \cong M_{2^\infty}(\mathbb{C})$.

Example 3.2.8. If $\mathcal{G}$ is the graph groupoid of a directed graph E as in Example 2.4.7, then the characteristic functions $\{1_{Z(v,v)} \mid v \in E^0\}$ and $\{1_{Z(e,s(e))} \mid e \in E^1\}$ constitute a Cuntz–Krieger family for E that generates $C^*(\mathcal{G})$. The homomorphism $C^*(E) \to C^*(\mathcal{G})$ induced by this family is an isomorphism.

Remark 3.2.9 (Unitary representations)**.** To make sense of a unitary representation of $\mathcal{G}$, we proceed, very roughly, as follows. A *unitary representation* of $\mathcal{G}$ is a triple $(\mathcal{H}, \mu, U)$ where μ is a Borel measure on $\mathcal{G}^{(0)}$, $\mathcal{H} = \bigsqcup_{x \in \mathcal{G}^{(0)}} \mathcal{H}_x$ is a μ-measurable field of Hilbert spaces, and $U = \{U_\gamma \mid \gamma \in \mathcal{G}\}$ is a family of unitary operators $U_\gamma : \mathcal{H}_{s(\gamma)} \to \mathcal{H}_{r(\gamma)}$ satisfying $U_\alpha U_\beta = U_{\alpha\beta}$, $U_{\alpha^{-1}} = U_\alpha^*$, and $\gamma \mapsto \big(U_\gamma \xi(s(\gamma)) \mid \eta(r(\gamma))\big)$ is measurable for each pair ξ, η of measurable sections of $\mathcal{H}$. Every unitary representation $(\mathcal{H}, \mu, U)$ of $\mathcal{G}$ induces a *-representation $\pi_{(\mathcal{H},\mu,U)}$ of $C_c(\mathcal{G})$ on the direct integral $\int_{\mathcal{G}^{(0)}}^\oplus \mathcal{H}_x \, d\mu(x)$ characterised by

$$\big(\pi_{(\mathcal{H},\mu,U)}(f)\xi \mid \eta\big) = \int_{\mathcal{G}^{(0)}} \sum_{\gamma \in \mathcal{G}_x} \big(f(\gamma)U_\gamma\xi(x) \mid \xi(r(\gamma))\big) \, d\mu(x).$$

This representation is called the *integrated form* of $(\mathcal{H}, \mu, U)$. Renault's *Disintegration Theorem* [37, Theorem II.1.21] says that every representation is unitarily equivalent to the integrated form of a unitary representation.

3.3 The reduced C^*-algebra

To show that the map $\pi_{\max} : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ of Theorem 3.2.3 is injective, we need to construct a representation that is injective on $C_c(\mathcal{G})$.

If $\mathcal{G}$ were a group, we would use the regular representation of $\mathcal{G}$. We aim to do the same thing for groupoids, but there are multiple regular representations to consider.

Proposition 3.3.1. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. For each $x \in \mathcal{G}^{(0)}$, there is a *-representation $\pi_x : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_x))$ such that*

$$\pi_x(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma}.$$

This π_x is called the regular representation *of $C_c(\mathcal{G})$ associated to x.*

For each $\eta \in \mathcal{G}$ the map $U_\eta : \ell^2(\mathcal{G}_{s(\eta)}) \to \ell^2(\mathcal{G}_{r(\eta)})$ given by $U_\eta \delta_\gamma = \delta_{\gamma\eta^{-1}}$ is a unitary operator and we have $\pi_{r(\eta)} = U_\eta \pi_{s(\eta)} U_\eta^*$.

Proof. The first assertion is relatively straightforward to check using that the formula given for $\pi_x(f)\delta_\gamma$ is really just the formula for $f * \delta_\gamma$ if the convolution product is extended to not-necessarily-continuous functions.

The operators U_η are certainly unitary, and for $f \in C_c(\mathcal{G})$,

$$U_\eta \pi_{s(\eta)}(f)U_\eta^* \delta_\gamma = U_\eta \pi_{s(\eta)}(f)\delta_{\gamma\eta}$$
$$= \sum_{\alpha \in \mathcal{G}_{r(\gamma\eta)}} f(\alpha)U_\eta \delta_{\alpha\gamma\eta} = \sum_{\alpha \in \mathcal{G}_{r(\gamma\eta)}} f(\alpha)\delta_{\alpha\gamma} = \pi_{r(\eta)}(f)\delta_\gamma. \qquad \square$$

Definition 3.3.2. Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. The *reduced C^*-algebra*, denoted $C_r^*(\mathcal{G})$, of $\mathcal{G}$ is the completion of

$$\Big(\bigoplus_{x \in \mathcal{G}^{(0)}} \pi_x \Big)(C_c(\mathcal{G})) \subseteq \bigoplus_{x \in \mathcal{G}^{(0)}} \mathcal{B}(\ell^2(\mathcal{G}_x)).$$

We write $\| \cdot \|_r$ for the C^*-norm on $C_r^*(\mathcal{G})$.

The universal property of $C^*(\mathcal{G})$ yields a homomorphism, which we denote π_r, from $C^*(\mathcal{G})$ to $C_r^*(\mathcal{G})$ such that $\pi_r \circ \pi_{\max} = \bigoplus_x \pi_x$. In particular, $\| \cdot \|_r \leq \| \cdot \|$ on $C_c(\mathcal{G})$.

Proposition 3.3.3. *Let $\mathcal{G}$ be a second-countable locally compact Hausdorff étale groupoid. There is an injective, norm-decreasing map $j : C_r^*(\mathcal{G}) \to C_0(\mathcal{G})$ such that*

$$j(a)(\gamma) = \big(\pi_{s(\gamma)}(a)\delta_{s(\gamma)} \mid \delta_\gamma \big)$$

for all $a \in C_r^*(\mathcal{G})$ and $\gamma \in \mathcal{G}$. For $f \in C_c(\mathcal{G})$, we have $j(f) = f$.

Proof. For $f \in C_c(\mathcal{G})$, we have

$$j(f)(\gamma) = \Big(\sum_{\alpha \in \mathcal{G}_{r(\gamma)}} f(\alpha)\delta_{\alpha s(\gamma)} \mid \delta_\gamma \Big) = f(\gamma).$$

The Cauchy–Schwarz inequality shows that j is norm-decreasing from $\|\cdot\|_r$ to $\|\cdot\|_\infty$. An $\varepsilon/3$-argument shows that each $j(a)$ belongs to $C_0(\mathcal{G})$. If $a \in C_r^*(\mathcal{G})$ is nonzero, then there exists $x \in \mathcal{G}^{(0)}$ such that $\pi_x(a) \neq 0$, and then there exist $\alpha, \beta \in \mathcal{G}_x$ such that $\big(\pi_x(a)\delta_\alpha \mid \delta_\beta \big) \neq 0$. Now Proposition 3.3.1 gives

$$j(a)(\beta\alpha^{-1}) = \big(\pi_{r(\alpha)}(a)\delta_{r(\alpha)} \mid \delta_{\beta\alpha^{-1}} \big) = \big(U_\alpha^* \pi_{r(\alpha)}(a)U_\alpha\delta_\alpha \mid \delta_\beta \big) = \big(\pi_{s(\alpha)}\delta_\alpha \mid \delta_\beta \big) \neq 0.$$

Hence j is injective. $\square$

Corollary 3.3.4. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. Then the homomorphisms $\pi_{\max} : C_c(\mathcal{G}) \to C^*(\mathcal{G})$ and $\bigoplus_x \pi_x : C_c(\mathcal{G}) \to C_r^*(\mathcal{G})$ are both injective. For $f \in C_c(\mathcal{G})$, we have $\|f\|_\infty \leq \|f\|_r \leq \|f\|$. If f is supported on a bisection, then we have equality throughout.*

Proof. That $\|f\|_r \leq \|f\|$ is by definition of the universal norm. That $\|f\|_\infty \leq \|f\|_r$ follows from Proposition 3.3.3. Since $j(f) = f$ for all $f \in C_c(\mathcal{G})$, it follows that $\pi_{\max}$ and $\bigoplus_x \pi_x$ are injective on $C_c(\mathcal{G})$. If f is supported on a bisection, then we have $\|f\|^2 = \|f^* * f\|$. Since $f^* * f \in C_c(\mathcal{G}^{(0)})$, the uniqueness of the *-algebra norm on $C_c(\mathcal{G}^{(0)})$ gives $\|f\|^2 = \|f^* * f\|_\infty$, and this is precisely $\|f\|_\infty^2$ by Lemma 3.1.4. $\square$

Using the above Corollary, we see that we can apply the Stone–Weierstrass theorem to establish surjectivity of a homomorphism into either $C^*(\mathcal{G})$ or $C_r^*(\mathcal{G})$.

Corollary 3.3.5. *Suppose that $\mathcal{G}$ is a second-countable locally compact Hausdorff étale groupoid. Let A be a C^*-algebra, and suppose that $\pi : A \to C^*(G)$ (or $\pi : A \to C_r^*(\mathcal{G})$) is a homomorphism. Suppose that for each open bisection $U \subseteq G$ and each pair of distinct points $\beta, \gamma \in U$, there exists $a \in A$ such that $\pi(a) \in C_0(U)$, $\pi(a)(\beta) = 0$ and $\pi(a)(\gamma) = 1$. Then π is surjective.*

Proof. A straightforward application of the Stone–Weierstrass theorem shows that $\pi(A)$ contains $C_0(\mathcal{G}^{(0)})$. Using this and the convolution formula in Lemma 3.1.4, we see that for $U \subseteq \mathcal{G}$ an open bisection, the set $\pi(A) \cap C_0(U)$ is closed under pointwise multiplication (identify $C_0(U)$ with $C_0(r(U))$), and then note that if $f, g \in \pi(A) \cap C_0(U)$, then $f \circ r^{-1} \in C_0(r(U)) \subseteq C_0(\mathcal{G}) \subseteq \pi(A)$, and $f \cdot g = (f \circ r^{-1}) * g$. So another application of the Stone–Weierstrass theorem combined with the fact that $\|\cdot\|_{C^*(\mathcal{G})}$ agrees with $\|\cdot\|_\infty$ on $C_c(U)$ shows that $C_c(U) \subseteq \pi(A)$. Now Lemma 3.1.3 shows that $C_c(\mathcal{G}) \subseteq \pi(A)$. Since π is a C^*-homomorphism, it has closed range, and we deduce that $C^*(\mathcal{G}) \subseteq \pi(A)$. $\square$

Example 3.3.6. If $\mathcal{G}$ is a group, then there is just one unit e, and the regular representation π_e is the usual left-regular representation of the group algebra. So $C_r^*(\mathcal{G})$ is the usual reduced C^*-algebra. In particular, $C_r^*(\mathcal{G}) = C^*(\mathcal{G})$ if and only if $\mathcal{G}$ is amenable.

Example 3.3.7. Let R be a discrete equivalence relation. For distinct equivalence classes $S, T \subseteq R^{(0)}$ under R, we have $C_c(S \times S) \perp C_c(T \times T)$ inside $C_c(R)$, and so $C_c(R)$ is the direct sum of the *-subalgebras $C_c(S)$. It follows that the completions of these *-subalgebras are direct summands in each of $C^*(R)$ and $C_r^*(R)$. For a fixed S, the elements $\{1_{(s,s')} \mid s, s' \in S\}$ form a complete set of nonzero matrix units indexed by S, and generate both $C^*(S \times S)$ and $C_r^*(S \times S)$ as C^*-algebras. Since $\mathcal{K}(\ell^2(S))$ is the unique C^*-algebra generated by a family of nonzero matrix units indexed by S, we deduce that $C_r^*(S \times S) \cong C^*(S \times S) \cong \mathcal{K}(\ell^2(S))$.

Example 3.3.8. If $\mathcal{G}$ is the transformation groupoid for an action of a discrete group Γ on a compact Hausdorff space X, then the unit space $\mathcal{G}^{(0)}$ is just X. Fix $x \in X = \mathcal{G}^{(0)}$. The isomorphism $\omega : C_c(\mathcal{G}) \to C_c(\Gamma, C(X))$ of Example 3.1.7 intertwines the regular representation π_x of $C_c(\mathcal{G})$ with the induced representation of $C_c(\Gamma, C(X))$ associated the character of $C(X)$ given by evaluation at x, denoted $\text{Ind} \, \text{ev}_x$ in [52, Example 2.4.3]. So as discussed in that example, since $\bigoplus_x \text{ev}_x$ is a faithful representation of $C_0(X)$, we see that $C_r^*(\mathcal{G})$ is isomorphic to the reduced crossed product $C(X) \times_{\alpha,r} \Gamma$.

Remark 3.3.9. There is an alternative, slightly slicker, approach to defining $C_r^*(\mathcal{G})$. Define $\langle \cdot, \cdot \rangle_{C_0(\mathcal{G}^{(0)})} : C_c(\mathcal{G}) \times C_c(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$ by $\langle f, g \rangle_{C_0(\mathcal{G}^{(0)})} := (f^* * g)|_{\mathcal{G}^{(0)}}$. It is straightforward to check that this is a $C_0(\mathcal{G}^{(0)})$-valued inner-product (in particular, positive definite) on $C_c(\mathcal{G})$, so we can form the corresponding Hilbert-module completion $X_{\mathcal{G}}$. Now the action of $C_c(\mathcal{G})$ on itself by left multiplication extends to a homomorphism $L : C_c(\mathcal{G}) \to \mathcal{L}(X_{\mathcal{G}})$, the C^*-algebra of adjointable operators on $X_{\mathcal{G}}$. It is a fairly straightforward exercise, if you are familiar with Hilbert modules, to verify that $\|L(f)\| = \|f\|_r$ for all $f \in C_c(\mathcal{G})$. So the completion of $L(C_c(\mathcal{G}))$ in $\mathcal{L}(X_{\mathcal{G}})$ is isomorphic to $C_r^*(\mathcal{G})$; indeed, we could have taken this as the definition of $C_r^*(\mathcal{G})$.

3.4 Equivalence of groupoids

Renault's notion of equivalence of groupoids closely reflects Morita equivalence for C^*-algebras.

Definition 3.4.1. Suppose that $\mathcal{G}$ is an étale groupoid. A *left $\mathcal{G}$-space* is a locally compact Hausdorff space X endowed with a continuous map $r : X \to \mathcal{G}^{(0)}$ and a continuous map $\cdot : \mathcal{G} * X := \{(\gamma, \xi) \in \mathcal{G} \times X \mid s(\gamma) = r(\xi)\} \to X$ such that

1. $r(\gamma \cdot \xi) = r(\gamma)$ for all $(\gamma, \xi) \in \mathcal{G} * X$;

2. $r(\xi) \cdot \xi = \xi$ for all $\xi \in X$; and

3. $\alpha \cdot (\beta \cdot \xi) = (\alpha\beta) \cdot \xi$ for all $(\beta, \xi) \in \mathcal{G} * X$ and $\alpha \in \mathcal{G}_{r(\beta)}$.

A *right $\mathcal{G}$-space* is defined similarly, but with a map $s : X \to \mathcal{G}^{(0)}$ and the roles of s and r reversed in (1)–(3).

We say that X is a *proper* left $\mathcal{G}$-space if the map $(\gamma,\xi) \mapsto (\gamma \cdot \xi, \xi)$ from $\mathcal{G} * X \to X \times X$ is a proper map. It is *free* if $\gamma \cdot \xi = \xi$ implies $\gamma = r(\xi)$.

If $\mathcal{G}$ and $\mathcal{H}$ are two étale groupoids, then a $\mathcal{G}$–$\mathcal{H}$-equivalence is a locally compact Hausdorff space X that is simultaneously a free and proper left $\mathcal{G}$-space and a free and proper right $\mathcal{H}$ space such that the left and right actions commute, and such that $r : X \to \mathcal{G}^{(0)}$ and $s : X \to \mathcal{H}^{(0)}$ are open maps and induce homeomorphisms $\tilde{r} : X/\mathcal{H} \cong \mathcal{G}^{(0)}$, and $\tilde{s} : \mathcal{G}\backslash X \to \mathcal{H}^{(0)}$.

Lemma 3.4.2. *Suppose that $\mathcal{G}, \mathcal{H}$ are second-countable locally compact Hausdorff étale groupoids. Let X be a $\mathcal{G}$–$\mathcal{H}$-equivalence. If $\xi, \eta \in X$ satisfy $r(\xi) = r(\eta)$, then there is a unique element $[\xi,\eta]_\mathcal{H} \in \mathcal{H}$ such that $\xi \cdot [\xi,\eta]_\mathcal{H} = \eta$. Likewise, if $s(\xi) = s(\eta)$, then there is a unique $_\mathcal{G}[\xi,\eta] \in \mathcal{G}$ such that $_\mathcal{G}[\xi,\eta] \cdot \eta = \xi$.*

Proof. It suffices to prove the first statement; the second is symmetric. That r descends to a homeomorphism $X/\mathcal{H} \to \mathcal{G}^{(0)}$ shows that there exists an element $\lambda \in \mathcal{H}$ such that $\xi \cdot \lambda = \eta$. Freeness shows that this λ is unique. $\square$

Given a $\mathcal{G}$–$\mathcal{H}$-equivalence X, there is a corresponding $\mathcal{H}$–$\mathcal{G}$-equivalence $X^* :=$ $\{\xi^* \mid \xi \in X\}$ defined by $r(\xi^*) = s(\xi)$, $s(\xi^*) = r(\xi)$, $\lambda \cdot \xi^* := (\xi \cdot \lambda^{-1})^*$ and $\xi^* \cdot \gamma = (\gamma^{-1} \cdot x)^*$. Clearly $X^{**} \cong X$ via the map $\xi^{**} \mapsto \xi$.

Proposition 3.4.3. *Suppose that $\mathcal{G}, \mathcal{H}$ are second-countable locally compact Hausdorff étale groupoids. Let X be a $\mathcal{G}$–$\mathcal{H}$-equivalence. Let $L := \mathcal{G} \sqcup X \sqcup X^* \sqcup \mathcal{H}$, with the relative topology. Then L is an étale groupoid with*

- *$L^{(0)} = \mathcal{G}^{(0)} \sqcup \mathcal{H}^{(0)}$,*

- *range and source maps inherited from those on $\mathcal{G}, X, X^*, \mathcal{H}$,*

- *multiplication inherited from multiplication in $\mathcal{G}$ and $\mathcal{H}$, the actions of $\mathcal{G}$ and $\mathcal{H}$ on X and X^* and with $\xi^* \eta := [\xi,\eta]_\mathcal{H}$ for $\xi, \eta \in X$ with $r(\xi) = r(\eta)$, and $\xi \eta^* := {}_\mathcal{G}[\xi,\eta]$ for $\xi, \eta \in X$ with $s(\xi) = s(\eta)$, and*

- *$\xi^{-1} = \xi^*$ and $(\xi^*)^{-1} = \xi$ for $\xi \in X$.*

We have $\mathcal{G}^{(0)} L \mathcal{G}^{(0)} = \mathcal{G}$ and $\mathcal{H}^{(0)} L \mathcal{H}^{(0)} = \mathcal{H}$.

Proof. The proof that L is a topological groupoid is routine but tedious. To see that it is étale, we show that $r : X \to \mathcal{G}^{(0)}$ is a local homeomorphism; that s is a local homeomorphism follows from a similar argument.

We already know that r is an open map, so we need only show that it is locally injective. So fix $\xi \in X$ and sequences $\xi_n, \xi_n' \to \xi$ such that $r(\xi_n) = r(\xi_n')$ for all n. We must show that $\xi_n = \xi_n'$ for large n. We have $[\xi_n,\xi_n']_\mathcal{H} \to [\xi,\xi]_\mathcal{H} = s(\xi)$. Since $s(\xi) \in \mathcal{H}^{(0)}$ and since $\mathcal{H}^{(0)}$ is open, we therefore have $[\xi_n,\xi_n']_\mathcal{H} \in \mathcal{H}^{(0)}$ for large n; that is $[\xi_n,\xi_n']_\mathcal{H} = s(\xi_n)$ for large n, and therefore $\xi_n' = \xi_n \cdot s(\xi_n) = \xi_n$ for large n. $\square$

We call the groupoid of Proposition 3.4.3 the *linking groupoid* of X.

Theorem 3.4.4. *Suppose that $\mathcal{G}, \mathcal{H}$ are second-countable locally compact Hausdorff étale groupoids. Let X be a $\mathcal{G}$–$\mathcal{H}$-equivalence. Let L be the linking groupoid of X. Then $P \coloneqq 1_{\mathcal{G}^{(0)}}$ and $Q \coloneqq 1_{\mathcal{H}^{(0)}}$ belong to $\mathcal{M}(C^*(L))$ and to $\mathcal{M}(C_r^*(L))$, and are complementary full projections. We have $PC^*(L)P \cong C^*(\mathcal{G})$ and $QC^*(L)Q \cong C^*(\mathcal{H})$, and similarly $PC_r^*(L)P \cong C_r^*(\mathcal{G})$ and $QC_r^*(L)Q \cong C_r^*(\mathcal{H})$. In particular, $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$ are Morita equivalent via the imprimitivity bimodule $PC^*(L)Q$ and likewise $C_r^*(\mathcal{G})$ and $C_r^*(\mathcal{H})$ are Morita equivalent via the imprimitivity bimodule $PC_r^*(L)Q$.*

Proof. Fix an increasing sequence K_n of compact subsets of $\mathcal{G}^{(0)}$ with $\bigcup_n K_n = \mathcal{G}^{(0)}$ and choose functions $e_n \leq 1$ in $C_0(\mathcal{G}^{(0)})$ with $e_n \equiv 1$ on K_n. For $f \in C_c(L)$, we have $e_n f = f|_{\mathcal{G}^{(0)}L}$ for large n (just take n large enough so that $r(\mathrm{supp}(f)) \subseteq K_n$). So the e_n converge strictly to a multiplier projection P with the property that $Pf = 1_{\mathcal{G}^{(0)}} * f$ for $f \in C_c(\mathcal{G})$. A similar argument gives Q, and it is clear that $P + Q = 1$. The map π_r is clearly nondegenerate, and its extension to $\mathcal{M}(C^*(\mathcal{G}))$ takes P and Q to projections in $\mathcal{M}(C_r^*(\mathcal{G}))$ with the same properties, and which we continue to call P and Q.

To see that $PC_r^*(L)P \cong C_r^*(\mathcal{G})$, first note that since $\mathcal{G}$ is an open subgroupoid of L, there is an inclusion of $C_c(\mathcal{G})$ in $C_c(L)$ that extends a function $f \in C_c(\mathcal{G})$ to L by defining $f(\eta) = 0$ for $\eta \notin L$. We just have to show that this inclusion is isometric for the reduced norms. For a fixed $x \in \mathcal{G}^{(0)}$, consider the regular representation π_x^L of $C_c(L)$ on $\ell^2(L_x)$. Let $R \in \mathcal{B}(\ell^2(L_x))$ be the orthogonal projection onto $\overline{\mathrm{span}}\{\delta_\eta \mid r(\eta) \in \mathcal{G}^{(0)}\}$. We have $R = \tilde{\pi}_x^L(P)$, and since $Pf = fP = f$ for $f \in C_c(\mathcal{G})$, we see that $\pi_x^L(f) = R\pi_x^L(f)R$ for $f \in C_c(\mathcal{G})$. Since $x \in \mathcal{G}^{(0)}$, the set $\{\eta \in L_x \mid r(\eta) \in \mathcal{G}^{(0)}\}$ is precisely $\mathcal{G}_x$. Using this, it is easy to see that $R\pi_x^L|_{C_c(\mathcal{G})}R$ is a copy of the regular representation $\pi_x^{\mathcal{G}}$ of $C_c(\mathcal{G})$. So $\|\pi_x^{\mathcal{G}}(f)\| = \|R\pi_x^L(f)R\| \leq \|\pi_x^L(f)\|$ for all $f \in C_c(\mathcal{G})$. This immediately shows that for $f \in C_c(\mathcal{G})$ we have $\|f\|_{C_r^*(\mathcal{G})} \leq \|f\|_{C_r^*(L)}$. For the reverse inequality, it suffices to show that if $y \in L^{(0)} \smallsetminus \mathcal{G}^{(0)}$ then there exists $x \in \mathcal{G}^{(0)}$ such that $\|\pi_y^L(f)\| = \|\pi_x^L(f)\|$ for all $f \in C_c(\mathcal{G})$. For this, first note that since $s : X \to \mathcal{H}^{(0)}$ induces a homeomorphism of $\mathcal{G}\backslash X$ onto $\mathcal{H}^{(0)}$, it is surjective, so we can fix $\eta \in X$ with $s(\eta) = y$. By Proposition 3.3.1, the representation π_y^L is unitarily equivalent to $\pi_{r(\eta)}^L$, and so $x \coloneqq r(\eta) \in \mathcal{G}^{(0)}$ has the desired property. An identical argument shows that $QC^*(L)Q \cong C_r^*(\mathcal{H})$.

We now turn to universal C^*-algebras; again, it suffices to show that the inclusion $C_c(\mathcal{G}) \hookrightarrow C_c(L)$ is isometric for the universal norm. Certainly the inclusion $C_c(\mathcal{G}) \hookrightarrow C_c(L) \hookrightarrow C^*(L)$ determines a $*$-representation of $C_c(\mathcal{G})$, and so the universal property of $C^*(\mathcal{G})$ shows that there is a homomorphism $C^*(\mathcal{G}) \to C^*(L)$ that agrees with the canonical inclusion of $C_c(\mathcal{G})$. Thus the inclusion is norm-decreasing. For the reverse inequality, the rough idea is to use that the pair $\left(C_c(L\mathcal{G}^{(0)}), C_c(\mathcal{G}^{(0)}L)\right)$ is a $*$-Morita context in the sense of Ara [3] between $C_c(L)$ and $C_c(\mathcal{G})$. Fix a nondegenerate $*$-representation π of $C_c(\mathcal{G})$ in $\mathcal{B}(\mathcal{H})$. It suffices to show that there is a $*$-representation $\tilde{\pi}$ of $C_c(L)$ such that

$$\|\tilde{\pi}(f)\| \geq \|\pi(f)\| \quad \text{for all } f \in C_c(\mathcal{G}). \tag{3.2}$$

For this, define a positive semidefinite sesquilinear form on $C_c(L\mathcal{G}^{(0)}) \odot \mathcal{H}$ by $(f \odot h \mid g \odot k) = (h \mid \pi(f^*g)k)$. Let $\widetilde{\mathcal{H}}$ denote the Hilbert space obtained by quotienting out the space of vectors satisfying $(\xi \mid \xi) = 0$ and then completing in the norm coming from the inner-product. For $f \in C_c(L\mathcal{G}^{(0)})$ and $h \in \mathcal{H}$, we write $f \otimes_{C_c(\mathcal{G})} h$ for the image of $f \odot h$ in $\widetilde{\mathcal{H}}$. It's not hard to check that $\tilde{\pi}(f)(g \otimes_{C_c(\mathcal{G})} h) :=$ $(fg) \otimes_{C_c(\mathcal{G})} h$ defines a *-representation of $C_c(L)$ on $\widetilde{\mathcal{H}}$. To establish (3.2), fix $f \in C_c(\mathcal{G})$ and $\varepsilon > 0$, and choose $h \in \mathcal{H}$ such that $\|h\| = 1$ and $\|\pi(f)h\| > \|\pi(f)\| - \varepsilon$. Fix an increasing approximate identity e_n for $C_0(\mathcal{G}^{(0)})$ in $C_c(\mathcal{G}^{(0)})$ as in the first paragraph of the proof. Then $\pi(e_n)h \to h$ because π is nondegenerate. It then follows from the definition of the inner product on $\widetilde{\mathcal{H}}$ that $e_n \otimes_{C_c(\mathcal{G})} h$ is Cauchy in $\widetilde{\mathcal{H}}$ and so converges to some $\tilde{h}$. Since

$$\left(e_n \otimes_{C_c(\mathcal{G})} h \mid e_n \otimes_{C_c(\mathcal{G})} h\right) = \left(h \mid \pi(e_n^* e_n)h\right) \to 1,$$

we have $\|\tilde{h}\| = 1$. Since $fe_n = f$ for large n, we have

$$\begin{aligned}
\|\tilde{\pi}(f)\tilde{h}\|^2 &= \lim_n \|\pi(fe_n) \otimes_{C_c(\mathcal{G})} h\|^2 \\
&= \lim_n \left(h \mid \pi((fe_n e_n^* f^*))h\right) = \left(h \mid \pi(ff^*)h\right) = \|\pi(f)h\|^2
\end{aligned}$$

So $\|\tilde{\pi}(f)\tilde{h}\| = \|\pi(f)h\| > \|\pi(f)\| - \varepsilon$, and since $\|\tilde{h}\| = 1$, we deduce that $\|\tilde{\pi}(f)\| > \|\pi(f)\| - \varepsilon$. Letting $\varepsilon \to 0$ gives (3.2). $\qquad\square$

Remark 3.4.5. Recall that, by Brown's theorem [9], if A is a σ-unital C^*-algebra, and P is a multiplier projection of A, then $PAP \otimes \mathcal{K} \cong APA \otimes \mathcal{K}$, and then the Brown–Green–Rieffel theorem [10] says that σ-unital C^*-algebras A and B are Morita equivalent if and only if they are stably isomorphic. We now have a version of equivalence for groupoids, and we know that the (discrete) full equivalence relation $R_{\mathbb{N}} := \mathbb{N} \times \mathbb{N}$ has C^*-algebra isomorphic to $\mathcal{K}$. It is also not too difficult to see, using universal properties, that if $\mathcal{G}$ and $\mathcal{H}$ are étale groupoids, then $\mathcal{G} \times \mathcal{H}$ is a groupoid, and that $C^*(\mathcal{G} \times \mathcal{H}) \cong C^*(\mathcal{G}) \otimes C^*(\mathcal{H})$. So it's not unreasonable to ask whether Brown's theorem and the Brown–Green–Rieffel theorem could carry over to groupoids. The answer is a qualified yes. Specifically, if $\mathcal{G}$ and $\mathcal{H}$ are étale groupoids such that $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$ are totally disconnected as topological spaces, then for every compact open $K \subseteq \mathcal{G}^{(0)}$, we have $K\mathcal{G}K \times R_{\mathbb{N}} \cong \mathcal{G}K\mathcal{G} \times R_{\mathbb{N}}$; and $\mathcal{G}$ and $\mathcal{H}$ are equivalent if and only if $\mathcal{G} \times R_{\mathbb{N}} \cong \mathcal{H} \times R_{\mathbb{N}}$; the proof follows, almost exactly, the proofs of Brown's theorem and the Brown–Green–Rieffel theorem [12].

Chapter 4

Fundamental structure theory

In this section, we will discuss the structural properties of $C^*(\mathcal{G})$. When is it nuclear and when does it satisfy the UCT? When is it/isn't it simple, and more generally what is its ideal structure? When is it purely infinite?

4.1 Amenability, nuclearity and the UCT

The theory of amenability for groupoids is complicated; it could easily be a five-hour course all by itself. So we are going to skate over the top of it here. Most of what appears here is taken from [2].

Recall that a discrete group Γ is *amenable* if it admits a finitely additive probability measure μ with the property that $\mu(gA) = \mu(A)$ for all $A \subseteq \Gamma$ and $g \in \Gamma$. A discrete group Γ is amenable if and only if $C^*(\Gamma) = C_r^*(\Gamma)$.

Amenability for groupoids is intended as an analogue of amenability for groups, but unfortunately, the analogies are not so well behaved as we might like. There are a number of notions of amenability for groupoids, two of the most prominent being measurewise amenability and topological amenability. These two coincide for étale groupoids, by results of Anantharaman-Delaroche–Renault [2], but as we shall see, they are not equivalent to coincidence of the full and reduced C^*-algebras, even for group bundles.

We need some set-up. If $\mathcal{G}$ is an étale groupoid, then a *continuous system of probability measures for* $\mathcal{G}$ is a system $\{\lambda^x \mid x \in \mathcal{G}^{(0)}\}$ of Radon probability measures λ^x on $\mathcal{G}$ with the support of λ^x contained in $\mathcal{G}^x$ such that for $f \in C_c(\mathcal{G})$, the function $x \mapsto \int_{\mathcal{G}} f \, d\lambda^x$ is continuous.

Remark 4.1.1. Since each $\mathcal{G}^x$ is discrete, a Radon probability measure λ^x on $\mathcal{G}^x$ amounts to a function $\lambda^x : \mathcal{G}^x \to [0, \infty)$ with $\sum_\gamma w^x(\gamma) = 1$.

An *approximate invariant continuous mean* for $\mathcal{G}$ is a net λ_i of continuous systems of probability measures for $\mathcal{G}$ such that the net $\left(M_i : \gamma \mapsto \|\lambda_i^{r(\gamma)}(\gamma \cdot) - \right.$

$\lambda_i^{s(\gamma)}(\cdot)\|_1\big)_i$ of functions from $\mathcal{G}$ to $[0, \infty)$ has the property that $M_i|_K \to 0$ uniformly for every compact $K \subseteq \mathcal{G}$.

Definition 4.1.2. Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is *(topologically) amenable* if $\mathcal{G}$ admits an approximate invariant continuous mean.

There are a number of equivalent formulations of amenability, particularly in the setting of étale groupoids. Perhaps one of the most useful is the following:

Lemma 4.1.3 ([2, Proposition 2.2.13]). *Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}$ is amenable if and only if there is a sequence $(h_i)_{i=1}^{\infty}$ in $C_c(\mathcal{G})$ such that*

1. *the maps $x \mapsto \sum_{\gamma \in \mathcal{G}^x} |h_i(\gamma)|^2$ (indexed by i) converge uniformly to 1 on every compact subset of $\mathcal{G}^{(0)}$; and*

2. *the maps $\alpha \mapsto \sum_{\gamma \in \mathcal{G}^{r(\alpha)}} \big|h_i(\alpha^{-1}\gamma) - h_i(\gamma)\big|$ (indexed by i) converge uniformly to 0 on every compact subset of $\mathcal{G}$.*

The key point of amenability is the following:

Theorem 4.1.4 ([2, Proposition 6.1.8]). *Let $\mathcal{G}$ be an étale groupoid, and suppose that $\mathcal{G}$ is amenable. Then $\pi_r : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ is injective.*

From [52, Theorem 3.1.1], we know that if Γ is a discrete amenable group acting on a locally compact Hausdorff space X and $\mathcal{G}$ is the transformation groupoid, then $C^*(\mathcal{G}) \cong C_0(X) \rtimes \Gamma$ and $C_r^*(\mathcal{G}) \cong C_0(X) \rtimes_r \Gamma$ coincide. Indeed, if Γ if amenable, then so is $\mathcal{G}$: just pull back a mean on Γ to each $\mathcal{G}^x \cong \{x\} \times \Gamma$ to obtain a (constant) approximately invariant continuous mean. It is possible for $\mathcal{G}$ to be amenable even when Γ is not: for example, the transformation groupoid of the free group acting on its boundary.

Amenability also has some other very important consequences.

Theorem 4.1.5 ([2, Corollary 6.2.14 and Theorem 3.3.7]). *If $\mathcal{G}$ is an étale groupoid, then the following are equivalent: $\mathcal{G}$ is amenable; $C^*(\mathcal{G})$ is nuclear; $C_r^*(\mathcal{G})$ is nuclear.*

It is also possible to study the nuclear dimension of a groupoid C^*-algebra in terms of dynamical properties of the groupoid. For example, if $\mathcal{G}$ is a transformation groupoid, then finite Rokhlin dimension of the action as discussed in [46, Section 5.1] implies finite nuclear dimension for $C^*(\mathcal{G})$. For more general étale groupoids there is a generalisation of Rokhlin dimension, called *dynamic asymptotic dimension* [23] (see [46, Section 6.2]) which, for principal groupoids, guarantees finite nuclear dimension for the associated C^*-algebra.

Corollary 4.1.6. *If $\mathcal{G}$ and $\mathcal{H}$ are equivalent étale groupoids, then $\mathcal{G}$ is amenable if and only if $\mathcal{H}$ is.*

Proof. A direct groupoid-theoretic proof of this can be found in [2], but we will take a shortcut: since $\mathcal{G}$ and $\mathcal{H}$ are equivalent, $C^*(\mathcal{G})$ and $C^*(\mathcal{H})$ are Morita equivalent, and therefore $C^*(\mathcal{G})$ is nuclear if and only if $C^*(\mathcal{H})$ is nuclear, so the result follows from Theorem 4.1.5. □

A beautiful result of Tu also relates amenability to the UCT.

Theorem 4.1.7. *Let $\mathcal{G}$ be an étale groupoid. If $\mathcal{G}$ is amenable, then $C_r^*(\mathcal{G})$ satisfies the UCT.*

Remark 4.1.8. It is a very important question whether every nuclear C^*-algebra satisfies the UCT. The previous two theorems say that every nuclear groupoid C^*-algebra (associated to an étale groupoid) satisfies the UCT. In fact, results of Barlak and Li [4] show that this result extends—significantly—to twisted groupoid C^*-algebras. Moreover, results of Renault [39] show that we can characterise twisted C^*-algebras associated to étale effective groupoids amongst arbitrary C^*-algebras in purely C^*-algebraic terms: they are the ones that admit a *Cartan subalgebra*. We will discuss this further in Chapter 5.

Generally speaking, checking amenability using the definition is hard work. Fortunately, there is a fairly extensive bag of tricks available, and usually the best approach is to see if any of them apply or can be adapted to the example at hand. Theorem 4.1.5 certainly belongs to this bag; we'll list a few more that come up particularly frequently.

Proposition 4.1.9. *If $\mathcal{G}$ is a principal étale groupoid and is an F_σ set in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, then the orbit space $\mathcal{G}^{(0)}/\mathcal{G}$ is a T_0 space if and only if each orbit $[x] := \{r(\gamma) \mid s(\gamma) = x\}$ is locally closed (that is, each $[x]$ is relatively open in its closure), and these equivalent conditions imply that $\mathcal{G}$ is amenable.*

Proof. Since $\mathcal{G}$ is principal, it is algebraically isomorphic to $\mathcal{R}(\mathcal{G})$, and so $\mathcal{R}(\mathcal{G})$ is an F_σ in $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$. So all of the conditions (1)–(14) in the Ramsay–Mackey–Glimm dichotomy [36, Theorem 2.1] are equivalent, and in particular (4) $\iff$ (5) of that theorem shows that $\mathcal{G}^{(0)}/\mathcal{G}$ is T_0 if and only if each $[x]$ is locally closed. It then follows from [2, Examples 2.1.4(2)] that $\mathcal{G}$ is a proper Borel groupoid, and therefore amenable by [2, Examples 3.2.2(2) and Theorem 3.3.7]. □

It follows, in particular, that every discrete equivalence relation is amenable (though we could also deduce this from nuclearity of $\mathcal{K}$).

Proposition 4.1.10 ([2, Proposition 5.3.37]). *Suppose that $\mathcal{G}$ is an étale groupoid, and that for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a closed subgroupoid of $\mathcal{G}$ with $\mathcal{G}^{(0)} \subseteq \mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ for all n. Further suppose that each $\mathcal{G}_{n+1}$ is a proper $\mathcal{G}_n$-space, and that $\mathcal{G} = \bigcup_n \mathcal{G}_n$. If each $\mathcal{G}_n$ is amenable, then $\mathcal{G}$ is amenable.*

As an example of this, consider the groupoid $\mathcal{R}_{2^\infty}$ of Example 2.3.5. For each n, let $\mathcal{G}_n := \{(vx, wx) \mid |v| = |w| = n, x \in X\}$. Then each $\mathcal{G}_n$ is closed (in fact

clopen) in $\mathcal{G}$ and contains $\mathcal{G}^{(0)}$, and the $\mathcal{G}_n$ are nested. In a fixed $\mathcal{G}_n$ each orbit is finite, of size 2^n, and so $\mathcal{G}^{(0)}/\mathcal{G}_n$ is a standard Borel space. So [2, Examples 2.1.4(2), Examples 3.2.2(2) and Theorem 3.3.7] above show that each $\mathcal{G}_n$ is amenable. Since $\mathcal{G} = \bigcup_n \mathcal{G}_n$, we deduce that $\mathcal{G}$ is amenable.

Proposition 4.1.11 ([40, Corollary 4.5]). *Suppose that $\mathcal{G}$ is an étale groupoid and $c : \mathcal{G} \to \Gamma$ is a continuous homomorphism into a discrete amenable group. If the clopen subgroupoid $\ker(c) \subseteq \mathcal{G}$ is amenable, then $\mathcal{G}$ is amenable.*

The above result for Γ a discrete *abelian* group was first proved by Spielberg in [44, Proposition 9.3]. Spielberg's proof passed through C^*-algebra theory, proving that $C^*(\mathcal{G})$ is nuclear, and deducing that $\mathcal{G}$ is amenable from Theorem 4.1.5. The Renault–Williams proof, by contrast, is entirely groupoid theoretic. Moreover, the Renault–Williams result is more general even than the one stated above: see [40, Theorem 4.2].

Example 4.1.12. Deaconu–Renault groupoids for actions of $\mathbb{N}^k$ are amenable: if $\mathcal{G}$ is a Deaconu–Renault groupoid over $\mathbb{N}^k$, then the map $c(x, p - q, y) := p - q$ is a continuous cocycle into the abelian, and hence amenable, group Γ. An argument very similar to the one used above to see that the $\mathcal{G}_n$ in Example 2.3.5 are amenable, shows that for each $n \in \mathbb{N}^k$ the subgroupoid $\mathcal{G}_n := \{(x, 0, y) \mid T^n x = T^n y\}$ is a proper Borel groupoid, and then that $\ker(c) = \bigcup \mathcal{G}_n$ is amenable (see [43] for details).

Example 4.1.13. It follows that every graph groupoid is amenable, since it is the Deaconu–Renault groupoid of the shift map on E^∞.

Proposition 4.1.14. *If $\mathcal{G}$ is an amenable étale groupoid and $\mathcal{H}$ is an open or a closed subgroupoid of $\mathcal{G}$, then $\mathcal{H}$ is also amenable.*

The rough idea here is to verify that an approximate invariant continuous mean for $\mathcal{G}$ restricts to one for $\mathcal{H}$.

We finish the section with an example due to Willett that shows that, unlike the situation for groups, in the setting of groupoids it is not the case that amenability is equivalent to coincidence of the full and reduced C^*-algebras.

Example 4.1.15 ([50, Lemma 2.8]). Let $\mathbb{F}_2$ denote the free group on two generators. For $n \in \mathbb{N}$, let K_n denote the intersection of all the normal subgroups of $\mathbb{F}_2$ that have index at most n in $\mathbb{F}_2$. Willett shows that $\mathbb{F}_2$ is the infinite union of the K_n. For each n, let $\Gamma_n := \mathbb{F}_2/K_n$, and let $\Gamma_\infty = \mathbb{F}_2$. Let $\mathcal{G}^{(0)} := \mathbb{N} \cup \{\infty\}$, the 1-point compactification of $\mathbb{N}$, and let $\mathcal{G}$ be the group bundle $\bigcup_{x \in \mathcal{G}^{(0)}} \Gamma_x \times \{x\}$. For each $\gamma \in \mathcal{G}_\infty = \mathbb{F}_2$, and each $n \in \mathbb{N}$, let $W(\gamma, n) := \{(\pi_m(\gamma), m) \mid m \geq n\}$. Then

$$\{W(\gamma, n) \mid \gamma \in \mathbb{F}_2, n \in \mathbb{N}\} \cup \{\{(\gamma, n)\} \mid n \in \mathbb{N}, \gamma \in \Gamma_n\}$$

is a basis for a locally compact Hausdorff topology on $\mathcal{G}$ under which it is étale. (Each fibre $\mathcal{G}_x$ is discrete in the relative topology.) This groupoid is not amenable,

because an approximate invariant mean on $\mathcal{G}$ would restrict to an invariant mean on $\mathbb{F}_2$. However, Willet proves that $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$ by showing that the universal norm of $f \in C_c(\mathcal{G})$ is given by $\sup_{n \in \mathbb{N}} \|f|_{\Gamma_n}\|$.

It is still not known, for example, whether a minimal, or even transitive, action for which the full and reduced C^*-algebras coincide must have an amenable transformation groupoid.

4.2 Effective groupoids and uniqueness

Recall that an action of a discrete group Γ on a locally compact Hausdorff space X is *effective* if, for each $g \in \Gamma$, the set $\{x \in X \mid g \cdot x = x\}$ has empty interior.

In the corresponding transformation groupoid, the basic open sets are the bisections of the form $\{g\} \times U$ where U ranges over a base for the topology on $\mathcal{G}^{(0)}$. So we can reinterpret effectiveness of an action in terms of the transformation groupoid as follows: the action of Γ on X is effective if and only if the interior of the isotropy in $\mathcal{G}$ is equal to $\mathcal{G}^{(0)}$. This leads us to a definition.

Definition 4.2.1. Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is *effective* if $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$.

A Baire category argument shows that an action of a countable discrete group is effective if and only if the points in X at which the isotropy is trivial are dense in X. We will need the equivalent for second-countable étale groupoids. We first need a technical lemma that will come up again later.

Lemma 4.2.2. *Let $\mathcal{G}$ be an étale groupoid, and suppose that $\gamma \in \mathcal{G}$ satisfies $r(\gamma) \neq s(\gamma)$ and that U is a bisection containing γ. Then there is an open neighbourhood V of $s(\gamma)$ such that $r(UV) \cap V = \varnothing$.*

Proof. We prove the contrapositive. That is, we suppose that U is an open bisection and that $\gamma \in U$, and that for every neighbourhood V of $s(\gamma)$, we have $r(UV) \cap V \neq \varnothing$. Choose a descending neighbourhood base V_i at $s(\gamma)$ with each $V_i \subseteq s(U)$. Since each $r(UV_i) \cap V_i$ is nonempty, for each i we can choose $\gamma_i \in UV_i$ with $r(\gamma_i) \in V_i$. Since $s|_U$ is a homeomorphism, the UV_i form a neighbourhood base at γ, and so $\gamma_i \to \gamma$. In particular, $r(\gamma) = \lim_i r(\gamma_i)$. Since each $r(\gamma_i) \in V_i$ and the V_i are a neighbourhood base at $s(\gamma)$, we deduce that $r(\gamma_i) \to s(\gamma)$; so $s(\gamma) = r(\gamma)$. □

Lemma 4.2.3 ([39, Proposition 3.6]). *Let $\mathcal{G}$ be a second-countable étale groupoid. Then $\mathcal{G}$ is effective if and only if $\{x \mid \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$.*

Proof. First suppose that $\mathrm{Iso}(\mathcal{G})^\circ \neq \mathcal{G}^{(0)}$. Then there is an open $U \subseteq \mathrm{Iso}(\mathcal{G})$ that is not contained in $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is closed, $U \smallsetminus \mathcal{G}^{(0)}$ is open and nonempty, so we can assume that U has trivial intersection with $\mathcal{G}^{(0)}$. Since s is an open map,

$s(U)$ is an open subset of $\mathcal{G}^{(0)}$ such that $\mathcal{G}_x^x \smallsetminus \{x\} \supseteq Ux$ is nonempty for all $x \in U$. So $\{x \mid \mathcal{G}_x^x = \{x\}\}$ is not dense in $\mathcal{G}^{(0)}$.

Now suppose that $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$. Let U be an open bisection that does not intersect $\mathcal{G}^{(0)}$. We claim that $U \smallsetminus \mathrm{Iso}(\mathcal{G})$ is open. To see this, suppose that $\gamma \in U \smallsetminus \mathrm{Iso}(\mathcal{G})$. We have $r(\gamma) \neq s(\gamma)$, and so Lemma 4.2.2 shows that there is an open neighbourhood V of $s(\gamma)$ contained in $s(U)$ such that $r(UV) \cap V$ is empty. Now UV is an open neighbourhood of γ in $U \smallsetminus \mathrm{Iso}(\mathcal{G})$. Since $U \cap \mathrm{Iso}(\mathcal{G})$ has empty interior, we see that $U \smallsetminus \mathrm{Iso}(\mathcal{G})$ is dense in U. It follows that $A_U :=$ $r(U \smallsetminus \mathrm{Iso}(\mathcal{G})) \cup (\mathcal{G}^{(0)} \smallsetminus \overline{r(U)})$ is an open dense subset of $\mathcal{G}^{(0)}$. By definition, we have $A_U \subseteq \mathcal{G}^{(0)} \smallsetminus \{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x \cap U \neq \varnothing\}$.

Now take a countable cover $\{U_i\}$ of $\mathcal{G} \smallsetminus \mathcal{G}^{(0)}$ by open bisections. By the preceding paragraph, the sets $A_{U_i} \subseteq \mathcal{G}^{(0)}$ are open dense sets. So the Baire category theorem implies that $\bigcap_i A_{U_i}$ is dense in $\mathcal{G}^{(0)}$. By construction of the A_{U_i}, we have $\mathcal{G}_x^x \cap U_j = \varnothing$ for every $x \in \bigcap_i A_{U_i}$ and every $j \in \mathbb{N}$. Since the U_i cover $\mathcal{G} \smallsetminus \mathcal{G}^{(0)}$, it follows that $\bigcap_i A_i \subseteq \{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x = \{x\}\}$, and so the latter is dense as claimed. □

Remark 4.2.4. In the literature, the condition that $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x = \{x\}\}$ is dense has gone by many names, including "topologically principal," and "topologically free;" but both of these terms have also been used elsewhere for different concepts. So one has to be careful with these terms in the literature: in any given article, check what definition is being used.

Since, in an étale groupoid, the unit space is a clopen subset of $\mathcal{G}$, the map $f \mapsto f|_{\mathcal{G}^{(0)}}$ is a map from $C_c(\mathcal{G})$ to $C_c(\mathcal{G}^{(0)})$. We regard $C_c(\mathcal{G}^{(0)})$ as an abelian subalgebra of $C_c(\mathcal{G})$. We will see later that this restriction map extends to a faithful conditional expectation of $C_r^*(\mathcal{G})$ onto $C_0(\mathcal{G}^{(0)})$. But to exploit this, we need some preliminary work.

Lemma 4.2.5. *Let $\mathcal{G}$ be an effective étale groupoid, and suppose that $\pi : C_c(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ is a $*$-representation that is injective on $C_c(\mathcal{G}^{(0)})$. For each $f \in C_c(\mathcal{G})$ and each $\varepsilon > 0$, there exists $h \in C_c(\mathcal{G}^{(0)})$ such that $\|h\| = 1$, $hfh = hf|_{\mathcal{G}^{(0)}}h \in C_c(\mathcal{G}^{(0)})$ and $\|\pi(hfh)\| \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$. In particular, $\|\pi(f|_{\mathcal{G}^{(0)}})\| \leq \|\pi(f)\|$.*

Proof. Fix $f \in C_c(\mathcal{G})$, and $\varepsilon > 0$. It suffices to show that $\|\pi(f)\| \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$.

Since π is injective on $C_c(\mathcal{G}^{(0)})$, we have $\|\pi(f|_{\mathcal{G}^{(0)}})\| = \|f_{\mathcal{G}^{(0)}}\|_\infty$. Thus, since $\mathcal{G}$ is effective, Lemma 4.2.3 shows that there exists $x \in \mathcal{G}^{(0)}$ such that $f(x) \geq \|f|_{\mathcal{G}^{(0)}}\| - \varepsilon$ and $\mathcal{G}_x^x = \{x\}$.

Let $f_0 := f|_{\mathcal{G}^{(0)}}$. Then $f - f_0 \in C_c(\mathcal{G})$, and so Lemma 3.1.3, shows that we can write $f - f_0 = \sum_{i=1}^n f_i$ where each f_i is supported on a bisection U_i that does not intersect $\mathcal{G}^{(0)}$. For each $i \neq 0$ such that $x \notin s(\mathrm{supp}(f_i))$, choose an open neighbourhood V_i of x in $\mathcal{G}^{(0)}$ such that $V_i \cap s(\mathrm{supp}(f_i)) = \varnothing$. Then for any $h \in C_c(V_i)$ we have $hf_ih = 0$. For each $i \neq 0$ such that $x \in s(\mathrm{supp}(f_i))$, the unique element γ of U_ix belongs to $\mathcal{G}_x \smallsetminus \{x\}$. Since $\mathcal{G}_x^x = \{x\}$, we deduce that $r(\gamma) \neq x$. By Lemma 4.2.2, we can choose a neighbourhood V_i of x with $V_i \subseteq s(U_i)$ and

$r(U_iV_i) \cap V_i = \varnothing$. In particular, $r(\mathrm{supp}(f_i)V_i) \cap V_i = \varnothing$. Hence Lemma 3.1.4 shows that for any $h \in C_c(V_i)$ we have $hf_ih = 0$.

Now let $V := \bigcap_{i=1}^n V_i$. Then V is an open neighbourhood of x and so we can choose $h \in C_c(V)$ such that $h(x) = 1$ and $\|h\|_\infty = 1$. Since π is injective on $C_c(\mathcal{G}^{(0)})$, it is isometric on $C_c(\mathcal{G}^{(0)})$. We have $h \in C_c(V_i)$ for each i, and so $hf_ih = 0$ for $i \neq 0$ by choice of the V_i. Hence $hfh = hf_0h$, giving

$$\|\pi(f)\| \geq \|\pi(hfh)\| = \|\pi(hf_0h)\| = \|hf_0h\| \geq (hf_0h)(x) = f(x) \geq \|\pi(f_0)\| - \varepsilon. \quad \square$$

Proposition 4.2.6. *Let $\mathcal{G}$ be an étale groupoid. There is a faithful conditional expectation $\Phi : C_r^*(\mathcal{G}^{(0)}) \to C_0(\mathcal{G}^{(0)})$ such that $\Phi(f) = f|_{\mathcal{G}^{(0)}}$ for all $f \in C_c(\mathcal{G})$. We have $j(\Phi(a)) = j(a)|_{\mathcal{G}^{(0)}}$ for all $a \in C_r^*(\mathcal{G})$.*

Proof. Proposition 3.3.3 shows that $\left\|f|_{\mathcal{G}^{(0)}}\right\|_\infty \leq \|f\|_r$ for every $f \in C_c(\mathcal{G})$. By Corollary 3.3.4, we have $\left\|f|_{\mathcal{G}^{(0)}}\right\|_\infty = \left\|f|_{\mathcal{G}^{(0)}}\right\|_r$, and we deduce that $f \mapsto f|_{\mathcal{G}^{(0)}}$ is a norm-decreasing idempotent linear map from $C_c(\mathcal{G})$ to $C_0(\mathcal{G}^{(0)})$. It therefore extends to a idempotent linear map Φ of norm 1 of $C_r^*(\mathcal{G})$ onto $C_0(\mathcal{G}^{(0)})$. By [5, Theorem II.6.10.2], this Φ is a conditional expectation. Since $j(\Phi(f)) = j(f|_{\mathcal{G}^{(0)}})$ for $f \in C_c(\mathcal{G})$, continuity of j and Φ give $j(\Phi(a)) = j(a)|_{\mathcal{G}^{(0)}}$ for all $a \in C_r^*(\mathcal{G})$.

To see that Φ is faithful, suppose that $a \neq 0$. Then there exist $x \in \mathcal{G}^{(0)}$ and $\gamma \in \mathcal{G}_x$ such that $\left(\pi_x(a^*a)\delta_\gamma \mid \delta_\gamma\right) \neq 0$. Applying the unitary equivalence between π_x and $\pi_{r(\gamma)}$ obtained in Proposition 3.3.1, we see that $\left(\pi_{r(\gamma)}(a^*a)\delta_{r(\gamma)} \mid \delta_{r(\gamma)}\right) \neq 0$. That is, $j(a^*a)(r(\gamma)) \neq 0$. Hence $j(\Phi(a^*a)) = \Phi(a^*a)|_{\mathcal{G}^{(0)}} \neq 0$, and we conclude that $\Phi(a^*a) \neq 0$. $\quad \square$

We can now prove our main theorem for this section.

Theorem 4.2.7. *Let $\mathcal{G}$ be an effective étale groupoid. If $\phi : C_r^*(\mathcal{G}) \to A$ is a C^*-homomorphism that is injective on $C_0(\mathcal{G}^{(0)})$, then it is injective on all of $C_r^*(\mathcal{G})$.*

Proof. By Lemma 4.2.5, we have $\|\pi(f)\| \geq \left\|\pi(f|_{\mathcal{G}^{(0)}})\right\|$ for all $f \in C_c(\mathcal{G})$, and so there is a well-defined linear map $\Psi : \pi(C_r^*(\mathcal{G})) \to \pi(C_0(\mathcal{G}^{(0)}))$ such that $\Psi(\pi(f)) = \pi(f|_{\mathcal{G}^{(0)}})$ for $f \in C_c(\mathcal{G})$. It follows from continuity that the faithful conditional expectation Φ of Proposition 4.2.6 satisfies $\Psi \circ \pi = \pi \circ \Phi$. Now we follow the standard argument, using injectivity of ϕ on $C_0(\mathcal{G}^{(0)})$ at the third implication:

$$\phi(a) = 0 \Rightarrow \Psi(\phi(a^*a)) = 0 \Rightarrow \phi(\Phi(a^*a)) = 0 \Rightarrow \Phi(a^*a) = 0 \Rightarrow a = 0. \quad \square$$

Remark 4.2.8. An equivalent restatement of Theorem 4.2.7 is that if $\mathcal{G}$ is a effective étale groupoid, then every nontrivial ideal of $C_r^*(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}^{(0)})$.

4.3 Invariant sets, ideals, and simplicity

Our aim in this section is to shed some light on the ideal structure of $C^*(\mathcal{G})$, and to characterise simplicity of $C^*(\mathcal{G})$ when $\mathcal{G}$ is an amenable étale groupoid. When $\mathcal{G}$ is not assumed amenable, things become more complicated: most of the statements in the section remain true for either the full or the reduced C^*-algebra, but typically not both, and care is required.

We will say that a subset U of $\mathcal{G}^{(0)}$ is *invariant* if $r(\mathcal{G}U) \subseteq U\}$. Observe that if $\mathcal{G}$ is a transformation groupoid, then an open $U \subseteq \mathcal{G}^{(0)}$ is invariant precisely if $C_0(U) \subseteq C_0(\mathcal{G}^{(0)})$ is an invariant ideal as in [52, Section 3.1.2].

Lemma 4.3.1. *Let $\mathcal{G}$ be an étale groupoid, and let I be an ideal of $C^*(\mathcal{G})$. Then there is an open invariant subset $\mathrm{supp}(I) \subseteq \mathcal{G}^{(0)}$ such that $I \cap C_0(\mathcal{G}^{(0)}) = \{f \in C_0(\mathcal{G}^{(0)}) \mid f(x) = 0$ for all $x \in \mathcal{G}^{(0)} \smallsetminus \mathrm{supp}(I)\} \cong C_0(\mathrm{supp}(I)).$*

Proof. Since $I \cap C_0(\mathcal{G}^{(0)})$ is an ideal of a commutative C^*-algebra, it has the form $I \cap C_0(\mathcal{G}^{(0)}) = C_0(\mathrm{supp}(I))$ for an open set $\mathrm{supp}(I) \subseteq \mathcal{G}^{(0)}$. We just have to show that this set is invariant. For this, suppose that $x \in \mathrm{supp}(I)$. Choose $f \in I \cap C_0(\mathcal{G}^{(0)})$ such that $f(x) \neq 0$. Fix $\gamma \in G_x$; we must show that $r(\gamma) \in \mathrm{supp}(I)$. For this, take an open bisection U containing γ and fix $h \in C_c(U)$ with $h(\gamma) = 1$. Lemma 3.1.4 shows that hfh^* is supported on $U\mathcal{G}^{(0)}U^{-1} = r(U) \subseteq \mathcal{G}^{(0)}$. So $hfh^* \in I \cap C_0(\mathcal{G}^{(0)})$. Since $(hfh)(r(\gamma)) = h(\gamma)f(x)h^*(\gamma^{-1}) = f(x) \neq 0$, we deduce that $r(\gamma) \in \mathrm{supp}(I)$. $\quad\square$

If U is an open invariant subset of $\mathcal{G}^{(0)}$, then $\mathcal{G}U$ is an open subgroupoid of $\mathcal{G}$ and so a locally compact Hausdorff étale groupoid in the relative topology. Similarly, $\mathcal{G} \smallsetminus \mathcal{G}U$ is a closed subgroupoid of $\mathcal{G}$, and hence again a locally compact étale groupoid in the subspace topology.

Proposition 4.3.2. *Let $\mathcal{G}$ be an étale groupoid, and let U be an open invariant subset of $\mathcal{G}^{(0)}$. Define $W := \mathcal{G}^{(0)} \smallsetminus U$. The inclusion $C_c(\mathcal{G}U) \hookrightarrow C_c(\mathcal{G})$ extends to an injective C^*-homomorphism $i_U : C^*(\mathcal{G}U) \hookrightarrow C^*(\mathcal{G})$. The image $I_U := i_U(C^*(\mathcal{G}U))$ is an ideal of $C^*(\mathcal{G})$, and is generated as an ideal by $C_c(U) \subseteq C_0(\mathcal{G}^{(0)})$. There is a homomorphism $\pi_U : C^*(\mathcal{G}) \to C^*(\mathcal{G}W)$ satisfying $\pi_U(f) = f|_{\mathcal{G}W}$ for all $f \in C_c(\mathcal{G})$. Moreover the sequence*

$$0 \longrightarrow C^*(\mathcal{G}U) \xrightarrow{\ i_U\ } C^*(\mathcal{G}) \xrightarrow{\ \pi_U\ } C^*(\mathcal{G}W) \longrightarrow 0$$

is exact.

Proof. The inclusion $C_c(\mathcal{G}U) \hookrightarrow C_c(\mathcal{G}) \hookrightarrow C^*(\mathcal{G})$ is a $*$-homomorphism, so the universal property of $C^*(\mathcal{G})$ shows that there is a homomorphism $i_U : C^*(\mathcal{G}U) \to C^*(\mathcal{G})$ as required.

To see that the image of i_U is an ideal, observe that if $f \in C_c(\mathcal{G}_U)$, and if $g \in C_c(\mathcal{G})$ is supported on a bisection, then $\mathrm{supp}(g * f) \subseteq \mathrm{supp}(g)\,\mathrm{supp}(f) \subseteq \mathcal{G}(\mathcal{G}U) = \mathcal{G}U$; similarly (or by taking adjoints) we have $\mathrm{supp}(f * g) \subseteq \mathcal{G}_U$. So $i_U(C_c(\mathcal{G}U))$ is an algebraic 2-sided ideal of $C^*(\mathcal{G})$, and hence I_U is an ideal by

continuity. Since $C_c(U)$ contains an approximate identity for $C^*(\mathcal{G}U)$, the ideal I_U is generated as an ideal by $C_c(U)$.

Since every *-homomorphism π of $C_c(\mathcal{G})$ into a C^*-algebra B can be composed with a faithful representation of B to obtain a *-representation of $C_c(\mathcal{G})$ that achieves the same norm on every element, it suffices to show that there is a *-homomorphism $\pi : C_c(\mathcal{G}) \to B$ for some C^*-algebra B such that $\|\pi(f)\| \geq \|f\|_{C^*(\mathcal{G}U)}$ for every $f \in C_c(\mathcal{G}_u)$. To see this observe that $C_c(\mathcal{G}U)$ is an algebraic ideal of $C_c(\mathcal{G})$, and so for each $f \in C_c(\mathcal{G})$, there is a linear map $\pi(f) : C_c(\mathcal{G}U) \to C_c(\mathcal{G}U)$ given by $\pi(f)g = f * g$. If we regard $C^*(\mathcal{G}U)$ as a Hilbert bimodule over itself with inner product $\langle a, b \rangle_{C^*(\mathcal{G}U)} = a^*b$, then $\langle \pi(f)a, b \rangle_{C^*(\mathcal{G}U)} = a^*\pi(f^*)b = \langle a, \pi(f^*)b \rangle_{C^*(\mathcal{G}U)}$; so $\pi(f)$ is an adjointable operator on $C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)}$. From this we see that π is a *-homomorphism into the algebra $\mathcal{L}(C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)})$ of adjointable operators on $C^*(\mathcal{G}U)_{C^*(\mathcal{G}U)}$. For $f \in C_c(\mathcal{G}U)$, we have

$$\|\pi(f)\| \geq \left\|\pi(f)\tfrac{f^*}{\|f\|}\right\| = \|f\|$$

as required.

The map $f \mapsto f|_W$ is a *-homomorphism of $C_c(\mathcal{G})$ onto $C_c(\mathcal{G}W)$ and hence determines a *-homomorphism from $C_c(\mathcal{G})$ to $C^*(\mathcal{G}W)$; so once again the universal property gives a homomorphism $\pi : C^*(\mathcal{G}) \to C^*(\mathcal{G}_W)$ that extends restriction of functions. Clearly $\ker \pi$ contains $C_c(\mathcal{G}U)$ and hence the image of i_U. In particular, π induces a homomorphism $\tilde{\pi} : C^*(\mathcal{G})/I_U \to C^*(\mathcal{G}W)$. To see that this homomorphism is injective, observe that since $C_c(\mathcal{G}U) \subseteq I_U$, if $f, g \in C_c(\mathcal{G})$ satisfy $f|_{\mathcal{G}W} = g|_{\mathcal{G}W}$, then $f - g \in I_U$. Hence there is a well-defined *-homomorphism $\phi : C_c(\mathcal{G}W) \to C^*(\mathcal{G})/I_U$ such that $\phi(f|_W) = f + I_U$ for all $f \in C_c(\mathcal{G})$. The universal property of $C^*(\mathcal{G}W)$ shows that ϕ extends to a homomorphism $\phi : C^*(\mathcal{G}W) \to C^*(\mathcal{G})/I_U$. The image of ϕ contains the image of $C_c(\mathcal{G})$ in the quotient and so ϕ is surjective. Since $\tilde{\pi}_U \circ \phi$ is the identity map on $C_c(\mathcal{G}W)$ we see that $\tilde{\pi}_U \circ \phi$ is the identity homomorphism; so surjectivity of ϕ ensures that $\tilde{\pi}_U$ is injective, and therefore that $\ker(\pi_U) = I_U$. □

The preceding proposition holds for general groupoids, but the proof requires the Disintegration Theorem. For reduced C^*-algebras, the corresponding maps i_U^r and π_U^r between reduced C^*-algebras exist, i_U^r is injective, π_U^r is surjective, and $\ker \pi_U^r$ contains the image of i_U^r (these statements are all relatively easy to prove using the properties of regular representations). But the sequence is not necessarily exact. The first example of this was given by Skandalis [38, Appendix, p. 35], but Willet's example, Example 4.1.15, also gives an instance of the failure of exactness: let $\mathcal{G}$ be Willett's groupoid. Since $\mathcal{G}$ is a group bundle, the set $\mathbb{N} \subseteq \mathbb{N} \cup \{\infty\} = \mathcal{G}^{(0)}$ is an open invariant subset. Since $\mathcal{G}\mathbb{N}$ is a (discrete) bundle of finite groups, it is amenable (just take normalised counting measure on each fibre), so $C^*(\mathcal{G}\mathbb{N}) = C_r^*(\mathcal{G}\mathbb{N})$, and Willett's result says that $C^*(\mathcal{G}) = C_r^*(\mathcal{G})$. So Proposition 4.3.2 shows that the sequence

$$0 \to C_r^*(\mathcal{G}\mathbb{N}) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C^*(\mathbb{F}_2) \to 0$$

is exact. Since $C_r^*(\mathbb{F}_2)$ is a proper quotient of $C^*(\mathbb{F}_2)$, we deduce that

$$0 \to C_r^*(\mathcal{G}\mathbb{N}) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathbb{F}_2) \to 0$$

is not exact.

We will say that $\mathcal{G}$ is *strongly effective* if $\mathcal{G}W$ is effective for every closed invariant subset of $\mathcal{G}^{(0)}$. This is a strictly stronger condition than effectiveness: Consider the action of $\mathbb{Z}$ on its own 1-point compactification $\mathbb{Z} \cup \{\infty\}$ determined by continuous extension of the translation action of $\mathbb{Z}$ on itself. The resulting transformation groupoid $\mathcal{G}$ is effective because the only point with nontrivial isotropy is ∞; but $\{\infty\}$ is a closed invariant subset of $\mathcal{G}^{(0)}$ and clearly $\mathcal{G}\{\infty\} \cong \mathbb{Z}$ is not effective.

Theorem 4.3.3. *Let $\mathcal{G}$ be an amenable étale groupoid. The map $U \mapsto I_U$ is an injection from the set of open invariant subsets of $\mathcal{G}^{(0)}$ to the set of ideals of $C^*(\mathcal{G})$. It is bijective if and only if $\mathcal{G}$ is strongly effective.*

Proof. If U, V are distinct open invariant sets, then I_U and I_V are distinct because $I_U \cap C_0(\mathcal{G}^{(0)}) = C_0(U) \neq C_0(V) = I_V \cap C_0(\mathcal{G}^{(0)})$. So $U \mapsto I_U$ is injective.

First suppose that $\mathcal{G}$ is strongly effective. Fix an ideal I of $C^*(\mathcal{G})$. We must show that $I = I_{\mathrm{supp}(I)}$. To see this, first observe that Proposition 4.3.2 shows that $I_{\mathrm{supp}(I)}$ is generated as an ideal by $C_c(\mathrm{supp}(I))$, which is a subset of $I \cap C_0(\mathcal{G}^{(0)})$ by definition. So $I_{\mathrm{supp}(I)} \subseteq I$. Thus the quotient map induces a homomorphism $\tilde{q} : C^*(\mathcal{G})/I_{\mathrm{supp}(I)} \to C^*(\mathcal{G})/I$. Let $W := \mathcal{G}^{(0)} \smallsetminus \mathrm{supp}(I)$. Proposition 4.3.2 gives an isomorphism $\tilde{\pi} : C^*(\mathcal{G})/I_{\mathrm{supp}(I)} \cong C^*(\mathcal{G}W)$ extending restriction of functions on $C_c(\mathcal{G})$. Since $I \cap C_0(\mathcal{G}^{(0)}) = C_0(\mathrm{supp}(I))$, we see that $\tilde{q} \circ \tilde{\pi}$ is injective on $C_0(W)$. Since $\mathcal{G}$ is strongly effective, $\mathcal{G}W$ is effective, and it then follows from Theorem 4.2.7 that $\tilde{\pi}$ is injective. So $I = I_{\mathrm{supp}\, I}$.

Now suppose that $\mathcal{G}$ is not strongly effective. Fix a closed invariant set $W \subseteq \mathcal{G}^{(0)}$ such that $\mathcal{H} := \mathcal{G}W$ is not effective. We will construct a representation ψ of $C^*(\mathcal{H})$ such that ψ is faithful on $C_0(\mathcal{H}^{(0)})$ but is not faithful on $C^*(\mathcal{H})$.

Recall that the *orbit* $[x]$ of $x \in \mathcal{H}^{(0)}$ is the set $\{r(\gamma) \mid \gamma \in \mathcal{H}_x\}$. For each $x \in \mathcal{H}^{(0)}$, there is a linear map $\epsilon_x : C_c(\mathcal{H}) \to \mathcal{B}(\ell^2([x]))$ given by $\epsilon_x(f)\delta_y = \sum_{\gamma \in \mathcal{H}_y} f(\gamma)\delta_{r(\gamma)}$. For $f, g \in C_c(\mathcal{H})$ and $y, z \in [x]$, we have

$$\left(\epsilon_x(f)\epsilon_x(g)\delta_y \mid \delta_z\right) = \sum_{\beta \in \mathcal{H}_y} g(\beta)\left(\epsilon_x(f)\delta_{r(\alpha)} \mid \delta_z\right) = \sum_{\beta \in \mathcal{H}_y, \alpha \in \mathcal{H}^z_{r(\beta)}} f(\alpha)g(\beta).$$

On the other hand,

$$\left(\epsilon_x(f * g)\delta_y \mid \delta_z\right) = \sum_{\gamma \in \mathcal{H}^z_y} (f * g)(\gamma) = \sum_{\gamma \in \mathcal{H}^z_y} \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta) = \sum_{\gamma \in \mathcal{H}^z_y, \alpha \in \mathcal{H}^z} f(\alpha)g(\alpha^{-1}\gamma).$$

The map $(\alpha, \beta) \mapsto (\alpha, \alpha\beta)$ is a bijection from $\{(\alpha, \beta) \mid \beta \in \mathcal{H}_y, \alpha \in \mathcal{H}^z_{r(\beta)}\}$ to $\{(\alpha, \gamma) \mid \gamma \in \mathcal{H}^z_y, \alpha \in \mathcal{H}^z\}$ (the inverse is $(\alpha, \gamma) \mapsto (\alpha, \alpha^{-1}\gamma)$). So the two sums are

equal, and therefore ϵ_x is multiplicative. Likewise,

$$\left(\epsilon_x(f^*)\delta_y \mid \delta_z\right) = \sum_{\gamma \in \mathcal{H}_y^z} f^*(\gamma) = \sum_{\gamma \in \mathcal{H}_y^z} \overline{f(\gamma^{-1})} = \sum_{\eta \in \mathcal{H}_z^y} \overline{f(\eta)} = \left(\delta_y \mid \epsilon_x(g)\delta_z\right).$$

So ϵ_x is a *-representation of $C_c(\mathcal{H})$. The universal property of $C^*(\mathcal{H})$ therefore shows that ϵ_x extends to a representation of $C^*(\mathcal{H})$. Let $\psi := \bigoplus_{x \in \mathcal{H}^{(0)}} \epsilon_x$. If $f \in C_0(\mathcal{H}^{(0)}) \smallsetminus \{0\}$, say $f(x) \neq 0$, then $\epsilon_x(f)\delta_x = f(x)\delta_x \neq 0$, so ψ is faithful on $C_0(\mathcal{H}^{(0)})$.

Since $\mathcal{H}$ is not effective, there is an open bisection U contained in $\mathrm{Iso}(\mathcal{H}) \smallsetminus \mathcal{H}^{(0)}$. Fix $f \in C_c(r(U))$, and define $\tilde{f} \in C_c(U)$ by

$$\tilde{f}(\gamma) = f(r(\gamma)) \quad \text{for all } \gamma \in U.$$

Since U and $r(U)$ are open in $\mathcal{H}$, we can regard f and $\tilde{f}$ as elements of $C_c(\mathcal{H})$. Since U is contained in $\mathrm{Iso}(\mathcal{H})$, for $x \in \mathcal{H}^{(0)}$ and $y \in [x]$, we have

$$\epsilon_x(\tilde{f})\delta_y = \begin{cases} \tilde{f}(Uy)\delta_y & \text{if } y \in r(U) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(y)\delta_y & \text{if } y \in r(U) \\ 0 & \text{otherwise} \end{cases}$$

$$= \epsilon_x(f)\delta_y.$$

So $f - \tilde{f} \in \ker(\psi)$, and $f - \tilde{f} \neq 0$ because $\mathrm{supp}(f) \subseteq \mathcal{G}^{(0)}$ and $\mathrm{supp}(\tilde{f}) \subseteq U \subseteq \mathcal{G} \smallsetminus \mathcal{G}^{(0)}$.

Now consider the ideals $I_{\mathcal{G}^{(0)} \smallsetminus W}$ and $\ker(\psi \circ \pi_{\mathcal{G}^{(0)} \smallsetminus W})$. We have just seen that they have identical intersection with $C_0(\mathcal{G}^{(0)})$ (namely $C_0(\mathcal{G}^{(0)} \smallsetminus W)$), but are not equal. So $U \mapsto I_U$ is not a bijection. $\qquad\qquad\square$

Remark 4.3.4.   • Whether or not $\mathcal{G}$ is amenable or strongly effective, the map $U \mapsto I_U$ is an injection from the collection of open invariant sets of $\mathcal{G}^{(0)}$ to the space of ideals of $C^*(\mathcal{G})$.

- We could replace amenability of $\mathcal{G}$ with the requirement that $C^*(\mathcal{G}W) = C_r^*(\mathcal{G}W)$ for every closed invariant W in both Proposition 4.3.2 and Theorem 4.3.3. Conversely, if there exists a closed invariant set W such that $C^*(\mathcal{G}W) \neq C_r^*(\mathcal{G}W)$, then $U \mapsto I_U$ is not surjective because the kernel of $\pi_r^{\mathcal{G}W} \circ \pi_{\mathcal{G}^{(0)} \smallsetminus W}$ is not in its range.

- In the non-amenable case, the map $U \mapsto \ker(\pi_U^r)$ remains an injection from open invariant sets to ideals of the reduced C^*-algebra. It is possible for this map to be bijective even if $\mathcal{G}$ is not effective: for example, the reduced C^*-algebra of the free group is simple.

We say that a groupoid $\mathcal{G}$ is *minimal* if for every $x \in \mathcal{G}^{(0)}$ the orbit $[x]$ is dense in $\mathcal{G}^{(0)}$.

Lemma 4.3.5. *Let $\mathcal{G}$ be a topological groupoid. Then $\mathcal{G}$ is minimal if and only if the only open invariant subsets of $\mathcal{G}^{(0)}$ are $\varnothing$ and $\mathcal{G}^{(0)}$.*

Proof. If $\mathcal{G}$ is minimal, then the only nonempty closed invariant subset of $\mathcal{G}^{(0)}$ is $\mathcal{G}^{(0)}$. Hence the only open invariant subsets of $\mathcal{G}^{(0)}$ are $\varnothing$ and $\mathcal{G}^{(0)}$.

Now suppose that $\mathcal{G}$ is not minimal, say $x \in \mathcal{G}^{(0)}$ satisfies $\overline{[x]} \neq \mathcal{G}^{(0)}$. We claim that $\overline{[x]}$ is invariant. To see this, suppose that $y \in \overline{[x]}$ and $\gamma \in \mathcal{G}_y$; we must show that $r(\gamma) \in \overline{[x]}$. Choose $\alpha_n \in \mathcal{G}_x$ with $r(\alpha_n) \to y$. Chose an open bisection U containing γ. Since $s(U)$ is an open neighbourhood of $s(\gamma) = y$, we have $r(\alpha_n) \in U$ for large n, so by passing to a subsequence we can assume that $r(\alpha_n) \in U$ for all n. For each n, let γ_n be the unique element of U such that $s(\gamma_n) = r(\alpha_n)$. Since $s(\gamma_n) \to s(\gamma)$ and since $s|_U$ is a homeomorphism, we have $\gamma_n \to \gamma$ and so $r(\gamma_n) \to r(\gamma)$. Now each $\gamma_n \alpha_n \in \mathcal{G}_x$ and we have $r(\gamma_n \alpha_n) = r(\gamma_n) \to r(\gamma)$, and we deduce that $r(\gamma) \in \overline{[x]}$. That is, $\overline{[x]}$ is $\mathcal{G}$-invariant as claimed. Hence $\mathcal{G}^{(0)} \smallsetminus \overline{[x]}$ is a nontrivial open invariant subset of $\mathcal{G}^{(0)}$. $\qquad\square$

We therefore obtain the following characterisation of simplicity [8].

Theorem 4.3.6. *Let $\mathcal{G}$ be an amenable étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is effective and minimal.*

Proof. First suppose that $\mathcal{G}$ is effective and minimal. Since it is minimal, the only nonempty closed invariant set is $\mathcal{G}^{(0)}$ and so $\mathcal{G}$ is (trivially) strongly effective. So Theorem 4.3.3 and Lemma 4.3.5 show that the only ideals of $C^*(\mathcal{G})$ are $I_\varnothing = \{0\}$ and $I_{\mathcal{G}^{(0)}} = C^*(\mathcal{G})$.

If $\mathcal{G}$ is not minimal, then Lemma 4.3.5 gives a nontrivial open invariant subset of $\mathcal{G}^{(0)}$ and so Theorem 4.3.3 gives a nontrivial ideal. Likewise, if $\mathcal{G}$ is not minimal, then $U \mapsto I_U$ is injective but not bijective by Theorem 4.3.3; so it is not surjective. Since $\{0\} = I_\varnothing$ and $C^*(\mathcal{G}) = I_{\mathcal{G}^{(0)}}$ are in its range, it follows that $C^*(\mathcal{G})$ has a nontrivial ideal. $\qquad\square$

We also obtain a sufficient condition for reduced C^*-algebras.

Proposition 4.3.7. *If $\mathcal{G}$ is an effective, minimal, étale groupoid, then $C_r^*(\mathcal{G})$ is simple.*

Proof. Let I be a nonzero ideal of $C_r^*(\mathcal{G})$; we must show that $I = C_r^*(\mathcal{G})$. By Theorem 4.2.7, we have $I \cap C_0(\mathcal{G}^{(0)}) \neq \{0\}$. Now if $q_I : C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G})/I$ is the quotient map, and $\pi_r : C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$ is the canonical surjection, then $J := \ker(q_I \circ \pi_r) \cap C_0(\mathcal{G}^{(0)}) \supseteq I \cap C_0(\mathcal{G}^{(0)})$ is nonzero. So Lemma 4.3.1 shows that $\mathrm{supp}(J)$ is a nonempty open invariant set. Since $\mathcal{G}$ is minimal, it follows that $\mathrm{supp}(J) = \mathcal{G}^{(0)}$ and so $C_0(\mathcal{G}^{(0)}) \subseteq J$. So $\pi_r(C_0(\mathcal{G}^{(0)})) = C_0(\mathcal{G}^{(0)})$ is contained in I, and we deduce that $I = C_r^*(\mathcal{G})$. $\qquad\square$

4.4 Pure infiniteness

In this section we briefly discuss a result of Anantharaman-Delaroche giving a sufficient condition for $C^*(\mathcal{G})$ to be purely infinite. There is no improving on her argument, so the treatment here is more or less exactly the same as in [1].

Definition 4.4.1 ([1, Definition 2.1])**.** Let $\mathcal{G}$ be an étale groupoid. We say that $\mathcal{G}$ is *locally contracting at* $x \in \mathcal{G}^{(0)}$ if for every open neighbourhood V of x, there is an open set $W \subseteq V$ and an open bisection U such that $\overline{W} \subseteq s(U)$ and $r(U\overline{W}) \subsetneq W$. We say that $\mathcal{G}$ is *locally contracting* if it is locally contracting at x for every $x \in \mathcal{G}^{(0)}$.

Theorem 4.4.2 ([1, Proposition 2.4])**.** *Suppose that $\mathcal{G}$ is an effective, locally contracting, étale groupoid. Then $C_r^*(\mathcal{G})$ is purely infinite.*

Proof. It suffices to show that for every $a \in A_+$, the hereditary subalgebra generated by a contains an infinite projection. So fix $a \in A_+$. We may assume without loss of generality that the faithful conditional expectation $\Phi : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$ of Proposition 4.2.6 satisfies $\|\Phi(a)\| = 1$. Since $C_c(\mathcal{G})$ is dense in $C^*(\mathcal{G})$, we can choose $b \in C_c(\mathcal{G}) \cap A^+$ such that $b \leq a$ and $\|a - b\| < \frac{1}{4}$. Since Φ is norm decreasing we deduce that $\|\Phi(a) - \Phi(b)\| < \frac{1}{4}$, and so $\|\Phi(b)\| > \frac{3}{4}$. Lemma 4.2.5 with $\varepsilon = (\|\Phi(b)\| - 3/4)/2$ gives a function $h \in C_c(\mathcal{G}^{(0)})$ such that $\|h\| = 1$, $hbh = h\Phi(b)h \in C_0(\mathcal{G}^{(0)})$, and $\|hbh\| > \frac{3}{4}$. Since $hbh \leq b \leq a$, it suffices to find an infinite projection p and a partial isometry w in $C^*(\mathcal{G})$ such that $wpw^* \leq h\Phi(b)h =: b_0$.

Using that $\mathcal{G}$ is locally contracting, we choose an open V with $\overline{V} \subseteq \{x \in \mathcal{G}^{(0)} \mid b_0(x) > 3/4\}$ and a bisection B with $\overline{V} \subseteq s(B)$ and $r(B\overline{V}) \subsetneq V$. Let $T_B : s(B) \to r(B)$ be the homeomorphism $T_B(s(\gamma)) = r(\gamma)$ for $\gamma \in B$. Then $T_B(\overline{V}) = r(B\overline{V})$ is a compact subset of V and is not all of V. So we can choose $k \in C_c(V)$ such that k is identically 1 on $T_B(\overline{V})$. Define $x \in C_c(BV)$ by $x(\gamma) = k(s(\gamma))$ for $\gamma \in BV$. We have $x^*x = k^2$, and in particular x^*x is identically 1 on $r(B\overline{V}) \supseteq r(\operatorname{supp} x)$. Hence $x^*xx = x$.

So x is a scaling element. There is a standard trick for constructing a projection from such an element: Define v in the minimal unitisation of $C_r^*(\mathcal{G})$ by $v = x + (1 - x^*x)^{1/2}$. We have

$$v^*v = x^*x + x^*(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x + (1 - x^*x).$$

Since $(1 - x^*x)x = 0$, every $(1 - x^*x)^n x = 0$ and then by continuity $f(1 - x^*x)x = 0$ for every $f \in C(\sigma(1 - x^*x))$. In particular, $(1 - x^*x)^{1/2}x = 0 = x^*(1 - x^*x)^{1/2}$, and so $v^*v = 1$. Consequently, vv^* is a projection. We compute

$$vv^* = xx^* + x(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x^* + (1 - x^*x)$$

$$= 1 + xx^* - x^*x + x(1 - x^*x)^{1/2} + (1 - x^*x)^{1/2}x^*$$

$$= 1 - (x^*x - xx^* - x(1 - x^*x)^{1/2} - (1 - x^*x)^{1/2}x^*).$$

So $p := x^*x - xx^* - x(1 - x^*x)^{1/2} - (1 - x^*x)^{1/2}x^* = 1 - vv^*$ is a projection in $C_r^*(\mathcal{G})$. We have $\Phi(p) = x^*x - xx^*$ which is nonzero, and so p is nonzero. Also,

since $r(\mathrm{supp}(x)) \subseteq s(\mathrm{supp}(x)) \subseteq V \subseteq \{z \in \mathcal{G}^{(0)} \mid b_0(z) > 3/4\}$, we see that p belongs to the hereditary subalgebra generated by b.

To see that p is an infinite projection, argue exactly as above, but with V replaced by a nonempty open subset of $\mathrm{supp}(x^*x) \smallsetminus \mathrm{supp}(xx^*)$ to obtain a scaling element y in $pC^*(\mathcal{G})p$. Then the calculations we performed above for v show that $w := y + (p - y^*y)^{1/2}$ is a partial isometry with $w^*w = p$ and $ww^* < p$. So p is infinite as required.   □

When $\mathcal{G}$ is also minimal, we can verify that $\mathcal{G}$ is locally contracting by verifying it at any one unit x.

Lemma 4.4.3. *Let $\mathcal{G}$ be an étale groupoid. If $\mathcal{G}$ is minimal, then $\mathcal{G}$ is locally contracting at some point x if and only if $\mathcal{G}$ is locally contracting.*

Proof. The "if" implication is trivial.

For the "only if", suppose that $\mathcal{G}$ is locally contracting at x and fix $y \in \mathcal{G}^{(0)}$. Fix an open neighbourhood V of y. Since $\mathcal{G}$ is minimal, there is an open bisection B such that $r(B) \subseteq V$ and $x \in s(B)$. Since $\mathcal{G}$ is locally contracting at x, there is an open W containing x with $\overline{W} \subseteq s(B)$ and an open bisection U such that $\overline{W} \subseteq s(U)$ and $r(U\overline{W}) \subsetneq W$. Now $r(BW) = BWB^{-1}$ is an open neighbourhood of y with $\overline{r(BW)} \subseteq V$, and BUB^{-1} is a bisection satisfying

$$r(BUB^{-1}\overline{r(BW)}) = r(BUB^{-1}B\overline{W}B^{-1}) = r(BU\overline{W}B^{-1}) = r(Br(U\overline{W})) \subsetneq r(BW).$$
                                                                                      □

Remark 4.4.4. Brown, Clark and Sierakowski [7] have proved that if $\mathcal{G}$ is an étale, effective, minimal groupoid, then $C_r^*(\mathcal{G})$ is purely infinite if and only if every element of $C_0(\mathcal{G}^{(0)})$ is infinite in $C_r^*(\mathcal{G})$.

Chapter 5

Cartan pairs, and Dixmier–Douady theory for Fell Algebras

In this chapter we first discuss the beautiful reconstruction theorem of Renault [39] that shows that an effective groupoid and twist can be recovered from the associated twisted groupoid algebra. This builds on previous work of Kumjian [26], and develops ideas that go back to Feldman and Moore in the context of von Neumann algebras [19, 20, 21]. We will then discuss an application of this theory to the classification of Fell algebras up to spectrum-preserving Morita equivalence [25].

5.1   Kumjian–Renault theory

The aim in this section is to outline Renault's construction for recovering an étale groupoid from its reduced C^*-algebra together with the canonical abelian subalgebra $C_0(\mathcal{G}^{(0)})$. This is a C^*-algebraic analogue of Feldman–Moore theory for von Neumann algebras of Borel equivalence relations.

We will omit almost all of the proofs in this section. The details are due to Kumjian and then Renault and can be found in [26, 39].

To get the most out of this theory, we need to introduce twisted groupoid C^*-algebras. In Renault's original work [37], twisted groupoid C^*-algebras were determined by continuous normalised 2-cocycles on $\mathcal{G}$; that is, continuous maps $\sigma : \mathcal{G}^{(2)} \to \mathbb{T}$ satisfying $\sigma(r(\gamma), \gamma) = 1 = \sigma(\gamma, s(\gamma))$ for all γ and satisfying the cocycle identity $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\beta, \gamma)\sigma(\alpha, \beta\gamma)$ for every composable triple (α, β, γ).

The twisted convolution algebra is then defined as $C_c(\mathcal{G}, \sigma) = C_c(\mathcal{G})$ as

a vector space, but with multiplication and involution given by $(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} \sigma(\alpha, \beta) f(\alpha) g(\beta)$, and $f^*(\gamma) = \sigma(\gamma^{-1}, \gamma) \overline{f(\gamma^{-1})}$. However, Kumjian subsequently observed that the notion of a twisted groupoid C^*-algebra that most naturally leads to an analogue of Feldman–Moore theory comes from a *twist*.

Definition 5.1.1. Let $\mathcal{G}$ be an étale groupoid. A *twist* over $\mathcal{G}$ is a sequence

$$\mathcal{G}^{(0)} \times \mathbb{T} \xrightarrow{\ i\ } \mathcal{E} \xrightarrow{\ \pi\ } \mathcal{G},$$

where $\mathcal{G}^{(0)} \times \mathbb{T}$ is regarded as a trivial group bundle with fibres $\mathbb{T}$, $\mathcal{E}$ is a locally compact Hausdorff groupoid, and i and π are continuous groupoid homomorphisms that restrict to homeomorphisms of unit spaces (we identify $\mathcal{E}^{(0)}$ with $\mathcal{G}^{(0)}$ via i) such that

- i is injective,

- $\mathcal{E}$ is a locally trivial $\mathcal{G}$-bundle in the sense that every point $\alpha \in \mathcal{G}$ has a bisection neighbourhood U on which there exists a continuous section $S : U \to \mathcal{E}$ satisfying $\pi \circ S = \mathrm{id}_U$, and such that the map $(\alpha, z) \mapsto i(r(\alpha), z) S(\alpha)$ is a homeomorphism of $U \times \mathbb{T}$ onto $\pi^{-1}(U)$;

- $i(\mathcal{G}^{(0)} \times \mathbb{T})$ is central in $\mathcal{E}$ in the sense that $i(r(\varepsilon), z)\varepsilon = \varepsilon i(s(\varepsilon), z)$ for all $\varepsilon \in \mathcal{E}$ and $z \in \mathbb{T}$; and

- $\pi^{-1}(\mathcal{G}^{(0)}) = i(\mathcal{G}^{(0)} \times \mathbb{T})$.

If $\mathcal{G} = \Gamma$ is a discrete group, then a twist over $\mathcal{G}$ as defined above is precisely a central extension of Γ.

Notation 5.1.2. If $\mathcal{E}$ is a twist over $\mathcal{G}$, $\varepsilon \in \mathcal{E}$ and $z \in \mathbb{T}$, we will write $z \cdot \varepsilon := i(r(\varepsilon), z)\varepsilon$, and $\varepsilon \cdot z = \varepsilon i(s(\varepsilon), z)$; so $z \cdot \varepsilon = \varepsilon \cdot z$ because $i(\mathcal{G}^{(0)} \times \mathbb{T})$ is central in $\mathcal{E}$.

The local triviality of $\mathcal{E}$ guarantees that π is a quotient map, so that $\mathcal{G}$ is topologically identical to the quotient of $\mathcal{E}$ by the action of $\mathbb{T}$ just described.

Lemma 5.1.3. *If $\mathcal{E} \to \mathcal{G}$ is a twist, and $\varepsilon, \delta \in \mathcal{E}$ satisfy $\pi(\varepsilon) = \pi(\delta)$, then there is a unique $z \in \mathbb{T}$ such that $z \cdot \varepsilon = \delta$.*

Proof. We have $\pi(\varepsilon^{-1}\delta) = s(\delta) \in \mathcal{G}^{(0)}$, so $\varepsilon^{-1}\delta = i(\{s(\delta)\} \times z)$ for some $z \in \mathbb{T}$; there is just one such z because i is injective. We then have $(z \cdot \varepsilon)^{-1}\delta = (\varepsilon^{-1}\delta)i(s(\delta), \overline{z}) = s(\delta)$. Multiplying on the right by $z \cdot \varepsilon$ gives the result. $\square$

Example 5.1.4. The cartesian-product groupoid $\mathcal{G} \times \mathbb{T}$ is a twist over $\mathcal{G}$ in the obvious way. This is called the trivial twist over $\mathcal{G}$.

Example 5.1.5. More generally, if σ is a continuous normalised 2-cocycle on $\mathcal{G}$, then $\mathcal{G} \times \mathbb{T}$ can be made into a groupoid $\mathcal{E}_\sigma$ with the usual unit space and range and source maps, but with multiplication and inversion given by $(\alpha, w)(\beta, z) =$

$(\alpha\beta, \sigma(\alpha\beta)wz)$ and $(\alpha, w)^{-1} = (\alpha, \overline{\sigma(\alpha^{-1}, \alpha)w})$. Since $\sigma(r(\gamma), \gamma)) = 1 = \sigma(\gamma, s(\gamma))$ for all γ, the set inclusion $\mathcal{G}^{(0)} \times \mathbb{T} \hookrightarrow \mathcal{E}_\sigma$ is a groupoid homomorphism, as is the projection $\pi : \mathcal{E}_\sigma \to \mathcal{G}$ given by $\pi(\gamma, z) = \gamma$. It is routine to check that $\mathcal{E}_\sigma$ is then a twist over Γ with respect to i and π.

Remark 5.1.6. We can recover the cohomology class of σ from the twist $\mathcal{E}_\sigma \to \mathcal{G}$ as follows: choose any continuous section S for σ. For $(\alpha, \beta) \in \mathcal{G}^{(2)}$, we have $\sigma(S(\alpha)S(\beta)S((\alpha\beta)^{-1})) = r(\alpha) \in \mathcal{G}^{(0)}$, and so Lemma 5.1.3 shows that there is a unique element $\omega(\alpha, \beta) \in \mathbb{T}$ such that $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = (r(\alpha), \omega(\alpha, \beta))$. The map ω defined in this way is a continuous 2-cocycle. If S' is another continuous section for σ, then $\omega^{-1}\omega'$ is equal to the 2-coboundary obtained from the 1-cochain b determined by $S(\alpha)^{-1}S'(\alpha) = (r(\alpha), b(\alpha))$. Thus the cocycles obtained from distinct choices of S are cohomologous. Taking $S(\gamma) = (\gamma, 1)$ for all γ, yields $\omega = \sigma$, so the cohomology class $[\sigma]$ of σ is equal to that of any cocycle obtained from a continuous section $S : \mathcal{G} \to \mathcal{E}_\sigma$.

More generally, if $\mathcal{E}$ is a twist over $\mathcal{G}$ that admits a continuous section $S : \mathcal{G} \to \mathcal{E}$ for the quotient map, then there is a 2-cocycle σ on $\mathcal{G}$ defined by $S(\alpha)S(\beta)S(\alpha\beta)^{-1} = i(s(\alpha), \sigma(\alpha, \beta))$. There is then an isomorphism $\mathcal{E} \cong \mathcal{E}_\sigma$ that is equivariant for i and q. So $\mathcal{E}$ is isomorphic to a twist coming from a cocycle. But it is not clear that every $\mathcal{E}$ admits a continuous section, so the notion of a twist is formally more general than that of a continuous 2-cocycle.

Definition 5.1.7. If $\mathcal{E}$ is a twist over the étale groupoid $\mathcal{G}$, then we write

$$\Sigma_c(\mathcal{G}; \mathcal{E}) := \{f \in C_c(\mathcal{E}) \mid f(z \cdot \varepsilon) = zf(\varepsilon) \text{ for all } \varepsilon \in \mathcal{E} \text{ and } z \in \mathbb{T}\}.$$

Remark 5.1.8. Each twist $\mathcal{E}$ over $\mathcal{G}$ determines a complex line bundle $\widetilde{\mathcal{E}}$ over $\mathcal{G}$ as follows: Define an equivalence relation $\sim$ on $\mathcal{E} \times \mathbb{C}$ by $(\delta, w) \sim (\varepsilon, z)$ if $\pi(\delta) = \pi(\varepsilon)$, $|w| = |z|$ and either $w = z = 0$ or $(w/|w|) \cdot \delta = (z/|z|) \cdot \varepsilon$. Then $\widetilde{\mathcal{E}} := \mathcal{E}/\sim$ is a line-bundle over $\mathcal{G}$ with respect to the fibre map $p : \widetilde{\mathcal{E}} \to \mathcal{G}$ given by $p([\delta, w]) = \pi(\delta)$.

For $\gamma \in \mathcal{G}$, any choice of $\delta \in \pi^{-1}(\gamma)$ determines a homeomorphism $\mathbb{T} \cong \pi^{-1}(\gamma) \subseteq \mathcal{E}$ given by $z \mapsto z \cdot \delta$. Since Haar measure on $\mathbb{T}$ is rotation invariant, the measure on $\pi^{-1}(\gamma)$ obtained by pulling back Haar measure on $\mathbb{T}$ is independent of our choice of $\delta \in \pi^{-1}(\gamma)$. We endow each $\mathcal{E}^x$ with the measure λ^x that agrees with this pulled back copy of Haar measure on $\pi^{-1}(\gamma)$ for each $\gamma \in \mathcal{G}^x$ (so each $\pi^{-1}(\gamma)$ has measure 1).

Lemma 5.1.9. *The space* $\Sigma_c(\mathcal{G}; \mathcal{E})$ *is a* **-algebra under the operations*

$$f * g(\varepsilon) = \int_{\mathcal{E}^{r(\varepsilon)}} f(\delta)g(\delta^{-1}\varepsilon) \, d\lambda^{r(\varepsilon)} \text{ and } f^*(\varepsilon) = \overline{f(\varepsilon^{-1})}.$$

For any $\varepsilon \in \mathcal{G}$, $f, g \in \Sigma_c(\mathcal{G}; \mathcal{E})$ *and any choice of (not necessarily continuous) section* $\alpha \mapsto \tilde{\alpha}$ *for* $\pi|_{\mathcal{G}^{r(\varepsilon)}}$, *we have*

$$f * g(\varepsilon) = \sum_{\beta \in \mathcal{G}^{r(\varepsilon)}} f(\tilde{\beta})g(\tilde{\beta}^{-1}\varepsilon). \tag{5.1}$$

There is an isomorphism

$$C_c(\mathcal{G}^{(0)}) \cong D_0 := \{f \in \Sigma_c(\mathcal{G}; \mathcal{E}) \mid \operatorname{supp}(f) \subseteq i(\mathcal{G}^{(0)} \times \mathbb{T})\}$$

that carries $f \in C_c(\mathcal{G}^{(0)})$ to the function $\tilde{f} : i(x, z) \mapsto z f(x)$.

Proof. We verify (5.1): if $\beta \in \mathcal{G}^{r(\varepsilon)}$ and $\delta, \delta' \in \pi^{-1}(\beta)$, then $\delta' = z\delta$ for some $z \in \mathbb{T}$, and hence

$$f(\delta)g(\delta^{-1}\varepsilon) = f(\bar{z} \cdot \delta')g(z \cdot (\delta')^{-1}\varepsilon) = \bar{z}f(\delta')zg((\delta')^{-1}\varepsilon) = f(\delta')g((\delta')^{-1}\varepsilon).$$

So each $\int_{\delta \in \pi^{-1}(\beta)} f(\delta)g(\delta^{-1}\varepsilon)d\lambda^{r(\beta)}(\delta)$ collapses to $f(\tilde{\beta})g(\tilde{\beta}^{-1}\varepsilon)$.

From here, that $\Sigma_c(\mathcal{G}; \mathcal{E})$ is a *-algebra follows from calculations similar to the ones that show that $C_c(\mathcal{G})$ is a *-algebra.

Since $x \mapsto i(x, 1)$ is a section for π on $\mathcal{G}^{(0)}$, the final assertion follows from (5.1). $\square$

Remark 5.1.10. Kumjian points out that there is an isomorphism of $\Sigma_c(\mathcal{G}; \mathcal{E})$ with the space of compactly supported continuous sections of the complex line bundle $\widetilde{\mathcal{E}}$ over $\mathcal{G}$ described in Remark 5.1.8. This isomorphism carries $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$ to the section $\tilde{f}$ given by $\tilde{f}(\alpha) = [\tilde{\alpha}, f(\tilde{\alpha})]$ for any choice of $\tilde{\alpha}$ in $\pi^{-1}(\alpha)$.

We define the regular representations π_x, $x \in \mathcal{G}^{(0)}$ of $C_c(\mathcal{G}; \mathcal{E})$ on the spaces $L^2(\mathcal{G}_x; \mathcal{E}_x)$ of square-integrable $\mathbb{T}$-equivariant functions on $\mathcal{E}_x$ by extension of the convolution formula. We define $C_r^*(\mathcal{G}; \mathcal{E})$ to be the completion of the (injective) image of $\Sigma_c(\mathcal{G}; \mathcal{E})$ in the direct sum of these representations, and $\|\cdot\|_r$ the C^*-norm in this C^*-algebra. Arguments very similar to the ones for untwisted algebras give the following:

Theorem 5.1.11. *For any $f \in \Sigma_c(\mathcal{G}; \mathcal{E})$, the set*

$$\big\{ \|\pi(f)\| \,\big|\, \pi \text{ is a }^*\text{-representation of } \Sigma_c(\mathcal{G}; \mathcal{E}) \big\}$$

is bounded above. Taking the supremum gives a pre-C^*-norm $\|\cdot\|$ on $\Sigma_c(\mathcal{G}; \mathcal{E})$, and we define $C^*(\mathcal{G}; \mathcal{E})$ to be the completion in this norm. We have $\|\cdot\|_\infty \leq \|\cdot\|_r \leq \|\cdot\|$ on $\Sigma_c(\mathcal{G}, \mathcal{E})$, with equality on functions supported on $\pi^{-1}(U)$ for any bisection U. If $\mathcal{G}$ is amenable, then $\|\cdot\|_r$ and $\|\cdot\|$ agree. The map $f \mapsto f|_{\mathcal{E}^{(0)}}$ from D_0 to $C_c(\mathcal{G}^{(0)})$ extends to an isomorphism of the completion of D_0, in either norm, with $C_0(\mathcal{G}^{(0)})$.

We will write D for the completion in $C^*(\mathcal{E}; \mathcal{G})$ and D_r for the completion in $C_r^*(\mathcal{G})$.

Remark 5.1.12. For the trivial twist $\mathcal{G} \times \mathbb{T}$, the map $\gamma \mapsto (\gamma, 1)$ is a continuous section for $\pi : \mathcal{G} \times \mathbb{T} \to \mathcal{G}$. The cocycle obtained from this section as in Example 5.1.5 is the trivial one. So we can use the formula (5.1) to see that $C_r^*(\mathcal{G}; \mathcal{G} \times \mathbb{T}) \cong C_r^*(\mathcal{G})$ in the canonical way.

In this section, we are interested in $C_r^*(\mathcal{G}; \mathcal{E})$ and the subalgebra D_r.

Proposition 5.1.13. *Let $\mathcal{G}$ be an effective étale groupoid and $\mathcal{E}$ a twist over $\mathcal{G}$. There is a faithful conditional expectation $\Phi : C_r^*(\mathcal{G};\mathcal{E}) \to D_r$ that extends restriction of functions in $\Sigma_c(\mathcal{G};\mathcal{E})$ to $i(\mathcal{G}^{(0)} \times \mathbb{T})$. This is the only conditional expectation from $C_r^*(\mathcal{G};\mathcal{E})$ to D_r.*

Proof sketch. The proof of existence follows the outline of Proposition 4.2.6. To see that Φ is the unique conditional expectation onto D_r, first observe that the expectation property says that if $\Psi : C_r^*(\mathcal{G};\mathcal{E}) \to D_r$ is a conditional expectation, then for any $a \in C_r^*(\mathcal{G};\mathcal{E})$ and any $b \in D_r$, we have

$$\Psi(ab) = \Psi(a)b = b\Psi(a) = \Psi(ba). \tag{5.2}$$

Arguing as in Lemma 4.2.5, we show that for each $a \in \Sigma_c(\mathcal{G})$ whose support does not intersect $i(\mathcal{G}^{(0)} \times \mathbb{T})$, and each unit $x \in \mathcal{G}^{(0)}$, we can find an element $h \in D_r$ such that $hah = 0$ and $h(x) > 0$. Using that $s(\mathrm{supp}(a))$ is compact and a partition-of-unity argument, we find finitely many h_i such that $a \sum_i h_i^2 = a$ and $h_i a h_i = 0$ for all i. This gives $\Psi(a) = \Psi(a \sum h_i^2) = \sum \Psi(h_i a h_i) = 0$. So Ψ agrees with Φ on the space $D_0^\perp$ of elements of $\Sigma_c(\mathcal{G};\mathcal{E})$ whose support does not intersect $i(\mathcal{G}^{(0)} \times \mathbb{T})$, and it agrees with Φ on D_0 because every conditional expectation is the identity map on its range. Since $\Sigma_c(\mathcal{G};\mathcal{E}) = D_0 + D_0^\perp$, we deduce that Φ and Ψ agree on all of $\Sigma_c(\mathcal{G};\mathcal{E})$, and so are equal. $\square$

As in the untwisted case, each element of $C_r^*(\mathcal{G};\mathcal{E})$ determines a $\mathbb{T}$-equivariant function $j \in C_0(\mathcal{E})$. One way to see this is to fix a section (we do not require continuity) $\gamma \mapsto \tilde{\gamma}$ for the map $\pi : \mathcal{E} \to \mathcal{G}$, so that $\pi(\tilde{\gamma}) = \gamma$ for all γ. If $\delta, \varepsilon \in \mathcal{E}$ satisfy $\pi(\delta) = \pi(\varepsilon)$, then there is a unique $[\delta, \varepsilon] \in \mathbb{T}$ such that $\delta = [\delta, \varepsilon] \cdot \varepsilon$. In particular, if $(\alpha, \beta) \in \mathcal{G}^{(2)}$, then $\pi(\tilde{\alpha}\tilde{\beta}) = \alpha\beta = \pi(\widetilde{\alpha\beta})$. It is not too hard to see that for each $x \in \mathcal{G}^{(0)}$ there is a representation of $\Sigma_c(\mathcal{G};\mathcal{E})$ on $\ell^2(\mathcal{G}_x)$ satisfying

$$\tilde{\pi}_x(f)\delta_\beta = \sum_{\alpha \in \mathcal{G}_{r(\beta)}} [\tilde{\alpha}\tilde{\beta}, \widetilde{\alpha\beta}] f(\tilde{\alpha})\delta_{\alpha\beta},$$

and this representation is unitarily equivalent to the regular representation π_x of $\Sigma_c(\mathcal{G};\mathcal{E})$. With this representation in hand, the argument of Proposition 3.3.3 carries across to the twisted setting.

If A is a C^*-algebra and B is a subalgebra of A, we shall say that $n \in A$ is a *normaliser* of B if $nBn^* \cup n^*Bn \subseteq B$. We write $N(B)$ for the collection of all normalisers of B. We say that B is *regular* in A if A is generated as a C^*-algebra by $N(B)$.

Proposition 5.1.14. *If $\mathcal{G}$ is an étale, effective groupoid, and $\mathcal{E}$ is a twist over $\mathcal{G}$, then D_r is a regular maximal abelian subalgebra of $C_r^*(\mathcal{G};\mathcal{E})$ that contains an approximate unit for $C^*(\mathcal{G};\mathcal{E})$.*

Proof sketch. Clearly D_r is an abelian algebra. To see that it is maximal abelian, suppose that a belongs to its complement. Then $j(a)$ must be nonzero at some

$\varepsilon \in \mathcal{E} \setminus \pi^{-1}(\mathrm{Iso}(\mathcal{G})) = \mathrm{Iso}(\mathcal{E})$. So we can choose $h \in D_0 \subseteq D_r$ such that $h(r(\varepsilon)) = 1$ and $h(s(\varepsilon)) = 0$. Now $j(ah)(\varepsilon) = 0$ whereas $j(ha)(\varepsilon) = a(\varepsilon) \neq 0$. To see that D_r is regular, we use the multiplication formula to see that if $n \in C_c(\pi^{-1}(U))$ for some bisection U of $\mathcal{G}$, then for $h \in C_c(\mathcal{G}^{(0)})$ (regarded as an element of D_0 using the isomorphism of Theorem 5.1.11) we have $\mathrm{supp}(nhn^*) \subseteq \mathrm{supp}(n)\,\mathrm{supp}(h)\,\mathrm{supp}(n)^{-1} \subseteq \pi^{-1}(UU^{-1}) \subseteq \pi^{-1}(\mathcal{G}^{(0)})$, and similarly for n^*hn.   □

Following Renault [39], we shall say that a pair (A, B) of C^*-algebras is a *Cartan pair* and say that B is a *Cartan subalgebra* of A if A is a C^*-algebra, B is a C^*-subalgebra of A containing an approximate unit for A, there is a faithful conditional expectation of A onto B, B is a maximal abelian subalgebra of A, and B is regular in A. We can reinterpret the preceding result as saying that $(C^*(\mathcal{G}; \mathcal{E}), D_r)$ is a Cartan pair. Our main objective here is to prove that every Cartan pair has this form.

Given a Cartan pair (A, B), and given $n \in N(B)$, we write

$$\mathrm{dom}(n) := \overline{\{\phi \in \widehat{B} \mid \phi(n^*n) > 0\}} \quad \text{and} \quad \mathrm{ran}(n) := \overline{\{\phi \in \widehat{B} \mid \phi(nn^*) > 0\}}.$$

We have $\mathrm{ran}(n) = \mathrm{dom}(n^*)$.

Proposition 5.1.15. *Let (A, B) be a Cartan pair of C^*-algebras. For each $n \in N(B)$, there is a homeomorphism $\alpha_n : \mathrm{dom}(n) \to \mathrm{ran}(n)$ satisfying $\alpha_n(\phi)(nbn^*) = \phi(bn^*n)$. There is an equivalence relation $\sim$ on $\{(n, \phi) \mid n \in N(B), \phi \in \mathrm{dom}(n)\}$ such that $(n, \phi) = (m, \psi)$ if and only if $\phi = \psi$ and there is a neighbourhood U of ϕ such that $\alpha_n|_U = \alpha_m|_U$. The set*

$$\mathcal{G}_{(A,B)} := \{[n, \phi] \mid n \in N(B), \phi \in \mathrm{dom}(n)\}$$

is a groupoid with unit space

$$\mathcal{G}^{(0)}_{(A,B)} = \{[b, \phi] \mid b \in B, \phi \in \mathrm{supp}(B)\},$$

and groupoid structure given by

$$r([n, \phi]) = [nn^*, \alpha_n(\phi)], \qquad s([n, \phi]) = [n^*n, \phi],$$
$$[m, \alpha_n(\phi)][n, \phi] = [mn, \phi] \quad \text{and} \quad [n, \phi]^{-1} = [n^*, \alpha_n(\phi)].$$

Proof sketch. Take the polar decomposition $n = v|n|$ of n in A^{**} and observe that if f belongs to the dense subalgebra $C_c(\mathrm{dom}(n)^\circ)$ of $I_{n^*n} := C_0(\mathrm{dom}(n)^\circ)$, then $|n|$ is invertible on $\mathrm{supp}(f)$, and so we can write $f = |n|g|n|^*$ for some $g \in I_{n^*n}$. Hence $vfv^* = v|n|g|n|^*v^* = aga^* \in B$. Applying the same reasoning to n^* we see that conjugation by v determines an isomorphism of commutative C^*-algebras $I_{n^*n} \cong I_{nn^*}$, and therefore induces a homeomorphism α_n between their spectra. It is clear that $\sim$ is an equivalence relation, and that r, s are well defined. To check that the multiplication is well defined, we use the observation that if v, w are the

partial isometries appearing in the polar decompositions of m, n, and if $\mathrm{dom}(m) = \mathrm{ran}(n)$, then vw is the partial isometry appearing in the polar decomposition of mn; and also that for $h \in C_c(\mathrm{supp}(n))$ and ϕ satisfying $\phi(h) \neq 0$, we have $[n, \phi] = [nh, \phi] = [(h \circ \alpha_n^{-1})n, \phi]$. It is easy to check that inversion is well-defined using that if $n = v|n|$, then $n^* = v^*|n^*|$. Associativity of multiplication comes from associativity of multiplication in A, and the inverse property follows directly from the definitions of r and s. □

Theorem 5.1.16. *The groupoid $\mathcal{G}_{(A,B)}$ becomes an étale groupoid under the topology with basic open sets $Z(n, U) := \{[n, \phi] \mid \phi \in U\}$ indexed by elements $n \in N(B)$ and open sets $U \subseteq \mathrm{dom}(n)$. If $A = C^*(\mathcal{G}; \mathcal{E})$ and $B = D_0$ for some twist $\mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E} \to \mathcal{G}$, then there is an isomorphism $\theta : \mathcal{G} \cong \mathcal{G}_{(A,B)}$ such that for any $\gamma \in \mathcal{G}$, any open bisection U containing γ whose closure is a compact bisection and any $n \in \Sigma_c(\mathcal{G}; \mathcal{E})$ that is nonzero everywhere on $\pi^{-1}(U)$, we have $\theta(\gamma) = [n, \widehat{s(\gamma)}]$.*

Proof sketch. It is fairly straightforward to check that the sets $Z(n, U)$ are a base for a locally compact Hausdorff topology, and that $\mathcal{G}_{(A,B)}$ is étale in this topology. If $A = C^*(\mathcal{G}; \mathcal{E})$ and $B = D_r$, then the defining property of α_n (namely $\alpha_n(\phi)(nhn^*) = \phi(hn^*n)$) shows that if n is nonzero on all of $\pi^{-1}(U)$, then $\alpha_n(\widehat{s(\gamma)}) = \widehat{r(\gamma)}$ for $\gamma \in U$. It follows from direct computations that θ is an algebraic isomorphism. The pre-image of a given $Z(n, U)$ under θ is the open bisection $\mathrm{supp}(n)U$, so θ is continuous. Moreover, for a given $\gamma \in \mathcal{G}$ we can choose a compact bisection K containing γ, an open bisection U containing K and an element $n \in C_c(U)$ that is identically 1 on K, and θ is then a continuous bijection of the compact set K onto $Z(n, K)$, and hence a homeomorphism between these sets. Since the interiors of compact bisections form a base for the topology on $\mathcal{G}$, it follows that θ is open. □

As in [39], we call $\mathcal{G}_{(A,B)}$ the *Weyl groupoid* of the Cartan pair (A, B).

Proposition 5.1.17. *Let (A, B) be a Cartan pair of C^*-algebras. There is an equivalence relation $\approx$ on the set $\{(n, \phi) \in N(B) \times \widehat{B} \mid \psi(n^*n) > 0\}$ such that $(m, \phi) \approx (n, \phi)$ if $\phi = \psi$, and there exist $b, b' \in B$ with $\phi(b), \phi(b') > 0$ and $mb = nb'$. The set*

$$\mathcal{E}_{(A,B)} := \{(n, \phi) \in N(B) \times \widehat{B} \mid \psi(n^*n) > 0\}/\approx$$

is a groupoid with unit space $\{[\![h, \phi]\!] \mid h \in C_0(\mathcal{G}^{(0)}), \phi \in \mathrm{dom}(h)\}$, range and source maps $r([\![n, \phi]\!]) = [\![nn^*, \phi]\!]$, $s([\![n, \phi]\!]) = [\![n^*n, \phi]\!]$, multiplication $[\![m, \alpha_n(\phi)]\!][\![n, \phi]\!] = [\![mn, \phi]\!]$, and inversion $[\![n, \phi]\!]^{-1} = [\![n^*, \alpha_n(\phi)]\!]$.

Proof sketch. This is relatively straightforward; the only potential sticking point is well-definedness of multiplication, but this follows from the fact that if $(n, \phi) = (n', \phi)$ and $(m, \alpha_n(\phi)) = (m', \alpha_n(\phi))$, say $nc = n'c'$ and $mb = m'b'$, then we can assume (by multiplying by some $h > 0$ supported on $\alpha_n(\mathrm{supp}(c) \cap \mathrm{supp}(c'))$) that $\mathrm{supp}(b) = \mathrm{supp}(b') \subseteq \mathrm{ran}(n)$, and then note that $bn = n(b \circ \alpha_n)$ and $b'n' = n'(b' \circ \alpha_{n'})$, so that $mn(b \circ \alpha_n c) = m'n'(b' \circ \alpha_{n'}c')$. □

Proposition 5.1.18. *Let (A, B) be a Cartan pair. There is a locally compact Hausdorff topology on $\mathcal{E}_{(A,B)}$ with basic open sets $\widehat{Z}(n, U) = \{[\![n, \phi]\!] \mid \phi \in U\}$ indexed by $n \in N(B)$ and open sets $U \subseteq \mathrm{dom}(n)$. The groupoid $\mathcal{E}_{(A,B)}$ is a topological groupoid in this topology. There is an injective continuous groupoid homomorphism $i_{(A,B)}$: $\widehat{B} \times \mathbb{T} \to \mathcal{E}_{(A,B)}$ given by $i_{(A,B)}(\phi, z) = [\![b, \phi]\!]$ for any $b \in B$ such that $\phi(b) = z$, and there is continuous surjective groupoid homomorphism $\pi_{(A,B)} : \mathcal{E}_{(A,B)} \to \mathcal{G}_{(A,B)}$ such that $\pi([\![n, \phi]\!]) = [n, \phi]$. The sequence $\widehat{B} \times \mathbb{T} \to \mathcal{E}_{(A,B)} \to \mathcal{G}_{(A,B)}$ is a twist over $\mathcal{G}_{(A,B)}$.*

Proof sketch. This is largely a matter of straightforward checking of details. The key point is that if $(m, \phi) \approx (n, \phi)$, say $mb = nb'$ where $b, b' \in B$, then by multiplying b, b' by some positive h with $\phi(h) = 1$, we can assume that $\mathrm{supp}(b) = \mathrm{supp}(b') = U$, say, and then $\alpha_m|_U = \alpha_{mb}|_U = \alpha_{nb'}|_U = \alpha_n|_U$. This shows that the map $[\![n, \phi]\!] \to [n, \phi]$ makes sense and is well defined. To see why $i_{(A,B)}$ is injective, and its image is precisely the kernel of $\pi_{(A,B)}$, observe that for any $b, c \in C_0(\mathcal{G}^{(0)})$ with $\phi(b), \phi(b') \neq 0$, we have $[b, \phi] = [c, \phi]$, but we have $[\![b, \phi]\!] = [\![c, \phi]\!]$ if and only if $\phi(b)/|\phi(b)| = \phi(c)/|\phi(c)|$: for if so, then $z = \phi(b)/|\phi(b)|$ satisfies $\phi(\bar{z}b), \phi(\bar{z}c) > 0$ and $b(\bar{z}c) = c(\bar{z}b)$. □

The following theorem, which is the desired Feldman–Moore-type theorem, in our setting, is due to Renault. For $\mathcal{G}$ principal, it was first proved by Kumjian [26].

Theorem 5.1.19 (Renault, [39, Theorem 5.9]). *Let $\mathcal{G}$ be an effective étale groupoid. Suppose that $\mathcal{E}$ is a twist over $\mathcal{G}$. Then there is an isomorphism $\zeta : \mathcal{E}_{C^*_r(\mathcal{G};\mathcal{E}),D_r} \to \mathcal{E}$ such that the diagram*

$$
\begin{array}{ccccc}
\widehat{D_r} \times \mathbb{T} & \xrightarrow{\ i_{(C^*_r(\mathcal{G};\mathcal{E}),D_r)}\ } & \mathcal{E}_{C^*_r(\mathcal{G};\mathcal{E}),D_r} & \xrightarrow{\ \pi_{(C^*_r(\mathcal{G};\mathcal{E}),D_r)}\ } & \mathcal{G}_{C^*_r(\mathcal{G};\mathcal{E}),D_r} \\
\Big\downarrow{\cong} & & \Big\downarrow{\zeta} & & \Big\downarrow{\theta} \\
\mathcal{G}^{(0)} \times \mathbb{T} & \xrightarrow{\quad i \quad} & \mathcal{E} & \xrightarrow{\quad \pi \quad} & \mathcal{G}
\end{array}
$$

commutes. In particular, the map

$$\left(\mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E} \to \mathcal{G}\right) \longmapsto \left(C^*_r(\mathcal{G};\mathcal{E}), D_r\right)$$

induces a bijection between isomorphism classes of twists and isomorphism classes of Cartan pairs of C^*-algebras.

Corollary 5.1.20. *If (A, B) is a Cartan pair, then there is only one conditional expectation of A onto B.*

Proof. We proved in Proposition 5.1.13 that the expectation extending restriction of functions is the unique expectation from $C^*_r(\mathcal{G};\mathcal{E})$ to D_r. □

In [4], Barlak and Li showed how to extend Tu's result (Theorem 4.1.7) that the C^*-algebras of amenable groupoids always belong to the UCT class to twisted groupoids. Building on this, work of Takeishi and van Erp–Williams on nuclearity of C^*-algebras of groupoid Fell bundles, and combining this with Theorem 5.1.19, Barlak and Li made substantial progress on "UCT question": does every nuclear C^*-algebra belong to the UCT class?

Theorem 5.1.21 (Barlak–Li [4, Theorem 1.1 and Corollary 1.2]). *If $\mathcal{G}$ is an étale groupoid and $\mathcal{E}$ is a twist over $\mathcal{G}$, and if $C_r^*(\mathcal{G}; \mathcal{E})$ is nuclear, then $C_r^*(\mathcal{G}; \mathcal{E})$ belongs to the UCT class. In particular, if A is a separable nuclear C^*-algebra that is Morita equivalent to a C^*-algebra with a Cartan subalgebra, then it is in the UCT class.*

Another interesting application of Theorem 5.1.19 is a beautiful theorem of Matsumoto and Matui [28]. Recall that the Cuntz–Krieger algebra of an irreducible $\{0,1\}$ matrix A (that is not a permutation matrix) agrees with the C^*-algebra of the graph whose adjacency matrix is A [17, 27], and therefore with the groupoid C^*-algebra associated to the graph groupoid described in Example 2.4.7. A fundamental result of Franks [22], building on his previous work with Bowen [6] and on work of Parry and Sullivan [32], says that the (two-sided) shift spaces of $\{0,1\}$-matrices A and B are flow equivalent if and only if $\operatorname{coker}(1 - A^t) \cong \operatorname{coker}(1 - B^t)$ and, if nonzero, $\det(1 - A^t)$ and $\det(1 - B^t)$ have the same sign (positive or negative). Building on previous work of Cuntz [14], Rørdam proved [41] that the Cuntz–Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are stably isomorphic if and only if $\operatorname{coker}(1 - A^t) = \operatorname{coker}(1 - B^t)$. So the Cuntz–Krieger algebra "forgets" some information about flow-equivalence. But, using Renault's results and previous work of Matui, Matsumoto and Matui proved part (1) and the equivalence of (2a) and (2b) in the following theorem in 2013. The final two equivalences, proved later, require the groupoid equivalence theorem of [12], and the results of [11].

Recall that if E is a directed graph, then the Tomforde stabilisation graph SE is obtained by attaching an infinite head at each vertex of E [47]. The resulting "stabilised" symbolic dynamical system can be described as follows: if X_E is the shift space of E (namely E^∞ endowed with the usual shift map), then X_{SE} is given as a space by $X_{SE} = X_E \times \{0, 1, 2, \dots\}$, and the dynamics $\overline{\sigma}$ on X_{SE} is given by

$$\overline{\sigma}(x, n) = \begin{cases} (x, n-1) & \text{if } n \geq 1 \\ (\sigma(x), 0) & \text{if } n = 1. \end{cases}$$

We will call $(X_{SE}, \overline{\sigma})$ the *stabilisation* of (X_E, σ_E), and say that (X_E, σ_E) and (X_F, σ_F) are *stably orbit equivalent* if their stabilisations are orbit equivalent.

Theorem 5.1.22. *Let A, B be irreducible $\{0,1\}$-matrices that are not permutation matrices.*

1. *The following are equivalent:*

    (a) The one-sided shift-spaces determined by A and B are continuously orbit equivalent;

    (b) The Deaconu–Renault groupoids of the one-sided shift maps associated to A and B are isomorphic;

    (c) There is an isomorphism $\mathcal{O}_A \cong \mathcal{O}_B$ that carries $C_0(\mathcal{G}_A^{(0)})$ to $C_0(\mathcal{G}_B^{(0)})$.

2. The following are equivalent

    (a) The two-sided shift-spaces determined by A and B are flow equivalent;

    (b) The Cartan pairs $(\mathcal{O}_A \otimes \mathcal{K}, C_0(\mathcal{G}_A^{(0)}) \otimes c_0)$ and $(\mathcal{O}_B \otimes \mathcal{K}, C_0(\mathcal{G}_B^{(0)}) \otimes c_0)$ are isomorphic as Cartan pairs;

    (c) The groupoids $\mathcal{G}_A \times \mathcal{R}_\mathbb{N}$ and $\mathcal{G}_B \times \mathcal{R}_\mathbb{N}$ are isomorphic.

    (d) The dynamical systems determined (X_E, σ_E) and (X_F, σ_F) are stably orbit equivalent.

5.2   A Dixmier–Douady theorem for Fell algebras

In what follows, given a C^*-algebra A, we shall write $\hat{A}$ for the space of unitary equivalence classes of irreducible representations of A. We give $\hat{A}$ the initial topology obtained from the quotient map from $\hat{A}$ to $\mathrm{Prim}(A)$, where $\mathrm{Prim}(A)$ is given the Jacobson topology.

    Recall that a C^*-algebra A is *liminary* or Type I if every irreducible representation $\pi : A \to \mathcal{B}(\mathcal{H})$ has image $\mathcal{K}(\mathcal{H})$. A positive element a of a liminary C^*-algebra is a *continuous trace element* if $\pi(a)$ has finite trace for every $\pi \in \hat{A}$ and the map $\pi \mapsto \mathrm{Tr}(\pi(a))$ is continuous. A *continuous-trace C^*-algebra* is a liminary C^*-algebra that is generated as an ideal by its continuous-trace elements. The spectrum $\hat{A}$ of a continuous-trace C^*-algebra is always Hausdorff. The Dixmier–Douady theorem [16] says that for a given locally compact Hausdorff space X, the continuous-trace C^*-algebras with spectrum X are classified up to spectrum-preserving Morita equivalence by the Dixmier–Douady invariant, which is an element of the cohomology group $H^3(X, \mathbb{Z})$; moreover, the invariant is exhausted in the sense that each class in $H^3(X, \mathbb{Z})$ occurs as the Dixmier–Douady invariant of some continuous-trace algebra with spectrum X.

    Raeburn and Taylor subsequently gave a very appealing description of the continuous-trace C^*-algebra with given Dixmier–Douady invariant $\delta \in H^3(X, \mathbb{Z})$ using groupoids:

Example 5.2.1 (Raeburn–Taylor [34])**.** Recall that a Čech 2-cocycle on a locally compact Hausdorff space X consists of a cover of X by open sets U_i and a collection of continuous $\mathbb{T}$-valued functions c_{ijk} defined on triple-overlaps $U_{ijk} = U_i \cap U_j \cap U_k$ such that $c_{iij}, c_{ijj} \equiv 1$ for each i, j, and such that on nonempty quadruple overlaps U_{ijkl} we have $c_{ijk} c_{ikl} = c_{ijl} c_{jkl}$. A coboundary is a cocycle of the form $(\delta b)_{ijk}(x) :=$

$b_{ij}(x)b_{ik}(x)^{-1}b_{jk}(x)$ for some collection of continuous functions $b_{ij} : U_{ij} \to \mathbb{T}$ defined on double-overlaps. The Čech cohomology group is the quotient $\check{H}(X, \mathbb{T})$ of the group of 2-cocycles by the subgroup of 2-coboundaries. Given a Čech 2-cocycle on X, we can form an equivalence relation R with unit space $\bigsqcup\{i\} \times U_i \times \{i\}$ and with elements $\{(i, x, j) \mid x \in U_{ij}\}$, where $r(i, x, j) = (i, x, i)$ and $s(i, x, j) = (j, x, j)$. We then construct a twist over R by putting $\mathcal{E} = R \times \mathbb{T}$ and defining multiplication on $\mathcal{E}$ by $\big((i, x, j), w\big)\big((j, x, k), z\big) = \big((i, x, k), c_{ijk}(x)wz\big)$. Note that this twist comes from a continuous 2-cocycle on R. Raeburn and Taylor [34] proved that the C^*-algebra of this twist has Dixmier–Douady invariant equal to the cohomology class of c.

In this section we will give a brief overview of how, using groupoids and the construction of the preceding section, we can obtain a version of the Dixmier–Douady theorem for Fell algebras based on the Raeburn–Taylor construction of the preceding example. The details appear in [25], though of course the ideas there owe a great deal to the previous work of Dixmier–Douady [16], Raeburn–Taylor [34], and the excellent monograph on Dixmier–Douady theory by Raeburn–Williams [35]. The details of the material in this section involve significant extra background and set-up, so I will give almost no proofs, and just touch on the main points of the construction.

Definition 5.2.2. A C^*-algebra is called a *Fell algebra* if it is liminary, and for every $[\pi] \in \hat{A}$, there exists $b \in A^+$ and a neighbourhood U of $[\pi]$ such that $\psi(b)$ is a rank-1 projection whenever $[\psi] \in U$.

Roughly speaking, this says that lots of elements of A have the same rank under nearby irreducible representations. So it should be related to the continuous-trace condition. Indeed, it turns out that a Fell algebra is a continuous-trace algebra if and only if it has Hausdorff spectrum. Theorem 3.3 of [25] says that A is a Fell algebra if and only if it is liminary and generated as an ideal by elements $a \in A_+$ such that aAa is abelian, and that this in turn happens if and only if there is a set S of ideals of A each element of which is Morita equivalent to a commutative C^*-algebra and such that $\bigcup S$ spans a dense subspace of A.

We now show how to obtain an equivalence relation from a Fell algebra.

Proposition 5.2.3. *Let A be a Fell algebra. Choose a sequence d_i of positive elements of A with $\|d_i\| = 1$ such that each $d_i A d_i$ is abelian, and such that the $A d_i A$ generate A. For each i, let $a_i := d_i \otimes \theta_{i,i} \in A \otimes \mathcal{K}$. Then $\sum_i a_i$ converges to an element a of $\mathcal{M}(A \otimes \mathcal{K})$. This element is full in the sense that $\overline{\text{span}}(A \otimes \mathcal{K})a(A \otimes \mathcal{K}) = A \otimes \mathcal{K}$. Moreover $C := a(A \otimes \mathcal{K})a$ and $D := \sum_i a_i(A \otimes \mathcal{K})a_i \cong \bigoplus_i d_i A d_i$ form a Cartan pair (C, D).*

By Corollary 5.1.20, if (C, D) is a Cartan pair, then there is a unique expectation $\Phi : A \to B$. Given $\phi \in \widehat{D}$, the composition $\phi \circ \Phi$ gives a pure state of C, and then the GNS construction yields an irreducible representation. So we obtain a well-defined map $\sigma : \widehat{D} \to \widehat{C}$, which we call the *spectral map*.

Proposition 5.2.4. *Let A be a Fell algebra, and choose a sequence d_i as in the preceding proposition. The Weyl groupoid $\mathcal{G}_{C,D}$ of Theorem 5.1.16 is isomorphic to the equivalence relation $R(\sigma)$ determined by the spectral map: $R(\sigma) = \{(\phi, \psi) \in \widehat{D} \mid \sigma(\phi) = \sigma(\psi)\}$.*

Using this, we are able to characterise diagonal-preserving Morita equivalence of Fell algebras in terms of groupoid equivalence.

We shall say that twists $\mathcal{E}_1 \to \mathcal{G}_1$ and $\mathcal{E}_2 \to \mathcal{G}_2$ are equivalent if there is a linking groupoid L for $\mathcal{G}_1$ and $\mathcal{G}_2$ and a twist $\mathcal{L}$ over L such that reduction of L and $\mathcal{L}$ to $\mathcal{G}_i^{(0)} \subseteq L^{(0)}$ yields a twist $\mathcal{G}_i^{(0)} \times \mathbb{T} \to \mathcal{G}_i^{(0)} \mathcal{L} \mathcal{G}_i^{(0)} \to \mathcal{G}_i^{(0)} L \mathcal{G}_i^{(0)}$ that is isomorphic to $\mathcal{G}_i^{(0)} \times \mathbb{T} \to \mathcal{E}_i \to \mathcal{G}_i$. This is the natural extension of the notion of groupoid equivalence to twists.

Proposition 5.2.5. *If (C_1, D_1) and (C_2, D_2) are Cartan pairs in which C_1 and C_2 are Fell algebras, then C_1 and C_2 are Morita equivalent if and only if the twists $\mathbb{T} \times \widehat{D}_i \to \mathcal{E}_{C_i, D_i} \to \mathcal{G}_{C_i, D_i}$ $(i = 1, 2)$ are equivalent twists.*

The Dixmier–Douady invariant of a continuous-trace C^*-algebra with spectrum X is an element of a cohomology group. For us, the collection of equivalence classes of twists will act as a proxy for this cohomology group. (Theorem 5.2.7 describes how these two groups are related in the continuous-trace setting.)

We say that twists $\mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{E} \to \mathcal{G}$ and $\mathcal{G}^{(0)} \times \mathbb{T} \to \mathcal{F} \to \mathcal{G}$ over the same étale groupoid $\mathcal{G}$ are *isomorphic* if there is a groupoid isomorphism $\zeta : \mathcal{E} \to \mathcal{F}$ such that the diagram

$$
\begin{array}{ccccc}
\mathcal{G}^{(0)} \times \mathbb{T} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} \\
\downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle =} \\
\mathcal{G}^{(0)} \times \mathbb{T} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}
\end{array}
$$

commutes.

For the following result, we need to describe the pullback construction for twists over a given relation R. Let R be an equivalence relation, and suppose that $\mathcal{E}$ and $\mathcal{E}'$ are twists over R. Define an equivalence relation $\sim$ on

$$\mathcal{E} \,_\pi\!\times_{\pi'} \mathcal{E}' := \{(\varepsilon, \varepsilon') \in \mathcal{E} \times \mathcal{E}' \mid \pi(\varepsilon) = \pi'(\varepsilon')\}$$

by $(\varepsilon, \varepsilon') \sim (\delta, \delta')$ if and only if there exists $z \in \mathbb{T}$ such that $z \cdot \varepsilon = \delta$ and $\overline{z} \cdot \varepsilon' = \delta'$.

The pullback $\mathcal{E} * \mathcal{E}'$ is defined as

$$\mathcal{E} * \mathcal{E}' := \left(\mathcal{E} \,_\pi\!\times_{\pi'} \mathcal{E}' \right)/\!\sim.$$

This is a twist over R with respect to the map $\pi * \pi' : \mathcal{E} * \mathcal{E}' \to R$ given by $(\pi * \pi')([\varepsilon, \varepsilon']) = \pi(\varepsilon)$, and the map $i * i' : R^{(0)} \times \mathbb{T} \to \mathcal{E} * \mathcal{E}'$ given by $(i * i')(x, z) := [i(x, z), i'(x, 1)]$.

Lemma 5.2.6. *Let R be a topological equivalence relation. Then the collection of isomorphism classes of twists over R becomes an abelian group Tw_R with identity element equal to the class of the trivial twist, and with group operation given by $[\mathcal{E}] + [\mathcal{E}'] := [\mathcal{E} * \mathcal{E}']$.*

If A is a Fell algebra, we write $\mathcal{S}$ for the sheaf of germs of continuous $\mathbb{T}$-valued functions on $\widehat{A}$. One can then form the sheaf cohomology group $H^2(\widehat{A}, \mathcal{S})$. If $\widehat{A}$ is Hausdorff, then $H^2(\widehat{A}, \mathcal{S})$ is isomorphic to $H^3(\widehat{A}, \mathbb{Z})$.

Theorem 5.2.7. *Let R be a topological equivalence relation. Then there is a homomorphism $\rho_R : \mathrm{Tw}_R \to H^2(\widehat{A}, \mathcal{S})$. If A is a Fell algebra, (C_1, D_1) and (C_2, D_2) are two Cartan pairs constructed as in Proposition 5.2.3 and $\mathcal{E} := \mathcal{E}_{C_1, D_1} \to \mathcal{G}_{C_1, D_1}$ and $\mathcal{E}' := \mathcal{E}_{C_2, D_2} \to \mathcal{G}_{C_2, D_2}$ are the twists obtained from these two pairs, then $\rho_R([\mathcal{E} \to R]) = \rho_{R'}([\mathcal{E}' \to R'])$.*

We denote the element $\rho_R([\mathcal{E} \to R]) \in H^2(\widehat{A}, \mathcal{S})$ obtained from any Cartan pair constructed as in Proposition 5.2.3 by $\delta(A)$. For our final result, we need to recall that if A is a C^*-algebra, then its spectrum $\widehat{A}$ is a locally compact locally Hausdorff space whose every open subset is again locally compact. Dixmier calls such spaces *quasi locally compact* [15], but I'm going to call them *locally locally compact*.

Theorem 5.2.8.   *1. Let A and A' be Fell algebras. Then A and A' are Morita equivalent if and only if there is a homeomorphism $\widehat{A} \cong \widehat{A'}$ such that the induced isomorphism $H^2(\widehat{A}, \mathcal{S}) \cong H^2(\widehat{A'}, \mathcal{S})$ carries $\delta(A)$ to $\delta(A')$.*

     2. If X is a locally locally compact, locally Hausdorff space, and $\delta \in H^2(X, \mathcal{S})$, then there exist a Fell algebra A and a homeomorphism $\widehat{A} \to X$ such that the induced isomorphism $H^2(\widehat{A}, \mathcal{S}) \cong H^2(X, \mathcal{S})$ carries $\delta(A)$ to δ.

More or less by definition of the invariant $\delta(A)$, the proof of Theorem 5.2.8(2) is very closely related to the Raeburn–Taylor construction: Take $\delta \in H^2(X, \mathcal{S})$, represent it by a Čech cocycle c defined on an open cover $X = \bigcup_{i \in I} U_i$ by Hausdorff neighbourhoods. Let $Y := \bigsqcup_{i \in I} \{i\} \times U_i$, and define $\psi : Y \to X$ by $\psi(i, x) = x$. The cocycle c then determines a continuous cocycle σ on $R(\psi)$ as in Example 5.2.1, and the C^*-algebra of the resulting twist is then a Fell algebra with invariant δ.

Remark 5.2.9. I have refrained from calling $\delta(A)$ the Dixmier–Douady invariant of A because, unfortunately, if A is a continuous-trace C^*-algebra, it is not clear that $\delta(A)$ is equal to the classical Dixmier–Douady invariant.

Bibliography

[1] C. Anantharaman-Delaroche, *Purely infinite C^*-algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), 199–225.

[2] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Foreword by Georges Skandalis and Appendix B by E. Germain, L'Enseignement Mathématique, Geneva, 2000, 196.

[3] P. Ara, *Morita equivalence for rings with involution*, Algebr. Represent. Theory **2** (1999), 227–247.

[4] S. Barlak and X. Li, *Cartan subalgebras and the UCT problem*, preprint 2015 (arXiv:1511.02697 [math.OA]).

[5] B. Blackadar, Operator algebras, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III, Springer-Verlag, Berlin, 2006, xx+517.

[6] R. Bowen and J. Franks, *Homology for zero-dimensional nonwandering sets*, Ann. of Math. (2) **106** (1977), 73–92.

[7] J. Brown, L.O. Clark and A. Sierakowski, *Purely infinite C^*-algebras associated to étale groupoids*, Ergodic Theory Dynam. Systems **35** (2015), 2397–2411.

[8] J. Brown, L.O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[9] L.G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), 335–348.

[10] L.G. Brown, P. Green and M.A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific J. Math. **71** (1977), 349–363.

[11] N. Brownlowe, T.M. Carlsen and M.F. Whittaker, *Graph algebras and orbit equivalence*, Ergodic Theory Dynam. Systems **37** (2017), 389–417.

[12] T.M. Carlsen, E. Ruiz, and A. Sims, *Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph C^*-algebras and Leavitt path algebras*, Proc. Amer. Math. Soc. **145** (2017), 1581–1592.

[13] J.B. Conway, A course in functional analysis, Springer–Verlag, New York, 1990, xvi+399.

[14] J. Cuntz, *The classification problem for the C^*-algebras $\mathcal{O}_A$*, Pitman Res. Notes Math. Ser., 123, Geometric methods in operator algebras (Kyoto, 1983), 145–151, Longman Sci. Tech., Harlow, 1986.

[15] J. Dixmier, C^*-algebras, Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15, North-Holland Publishing Co., Amsterdam, 1977, xiii+492.

[16] J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[17] M. Enomoto and Y. Watatani, *A graph theory for C^*-algebras*, Math. Japon. **25** (1980), 435–442.

[18] R. Exel, *Inverse semigroups and combinatorial C^*-algebras*, Bull. Braz. Math. Soc. (N.S.) **39** (2008), 191–313.

[19] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, Bull. Amer. Math. Soc. **81** (1975), 921–924.

[20] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[21] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. **234** (1977), 325–359.

[22] J. Franks, *Flow equivalence of subshifts of finite type*, Ergodic Theory Dynam. Systems **4** (1984), 53–66.

[23] E. Guentner, R. Willett and G. Yu, *Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^*-algebras*, Math. Ann. **367** (2017), 785–829.

[24] P. Hahn, *Haar measure for measure groupoids*, Trans. Amer. Math. Soc. **242** (1978), 1–33.

[25] A. an Huef, A. Kumjian and A. Sims, *A Dixmier–Douady theorem for Fell algebras*, J. Funct. Anal. **260** (2011), 1543–1581.

[26] A. Kumjian, *On C^*-diagonals*, Canad. J. Math. **38** (1986), 969–1008.

[27] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

[28] K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Kyoto J. Math. **54** (2014), 863–877.

[29] P.S. Muhly, J.N. Renault and D.P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), 3–22.

[30] W. Paravicini, *Induction for Banach algebras, groupoids and* KK$^{\text{ban}}$, J. K-Theory **4** (2009), 405–468.

[31] A.L.T. Paterson, Groupoids, inverse semigroups, and their operator algebras, Birkhäuser Boston Inc., Boston, MA, 1999, xvi+274.

[32] B. Parry and D. Sullivan, *A topological invariant of flows on 1-dimensional spaces*, Topology **14** (1975), 297–299.

[33] I. Raeburn, Graph algebras, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.

[34] I. Raeburn and J.L. Taylor, *Continuous trace C^*-algebras with given Dixmier–Douady class*, J. Austral. Math. Soc. Ser. A **38** (1985), 394–407.

[35] I. Raeburn and D.P. Williams, Morita equivalence and continuous-trace C^*-algebras, American Mathematical Society, Providence, RI, 1998, xiv+327.

[36] A. Ramsay, *The Mackey–Glimm dichotomy for foliations and other Polish groupoids*, J. Funct. Anal. **94** (1990), 358–374.

[37] J.N. Renault, A groupoid approach to C^*-algebras, Springer, Berlin, 1980, ii+160.

[38] J.N. Renault, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Theory **25** (1991), 3–36.

[39] J.N. Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bulletin **61** (2008), 29–63.

[40] J.N. Renault and D.P. Williams, *Amenability of groupoids arising from partial semigroup actions and topological higher rank graphs*, Trans. Amer. Math. Soc. **369** (2017), 2255–2283.

[41] M. Rørdam, *Classification of Cuntz–Krieger algebras*, K-Theory **9** (1995), 31–58.

[42] A. Sims and D.P. Williams, *Renault's equivalence theorem for reduced groupoid C^*-algebras*, J. Operator Theory **68** (2012), 223–239.

[43] A. Sims and D.P. Williams, *The primitive ideals of some étale groupoid C^*-algebras*, Algebr. Represent. Theory **19** (2016), 255–276.

[44] J. Spielberg, *Groupoids and C^*-algebras for categories of paths*, Trans. Amer. Math. Soc. **366** (2014), 5771–5819.

[45] M.M. Stadler and M. O'uchi, *Correspondence of groupoid C^*-algebras*, J. Operator Theory **42** (1999), 103–119.

[46] G. Szabo, *Notes on Rokhlin dimension*, to appear in the volume "Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension" in *Advanced Courses in Mathematics. CRM Barcelona.*.

[47] M. Tomforde, *Stability of C^*-algebras associated to graphs*, Proc. Amer. Math. Soc. **132** (2004), 1787–1795.

[48] J.-L. Tu, *La conjecture de Baum–Connes pour les feuilletages moyennables*, K-Theory **17** (1999), 215–264.

[49] J.-L. Tu, *Non-Hausdorff groupoids, proper actions and K-theory*, Doc. Math. **9** (2004), 565–597.

[50] R. Willett, *A non-amenable groupoid whose maximal and reduced C^*-algebras are the same*, Münster J. Math. **8** (2015), 241–252.

[51] D.P. Williams, Crossed products of C^*-algebras, American Mathematical Society, Providence, RI, 2007, xvi+528.

[52] D.P. Williams, *A primer on crossed products*, to appear in the volume "Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension" in *Advanced Courses in Mathematics. CRM Barcelona.*.