The Two-Phase Simplex Method – Tableau Format

<u>Example 1:</u> Consider the problem

min z =	$4x_1 + x_2 + x_3$		
s.t.	$2x_1 + x_2 + 2x_3$	=	4
	$3x_1 + 3x_2 + x_3$	=	3
	x_1, x_2, x_3	>=	0

There is no basic feasible solution apparent so we use the two-phase method. The artificial variables are y_1 and y_2, one for each constraint of the original problem. The Phase I objective is min $w = y_1 + y_2$. The starting tableau (in nonstandard form) is:

BASIS	x_1	x_2	x_3	y_1	y_2	RHS
y_1	2	1	2	1	0	4
y_2	3	3	1	0	1	3
(-z)	4	1	1	0	0	0
(-w)	0	0	0	1	1	0

We convert the tableau to standard form by zeroing out the coefficients of the basic variables in the *w*-row:

BASIS	x_1	x_2	x_3	y_1	y_2	RHS	Min Ratio
y_1	2	1	2	1	0	4	2
y_2	3	3	1	0	1	3	1
(-z)	4	1	1	0	0	0	
(-w)	-5	-4	-3	0	0	-7	

The coefficient of x_1 in the *w*-row is negative so we attempt to bring x_1 in to the basis. The minimum ratio test is min{4/2, 3/3} = 1 so y_2 leaves the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	y_1	y_2	RHS	Min Ratio
y_1	0	-1	1.33	1	-0.67	2	1.5
x_1	1	1	0.33	0	0.33	1	3
(-z)	0	-3	-0.33	0	-1.33	-4	
(-w)	0	1	-1.33	0	1.67	-2	

The coefficient of x_3 in the *w*-row is negative so we attempt to bring x_3 in to the basis. The minimum ratio test is min{2/(4/3), 1/(1/3)} = 1.5 so y_1 leaves the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	y_1	y_2	RHS
x_3	0	-0.75	1	0.75	-0.5	1.5
x_1	1	1.25	0	-0.25	0.5	0.5
(-z)	0	-3.25	0	0.25	-1.5	-3.5
(-w)	0	0	0	1	1	0

All the coefficients in the *w*-row are nonnegative, w = 0, and there are no artificial variables in the basis, so we are done with Phase I. Phase II begins with the tableau shown below.

BASIS	x_1	x_2	x_3	RHS	Min Ratio
x_3	0	-0.75	1	1.5	
x_1	1	1.25	0	0.5	0.4
(-z)	0	-3.25	0	-3.5	

The coefficient of x_2 in the *z*-row is negative so we attempt to bring x_2 in to the basis. The minimum ratio test is min{(1/2)/(5/4)} = 2/5 so x_1 leaves the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	RHS
x_3	0.6	0	1	1.8
x_2	0.8	1	0	0.4
(-z)	2.6	0	0	-2.2

All the coefficients in the *z*-row are nonnegative so we are done with Phase II. The optimum solution is x = (0, 0.4, 1.8) and z = 2.2.

<u>Example 2:</u> (This is the problem started in section on 10/02/03.) Consider the problem

$$
\begin{array}{lllll}
\min z = & 2x_1 + 6x_2 + x_3 + x_4 & & \\
\text{s.t.} & x_1 + 2x_2 + x_4 & = & 6 \\
 & x_1 + 2x_2 + x_3 + x_4 & = & 7 \\
 & x_1 + 3x_2 - x_3 + 2x_4 & = & 7 \\
 & x_1 + x_2 + x_3 & = & 5 \\
 & x_1, x_2, x_3, x_4 & >= & 0
\end{array}
$$

Again, there is no basic feasible solution apparent so we'll use the two-phase method. The artificial variables are y_1, y_2, y_3, and y_4, one for each constraint of the original problem. The Phase I objective is min $w = y_1 + y_2 + y_3 + y_4$. The starting tableau (in nonstandard form) is:

BASIS	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
y_1	1	2	0	1	1	0	0	0	6
y_2	1	2	1	1	0	1	0	0	7
y_3	1	3	-1	2	0	0	1	0	7
y_4	1	1	1	0	0	0	0	1	5
(-z)	2	6	1	1	0	0	0	0	0
(-w)	0	0	0	0	1	1	1	1	0

We convert the tableau to standard form by zeroing out the coefficients of the basic variables in the *w*-row:

BASIS	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Min Ratio
y_1	1	2	0	1	1	0	0	0	6	6
y_2	1	2	1	1	0	1	0	0	7	7
y_3	1	3	-1	2	0	0	1	0	7	7
y_4	1	1	1	0	0	0	0	1	5	5
(-z)	2	6	1	1	0	0	0	0	0	
(-w)	-4	-8	-1	-4	0	0	0	0	-25	

The coefficient of x_1 in the *w*-row is negative so we attempt to bring x_1 in to the basis. The minimum ratio test indicates that y_4 should leave the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Min Ratio
y_1	0	1	-1	1	1	0	0	-1	1	1
y_2	0	1	0	1	0	1	0	-1	2	2
y_3	0	2	-2	2	0	0	1	-1	2	1
x_1	1	1	1	0	0	0	0	1	5	5
(-z)	0	4	-1	1	0	0	0	-2	-10	
(-w)	0	-4	3	-4	0	0	0	4	-5	

The coefficient of x_2 in the *w*-row is negative so we attempt to bring x_2 in to the basis. The minimum ratio test indicates that y_1 should leave the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS	Min Ratio
x_2	0	1	-1	1	1	0	0	-1	1	
y_2	0	0	1	0	-1	1	0	0	1	1
y_3	0	0	0	0	-2	0	1	1	0	
x_1	1	0	2	-1	-1	0	0	2	4	2
(-z)	0	0	3	-3	-4	0	0	2	-14	
(-w)	0	0	-1	0	4	0	0	0	-1	

The coefficient of x_3 in the *w*-row is negative so we attempt to bring x_3 in to the basis. The minimum ratio test indicates that y_2 should leave the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	RHS
x_2	0	1	0	1	0	1	0	-1	2
x_3	0	0	1	0	-1	1	0	0	1
y_3	0	0	0	0	-2	0	1	1	0
x_1	1	0	0	-1	1	-2	0	2	2
(-z)	0	0	0	-3	-1	-3	0	2	-17
(-w)	0	0	0	0	3	1	0	0	0

All the coefficients in the *w*-row are nonnegative, w = 0, BUT there is artificial variable, y_3, in the basis, so we are not quite done with Phase I.

Observe that $y_3 = 0$. Consider the matrix obtained by removing the columns corresponding to the artificial variables and w-row from the tableau (as we would do to start Phase II):

BASIS	x_1	x_2	x_3	x_4	RHS
x_2	0	1	0	1	2
x_3	0	0	1	0	1
y_3	0	0	0	0	0
x_1	1	0	0	-1	2
(-z)	0	0	0	-3	-17

The shaded row is a zero vector. This indicates that the row associated with the artificial variable y_3 is not linearly independent of the other rows of the matrix. It is not too hard to discover that:

CONSTRAINT 1 + CONSTRAINT 2 – CONSTRAINT 4 = CONSTRAINT 3.

Thus, we can safely remove constraint 3 from the tableau without changing the feasible region. Phase II begins with the tableau shown below.

BASIS	x_1	x_2	x_3	x_4	RHS
x_2	0	1	0	1	2
x_3	0	0	1	0	1
x_1	1	0	0	-1	2
(-z)	0	0	0	-3	-17

The coefficient of x_4 in the *z*-row is negative so we attempt to bring x_4 in to the basis. The minimum ratio test indicates that x_2 leaves the basis. The pivot element is shaded.

BASIS	x_1	x_2	x_3	x_4	RHS
x_2	0	1	0	1	2
x_3	0	0	1	0	1
x_4	1	1	0	0	4
(-z)	0	3	0	0	-11

All the coefficients in the *z*-row are nonnegative so we are done with Phase II. The optimum solution is x = (4, 0, 1, 2) and z = 11.