RESEARCH PROBLEMS

1. Daihachiro Sato: *Function theory.*

A. Does there exist a transcendental entire (meromorphic) function which has (1) algebraic values at all algebraic points and has (2) transcendental values at all transcendental points? (The proposer constructed a transcendental entire function with condition (1) alone in [4], using the method similar to that in [1]. The question of the existence of a transcendental entire function with condition (2) alone is open.)

B. Let

$$\begin{aligned}\phi(r) &= \max_n \frac{r^n}{\Gamma(n+1)} \\ &= \exp\left\{r - \frac{1}{2}\log r - \frac{1}{2}\log 2\pi + \frac{1}{24r} + \sum_{k=2}^{m} \frac{C_k}{r^k} + O\left(\frac{1}{r^{m+1}}\right)\right\}.\end{aligned}$$

Find the coefficients C_k explicitly. Are there infinitely many k with $C_k = 0$ as in the case of Stirling's formula for $\Gamma(r)$? (The C_k can be calculated successively, but we want to have C_k as a function of k as in the case of Stirling's formula.) $\phi(r)$ gives precise dividing line for the growth $M(r)$ of Hurwitz entire functions (i.e., entire functions $f(z)$ with $f^{(n)}(0) = \text{integer}$, $n = 0, 1, 2, \cdots$) below which one finds only polynomials [2; 3; 5].

References

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4. D. Sato, *A simple example of transcendental entire function which together with all its higher derivatives assumes algebraic values at all algebraic points*, Proc. Amer. Math. Soc. **14** (1963), 996.

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2. J. R. Isbell: *Piecewise polynomial functions.*

Can every continuous piecewise polynomial function on the plane be generated from polynomials by lattice operations? (f is piecewise polynomial if it satisfies an equation $\prod_i (f(x)-P_i(x))\equiv 0$, where the P_i are finitely many polynomials.)

This is related to questions on the field of elementarily definable sets, particularly whether components are elementarily definable. (This is the field generated by sets $P(x)>0$, P an integral polynomial; see Tarski, *A decision method for elementary algebra and geometry*, Berkeley, 1951.) In n-space it is known that an elementarily definable set has only finitely many components. For the sets $P(x)>0$, John Nash notes there are no more components than the number of maxima of a suitable rational function P/Q. From some elementary formulas (Henriksen and Isbell, Pacific J. Math. **12** (1962), 533), the lattice combinations of polynomials form a ring.

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3. E. M. Horadam: *Number theory or numerical semigroups.*

Suppose given a finite or infinite sequence $\{p\}$ of real numbers (generalised primes) such that $1<p_1<p_2<p_3<\cdots$. Form the set $\{l\}$ of all possible p-products, i.e., products $p_1^{v_1}p_2^{v_2}\cdots$, where $v_1, v_2\cdots$ are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalised integers and suppose that no two generalised integers are equal if their v's are different.

Then arrange $\{l\}$ as an increasing sequence: $1=l_1<l_2<l_3<\cdots$. Let $[x]$ denote the number of generalised integers $\leqq x$.

A particular sequence of generalised integers is constructed with the following additional assumption. If

$$[l_n] = [l_r] + [l_s],$$

then

$$\left[\frac{l_n}{p}\right] \geqq \left[\frac{l_r}{p}\right] + \left[\frac{l_s}{p}\right], \tag{1}$$

for any generalised prime p, and any generalised integers l_n, l_r and l_s.

Given the first two generalised primes, this sequence must begin as follows:

$$1 < p_1 < p_1^2 < \cdots < p_1^k < p_2 < \cdots \text{ (k an integer $\geqq 1$).} \tag{2}$$

PROBLEM A. Does assumption (1) define the sequence (2) sufficiently to give the only possible positions in which succeeding generalised primes $p_3, p_4, \cdots$ may appear? The answer to this is probably yes.

PROBLEM B. If succeeding generalised primes $p_3, p_4, \cdots$ are placed in all positions open to them, can the general term of (2) be found explicitly?

PROBLEM C. Apart from the special case $1<p_1<p_2<p_1^2<\cdots$, is it possible for (2) to contain two consecutive numbers which are both generalised primes?

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4. David E. Daykin: *Space filling with unequal cubes.*

It is shown in [1] that a unit square can be dissected into unequal rational squares. Using such a dissected unit square one can fill the plane with unequal rational squares (see Figure 1). One can do the same thing without having a dissected rectangle (see Figures 2, 3). It

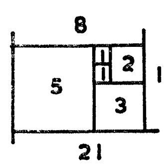

FIGURE 1

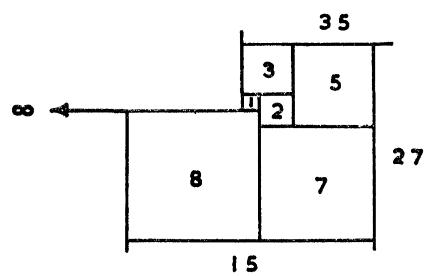

FIGURE 2

is also proved in [1] that a cube cannot be dissected into smaller unequal cubes. On the other hand C. A. Rogers has given an interesting space filling using cubes of just two sizes (cf. [2, p. 148]). We here ask, "Can space be filled with disjoint cubes, no two cubes being the same size, and the lengths of the edges of the cubes being integers?"

REFERENCES

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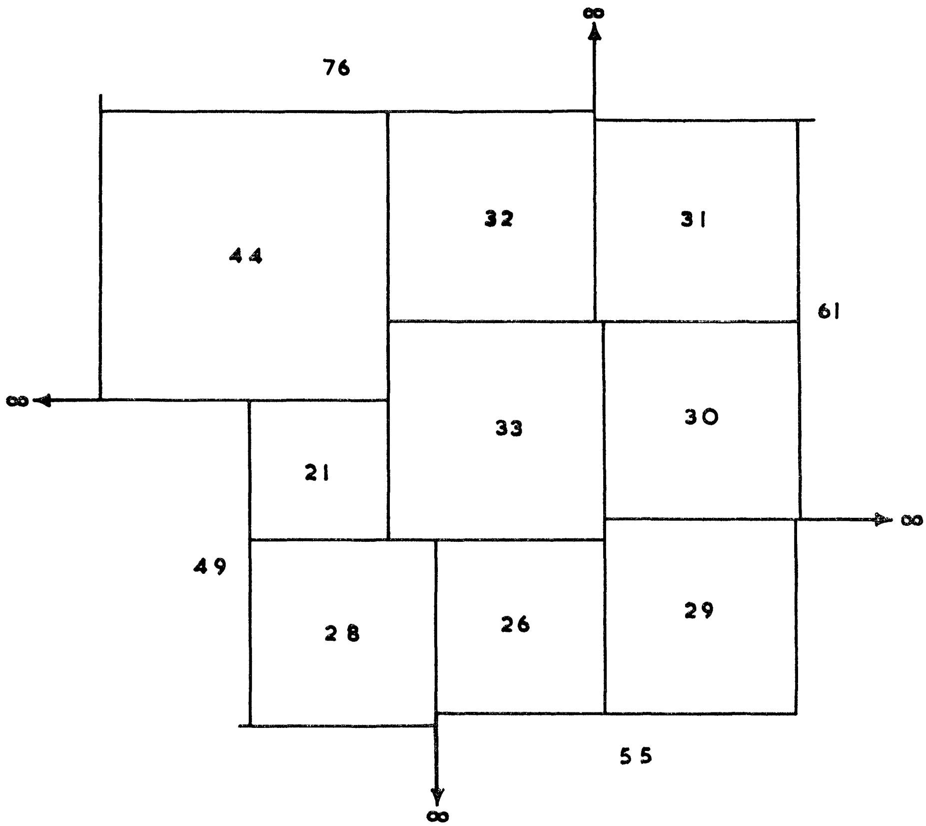

FIGURE 3

5. Robert R. Korfhage: *On a sequence of prime numbers.*

In the research problem entitled *Recursive function theory* (Bull. Amer. Math. Soc. **69** (1963), 737), Mullin raises a series of questions concerning prime sequences generated by following Euclid's scheme for proving the infinitude of the primes. We address ourselves to the third question, namely, whether or not the sequence generated in this manner, choosing at each step the highest prime factor, is monotone increasing. A short calculation on our IBM 7090 has shown that the sequence in question is 2, 3, 7, 43, 139, 50207, 340999, 3202139, 410353, · · · , and hence is not monotone. In fact, an examination of the table of prime factors given below shows that there is no way to choose the prime factors to form a monotone sequence, since at each stage there is at most one possible choice, namely the highest prime factor.

n	P_n	Prime Factors of $\prod_{i=1}^{n} P_i+1$
1	2	3
2	3	7
3	7	43
4	43	13, 139
5	139	5, 50207
6	50207	23, 1607, 340999
7	340999	5521, 3202139
8	3202139	5, 53, 199, 410353
9	410353	. . .

In view of this result, it seems natural to add the following questions to those proposed by Mullin. (i) Are any, or all, of the sets generated in this manner and choosing the prime factor at each stage in any way recursive? (ii) Do any, or all, of these sets contain all of the prime numbers?

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