JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 1, Number 1, January 1988

p-ADIC HODGE THEORY

GERD FALTINGS

1. INTRODUCTION

(a) We intend to describe a relationship between p-adic étale cohomology and Hodge cohomology for smooth algebraic varieties over a p-adic field K. The results are similar to classical Hodge theory over the complex numbers with Betti cohomology replaced by étale cohomology; however, there are some differences. For example, the Hodge filtration goes the other way. While in the classical case a holomorphic differential form defines a class in singular cohomology, giving an injection of the space of differentials into the cohomology group, in the p-adic case there is a surjection from cohomology to differentials. Also we are missing the $\mathbf{Z}$-structure on singular cohomology and the Gauss-Manin connection for families.

(b) Let us describe our results. Denote by V a complete discrete valuation ring of unequal characteristic with perfect residue field k of characteristic p and K its field of fractions. If X is a smooth proper K-scheme, there is a natural isomorphism (Chapter III, Theorem 4.1)

$$H^n\left(X\bigotimes_K \overline{K}, \mathbf{Z}_p\right)\bigotimes_{\mathbf{Z}_p} K^\wedge \cong \bigoplus_{a+b=n} H^a(X, \Omega^b_{X/K})\bigotimes_K \overline{K}^\wedge(-b).$$

Here the cohomology on the left is étale cohomology, " $\wedge$ " denotes the p-adic completion, " $(-b)$ " Tate twist, and the isomorphism is $\mathrm{Gal}(\overline{K}/K)$-equivariant. There is also a version with derived categories, or for open varieties which are the complement of a divisor with normal crossings in a complete X as above. Using corollaries one may derive algebraic proofs of degeneration of the Hodge spectral sequence and of Kodaira's vanishing theorem. However, there are better ways to do this using crystalline cohomology and Frobenius.

(c) If X extends to a smooth proper scheme over V, we have stronger statements. To describe them we need some notation. Denote by $\overline{V}$ the integral closure of V in $\overline{K}$, and by $\mathfrak{m}$ its maximal ideal. $\overline{V}$ is a valuation ring with nondiscrete value group $\mathbf{Q}$, so $\mathfrak{m}$ behaves differently from what we are used to. For example, $\mathfrak{m} = \mathfrak{m}^2$. An almost isomorphism of $\overline{V}$-modules is a map whose kernel and cokernel are annihilated by $\mathfrak{m}$.

Received by the editors February 27, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14F30.

We shall define an intermediate cohomology theory $\mathscr{H}^*(X)$ with a map $H^n(X \otimes_K \overline{K}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \overline{V}^\wedge \to \mathscr{H}^*(X)$, which turns out to be an almost isomorphism.

On the other hand $\mathscr{H}^*(X)$ is closely related to Hodge cohomology.

–In the sense of derived categories, it has a filtration with quotients isomorphic to $H^a(X, \Omega^b_{X/K}) \otimes_K \overline{V}^\wedge(-b)$ up to p-torsion of exponent $p^{1/(p-1)}$.

–There exists a natural transformation (§II, Theorem 3.1)

$$\bigoplus_{a+b=n} H^a(X, \Omega^b_{X/V}) \bigotimes_V \rho^b \overline{V}^\wedge(-b) \to \mathscr{H}^n(X),$$

where $\rho \in \overline{V}$ is a certain element related to the ramification of V over the Witt ring $W(k)$. If V is unramified (that is, p is a uniformizing element for V), its p-valuation is $1/(p-1)$. This transformation has an almost defined inverse up to ρ^n.

There are also versions for open varieties, or derived categories, etc.

(d) The method of proof follows ideas developed by J. Tate and J. M. Fontaine [Fo, T]. Basically étale cohomology is a global version of Galois cohomology. To study the latter we look at étale coverings of the generic fiber $X \otimes_V K$ and the normalizations of X in it. In general these are not étale over X as can be seen by the example of adjoining p-power roots of units. However, we show that this example is very typical of what is happening, in the sense that after adjoining enough such roots the remaining ramification can be controlled. This means in the case of good reduction that the rest is almost unramified, while for bad reduction the ramification is controlled by some finite fixed power p^e. After this we define $\mathscr{H}^*(X)$ as Galois cohomology of the p-adic completion of normalization of $\mathscr{O}_X$ in the maximal extension of the type described above. Our methods allow us to handle this infinite extension, and a study of differentials gives us the natural transformation relating it to Hodge cohomology.

(e) The paper is organized as follows. In §I we study the case of good reduction and prove the necessary results from commutative algebra. Loosely speaking, we redo the classical theory of étale coverings (SGA 1, 2) in the context of "almost-mathematics," where the results hold only up to $\mathfrak{m}$-torsion. One of the main difficulties is that we cannot use the usual finiteness results we are accostumed to. For the commutative algebra involved the reader might study [M].

In §II we construct the theory $\mathscr{H}^*(X)$. Here we need some simplicial methods for which [Fr] is a good reference. Besides this the proofs are standard applications of the machinery. Let us remark that we often have to refer to statements in the derived category. This means that we do not consider only cohomology groups but also the complexes which compute them. As those are naturally given by our constructions, this poses no problem.

Finally in §III we treat the case of bad reduction. Instead of working modulo $\mathfrak{m}$-torsion we formulate statements modulo a bounded p-power. The basic difficulty lies in showing the necessary facts from commutative algebra. We use a semiglobal method. Locally, schemes are elementary fibrations over a base of lower dimension, and these elementary fibrations are pullbacks from spaces with good reduction. We already know how to treat those, and the base is handled by induction. However, this method forces us to introduce some ideas (mainly due to M. Raynaud) from the theory of rigid spaces. We do not formulate them in this language, but the reader should know where they came from.

(f) Related results have been obtained in papers by S. Bloch and K. Kato, and by J. M. Fontaine and W. Messing. Both work only for good reduction, and the former assumes that the schemes involved are ordinary. However, the results of Fontaine and Messing are stronger than ours (in case they apply), as they give a rule for how to relate étale cohomology and crystalline cohomology. Our methods can be used to simplify some of their proofs, but it is too early to report about this now.

I thank the referee for pointing out some inaccuracies. The work was partially supported by NSF grant DMS-8502316.

I

1. Ramification theory for discrete valuation rings.

(a) We work with discrete valuation rings V whose fraction fields K have characteristic zero while the residue fields $k = V/\mathfrak{m}$ ($\mathfrak{m}$ = maximal ideal) have positive characteristic p. The valuations $v: K^* \to \mathbf{Q}$ will be normalized such that $v(p) = 1$. If $\alpha \in v(K^*)$ we denote by $p^\alpha V$ the ideal of V generated by an element of valuation α. Furthermore, we will assume that $[k: k^p] = p^d < \infty$. Let us also remark that all such V are excellent rings. This allows us to pass over to completions if needed. So we shall formulate our results for complete rings but use them sometimes also in a noncomplete case, where, however, passing over to completions poses no problem. So from now on let us assume that V is complete. By Ω_V we understand the universal finite V-module of differentials. Ω_V is finitely generated over V, and there exists a derivation $d: V \to \Omega_V$ which is universal for derivations into finitely generated V-modules. It is easy to see that such an Ω_V exists, and in fact it can be generated by $\leq d+1$ elements. This follows from the familiar exact sequence $\mathfrak{m}/\mathfrak{m}^2 \to \Omega_V \otimes_V k \to \Omega_k \to 0$.

If W is the normalization of V in a finite extension L of K, W is a local ring. The inverse different $p^{-\delta} W \subset L$ is the maximal fractional ideal which is mapped into V by the trace $\mathrm{tr}_{L/K}$. If W is generated by a single element w, with minimal equation $f(w) = 0$, δ is equal to the valuation of the derivative $f'(w)$. δ is additive for extensions. Assume $U \subset V \subset W$, then the different of W over U is the sum of the differents of W over V and of V over U. We also have the following lemma.

1.1. **Lemma.** *For any extension $V \subset W$, as above, the natural map $\Omega_V \otimes_V W \to \Omega_W$ is injective, and its cokernel $\Omega_{W/V}$ has the same length as $W/p^\delta W$.*

Proof. We may assume that L is a Galois extension of K and then reduce to the case of a totally ramified extension of prime degree. In this case W

can be generated by one element w, so $W \cong V[T]/f(T)$. Hence Ω_W is the quotient of $\Omega_V \otimes_V W \oplus WdT$ under the submodule generated by the derivative of $f(T)$. This element has second component $f'(w)dT$, and as $f'(w)$ is a nonzero-divisor in W the assertion follows. □

(b) One of the basic facts in our theory is the following. One very ramified extension kills all ramification. To be more precise, assume that we have given a family of extensions $V = V_0 \subset V_1 \subset V_2 \subset \cdots$, such that $\Omega_{V_n/V_{n-1}}$ has $(V_n/pV_n)^{d+1}$ as quotient.

1.2. **Theorem.** *If $V \subset W$ is any extension and W_n denotes the normalization of the tensor product $V_n \otimes_V W$, then the differents $\delta(W_n/V_n)$ of W_n over V_n (or more precisely of each factor of the semilocal ring W_n) converge to 0 for $n \to \infty$.*

Proof. Replacing V by some V_n we may assume that all W_n are integral domains. Look at the sequence of maps $\Omega_{W_n/V_n} \otimes_{W_n} W_{n+1} \to \Omega_{W_{n+1}/V_n} \to \Omega_{W_{n+1}/V_{n+1}}$. The kernel of the second map contains $\Omega_{V_{n+1}/V_n} \otimes_{V_{n+1}} W_{n+1}$, which has $(W_{n+1}/pW_{n+1})^{d+1}$ as quotient. As Ω_{W_{n+1}/V_n} is the direct sum of $d+1$ modules of the form $W_{n+1}/p^\alpha W_{n+1}$, the kernel of the second map contains the kernel of multiplication by p on Ω_{W_{n+1}/V_n}, and hence the composition of the two maps annihilates the kernel by p-multiplication on $\Omega_{W_n/V_n} \otimes_{W_n} W_{n+1}$. If we denote the different of W_n over V_n by δ_n, this kernel has length at least equal to that of $W_{n+1}/p^\beta W_{n+1}$, where $\beta = \min\{1, \delta_n/(d+1)\}$. Also it is clear that $p^{\delta_n - \delta_{n+1}}$ annihilates $W_{n+1}/(W_n \otimes_{V_n} V_{n+1})$, and so the cokernel of the composition of the two maps is annihilated by $p^{\delta_n - \delta_{n+1}}$. So its length is at most equal to that of W_{n+1} divided by the $(d+1)$st power of this. We derive that $\delta_n - \delta_{n+1} \geq \beta - (d+1)(\delta_n - \delta_{n+1})$. So if $\delta_n \geq d+1$, then $\delta_{n+1} \leq \delta_n - 1/(d+2)$, and otherwise $\delta_{n+1} \leq (1 - 1/(d+1)(d+2))\delta_n$. In any case $\delta_n \to 0$ for $n \to \infty$. □

A typical example of how we can obtain a sequence of V_n's as above goes as follows. Suppose V is the completion of $W(k)(T_1, \dots, T_d)$, where $W(k)$ denotes the Witt vectors associated to a perfect field k. Let V_n denote the extension of V generated by the p^nth roots of the T_i's together with the p^{n+1}st roots of unity. In more general situations V is usually a finite extension of a ring as above, and we can use the induced sequence of extensions.

(c) If we have a sequence of V_n's as above, let V_∞ denote their union. This is a subring of $\overline{V}$, the integral closure of V in the algebraic closure $\overline{K}$ in K. $\overline{V}$ is almost unramified over V_∞, as explained below.

2. Almost unramified extensions.

(a) We want to generalize some results about étale extensions to the case of almost unramified extensions. For this we always work over a normal base ring in which p is of the form $p = (\text{unit}) \cdot x^n$ for an infinite number of n's. We write $x = p^{1/n}$, which is well defined up to units. As we will be interested only in the ideal generated by x, this causes no problem. If $\alpha \in \mathbf{Q}$ satisfies $n \cdot \alpha \in \mathbf{Z}$ for some n as above, we also define p^α as the obvious element, unique up to units. We assume that multiplication by p is injective in all our rings. By $\mathfrak{m}$ we denote the ideal generated by all powers of p^ε, $\varepsilon > 0$.

2.1. **Definition.** Suppose A is a ring, B an A-algebra. B is called an almost étale covering of A if

(i) $B[1/p]$ is a projective $A[1/p]$-module of finite rank and an étale $A[1/p]$-algebra.
(ii) the trace map $\mathrm{tr}_{B/A}: B[1/p] \to A[1/p]$ maps B into A.
(iii) Let $e_{B/A} \in (B \otimes_A B)[1/p]$ denote the idempotent which corresponds to the (open) image of the closed diagonal immersion given on the level of rings by $(B \otimes_A B)[1/p] \to B[1/p]$. Then for any $\varepsilon > 0$, $p^\varepsilon e_{B/A}$ is in the image of $B \otimes_A B$.

Remarks. (i) If B is almost étale over A and C almost étale over B, this holds also for C over A.

(ii) If B is almost étale over A, so is its subalgebra $A + \mathfrak{m}B$.

(iii) Suppose B is almost étale over A. For some $\varepsilon > 0$, choose elements $x_i, y_i \in B$ such that $p^\varepsilon \cdot e_{B/A} = \sum x_i \otimes y_i$. Then for any $b \in B$, $p^\varepsilon \cdot b = \sum \mathrm{tr}_{B/A}(bx_i) \cdot y_i$. It follows that, up to the factor p^ε, B is a direct summand in a free A-module. Consequently B is almost projective and almost flat over A, which means that $\mathfrak{m}$ annihilates $\mathrm{Tor}_i^A(B, M)$ and $\mathrm{Ext}_A^i(B, M)$ for $i > 0$ and any A-module M.

(iv) We also can define almost étale extensions without assuming them to be finite. It is necessary that the extension be étale after inversion of p and that condition (iii) hold. However, there are some difficulties in proving the main results below in this context.

(v) If M is a $B \otimes_A B$-module, define the Hochschild cohomology $H^*(B/A, M)$ as cohomology of the usual complex. In degree n we have $B \otimes_A B$-homomorphisms $B \otimes_A B \otimes_A \cdots \otimes_A B \otimes_A B \to M$ ($n+2$ factors B), and the derivative is induced by the alternating sum of partial multiplications. (For example $b_0 \otimes b_1 \otimes b_2$ maps to $b_0 b_1 \otimes b_2 - b_0 \otimes b_1 b_2$, and $H^1(B/A, M)$ is given by derivations $B \to M$ modulo inner derivations). If $e_{B/A} = \sum x_i \otimes y_i$ were an element of $B \otimes_A B$ we would obtain a null-homotopy by $b_0 \otimes b_1 \otimes \cdots \otimes b_{n+1} \to \sum x_i \otimes y_i b_0 \otimes b_1 \otimes \cdots \otimes b_{n+1}$, and so Hochschild cohomology would vanish. The same argument gives that for B almost étale over A $\mathfrak{m}$ annihilates the Hochschild cohomology in positive degrees.

(b) We want to generalize the usual results of lifting over nilpotent ideals. This can be done provided we assume from the beginning that $B = A + \mathfrak{m}B$. Let us also remind the reader that we assumed that multiplication by p is injective in all rings involved. Thus the induction step where one usually replaces a nilpotent ideal $I \subset C$ by $I/I^2 \subset C/I^2$, and so on, has to be modified. Divide C/I^2 by its p-torsion, etc.

2.2. **Theorem.** *Suppose $B = A + \mathfrak{m}B$ is an almost étale covering of A, C an A-algebra, $I \subset C$ a nilpotent ideal, and $\phi\colon B \to C/I$ an A-algebra morphism. Then ϕ lifts uniquely to $B \to C$.*

Proof. We may assume that $I^2 = 0$. As B is almost A-projective, for any $\varepsilon > 0$ we can lift $p^\varepsilon\phi$ to an A-module map $\phi_\varepsilon\colon B \to C$.

$$b_0(p^\varepsilon\phi_\varepsilon(b_1b_2) - \phi_\varepsilon(b_1)\phi_\varepsilon(b_2))b_3 \in I$$

defines a class in $H^2(B/A, I)$, which is annihilated by $\mathfrak{m}$. Doubling ε and then enlarging it a little we may assume that this class vanishes, and then we can modify ϕ_ε so that it becomes multiplicative, i.e. such that $p^\varepsilon\phi_\varepsilon(xy) = \phi_\varepsilon(x)\phi_\varepsilon(y)$. Such a lifting is unique up to $H^1(B/A, I)$, hence up to p-torsion. As C has no such torsion, the different ϕ_ε glue together to give a multiplicative A-linear map $\phi_0\colon \mathfrak{m}B \to C$. We can extend to $B = A + \mathfrak{m}B$, because ϕ_0 maps p^ε to an element $x = p^\varepsilon + y$ $(y \in I)$ satisfying $x^2 = p^\varepsilon x$, hence $p^\varepsilon y = 0$. Uniqueness has already been established. □

2.3. **Theorem.** *Suppose $I \subset A$ is a nilpotent ideal, $\overline{B}$ an almost étale covering of $\overline{A} = A/I$, $\overline{B} = \overline{A} + \mathfrak{m}\overline{B}$. Then there exists an almost étale covering B of A such that $\overline{B} \cong B \otimes_A \overline{A}/(p\text{-torsion})$.*

Proof. We may assume that $I^2 = 0$. The proof proceeds similarly as before. We start with a small ε, which we enlarge at each step by a certain factor. First $\overline{B}$ is a direct summand in a free module $\overline{A}^r$ up to some p^ε. That is, there exists an $r \times r$ matrix $\overline{e}$ such that

$$\overline{e}^2 = p^\varepsilon\overline{e} \quad \text{and} \quad \overline{B} = \overline{A}^r/(p^\varepsilon - \overline{e})(\overline{A}^r) \quad \text{modulo } p\text{-torsion}.$$

Tripling ε we may assume that $\overline{e}$ lifts to an $r \times r$ matrix e with $e^2 = p^\varepsilon e$. Let $B_\varepsilon = A^r/(p^\varepsilon - e)(A^r)$. Then B_ε is A-projective up to p^ε, that is, p^ε annihilates all higher Ext-groups $\mathrm{Ext}^1_A(B_\varepsilon, M)$. Furthermore, $\overline{B}$ contains $B_\varepsilon \otimes_A \overline{A}/(p$-torsion), the quotient being annihilated by p^ε. Let $\overline{m}\colon \overline{B} \otimes_A \overline{B} \to \overline{B}$ denote the multiplication map. $p^{3\varepsilon}\overline{m}$ lifts to $m_\varepsilon\colon B_\varepsilon \otimes_A B_\varepsilon \to B_\varepsilon$. The obstruction to m_ε being associative, is given by

$$b_0[m_\varepsilon(b_1, m_\varepsilon(b_2, b_3)) - m_\varepsilon(m_\varepsilon(b_1, b_2), b_3)]b_4 \in \mathrm{Kern}(B_\varepsilon \to \overline{B})$$

and defines a class in $H^3(\overline{B}/\overline{A})$ with values in this module. (That we get a cycle amounts to a five-term identity well known in the study of associativity.) It follows that we can modify $p^{4\varepsilon}m_\varepsilon$ such that the associative law holds. Changing ε we may assume that this is already true for m_ε. Again the lifting is unique up to p-torsion, hence unique. It also follows that the product is commutative, as the commutator with a fixed element of B_ε is a derivation.

Now each $B_\varepsilon[1/p]$ is a lifting of the étale $\overline{A}[1/p]$-algebra $\overline{B}[1/p]$ and is a projective $A[1/p]$-module. It follows that all $B_\varepsilon[1/p]$ are isomorphic to a fixed algebra $B[1/p]$ (say), and we may imbed B_ε into $B[1/p]$ such that m_ε corresponds to $p^{3\varepsilon}$-product in $B[1/p]$ (use $p^{3\varepsilon}$-standard imbedding). Choose a $\delta \leq \varepsilon$. Then the map $B_\varepsilon \to B[1/p] \to B[1/p]/p^{-3\delta}B_\delta$ induces a derivation $\overline{B} \to I$-torsion in $B[1/p]/p^{-3\delta}B_\delta$. By Hochschild cohomology (this time we use H^1) the image is annihilated by $\mathfrak{m}$. Hence $B_\varepsilon \subset p^{-4\delta}B_\delta$. Consider the subalgebras $C_\varepsilon = A + p^{5\varepsilon}B_\varepsilon \subset B[1/p]$. If $\delta \leq \varepsilon/2$ then $C_\varepsilon \subset C_\delta$. The image of C_ε in $\overline{B}$ contains $p^{6\varepsilon} \cdot \overline{B}$. It follows that $p^{13\varepsilon} \cdot \overline{e}_{B/A}$ lies in $\overline{C}_\varepsilon \otimes_A \overline{C}_\varepsilon$, hence $p^{27\varepsilon} \cdot \overline{e}_{B/A}$ lifts to an idempotent in $C_\varepsilon \otimes_A C_\varepsilon$. If we choose a sequence ε_n with $\varepsilon_{n+1} \leq \varepsilon_n/2$, the correspondings C_ε's form an increasing sequence. If B denotes their union, $B \otimes_A B$ contains $\mathfrak{m}e_{B/A}$, hence B is the required lifting of $\overline{B}$. □

(c) We can also prove the usual results about cohomology and differentials.

2.4. **Theorem.** *Suppose B is an almost étale covering of A.*

(i) *The map $\Omega_A \otimes_A B \to \Omega_B$ is an almost isomorphism, that is, its kernel and cokernel are annihilated by $\mathfrak{m}$.*

(ii) *Suppose a finite group G operates on B, such that $B[1/p]$ is a Galois covering of $A[1/p]$ with group G. If M is any B-module with a semilinear G-action, then $\mathfrak{m}$ annihilates all higher cohomology $H^i(G, M)$, $i > 0$. The same holds for $M^G/\mathrm{tr}_G(M)$.*

Proof. For (i) we have to show that for any B-module M (which is allowed to have p-torsion contrary to our general assumptions), the restriction maps on derivations $\mathrm{Der}(B, M) \to \mathrm{Der}(A, M)$ is an almost isomorphism. This follows in the usual way using Hochschild cohomology $H^*(B/A, M)$.

For (ii), it suffices to show that $\mathfrak{m}$ annihilates $A/\mathrm{tr}_{B/A}(B)$. We may assume that $A = B^G$. Denote the A-ideal $\mathrm{tr}_{B/A}(B)$ by $\mathfrak{a}$. We show that $\mathfrak{a}B \supset \mathfrak{m}B$. Apply the norm $N_{B/A}$ to elements of both sides to derive the conclusion. However, if $p^\varepsilon \cdot e_{B/A} = \sum x_i \otimes y_i$, then $p^\varepsilon = \sum \mathrm{tr}_{B/A}(x_i) y_i$, hence everything follows. □

(d) Let us give some examples of almost étale coverings. First consider a discrete valuation ring V satisfying the assumptions of the first part and an increasing sequence V_n of finite extensions which kill ramification. Let V_∞ denote their union. If W is another finite extension of V, W_n the composite of W and V_n, and p^{δ_n} the different of W_n over V_n, it follows that p^{δ_n} makes e_{W_n/V_n} integral. As the δ_n converge to 0 for $n \to \infty$, it follows that the union W_∞ of the W_n is almost étale over V_∞. We derive that for any finite extension of the fraction field of V_∞ the normalization of V_∞ in this field is almost étale over it.

Another important example arises as follows. Suppose A is a normal integral domain with fraction field K of characteristic 0 such that $A = A^p + pA$ (modulo

p, Frobenius is surjective). If $u \in A^*$ is a unit and B denotes the normalization of A in $K(u^{1/p})$, then B is an almost étale covering of A. For the proof observe first that $B[1/p]$ is already étale over $A[1/p]$. So we may assume that A is local with p contained in the maximal ideal. For any $\alpha < 1/(p-1)$ we can find $x = x_\alpha \in A$ such that $u - x^p$ is divisible by $p^{p\alpha}$: Assume this holds for some α ($\alpha = 1/p$ works by assumption), and let $\beta \leq (\alpha+1)/p$. Dividing u by ${x_\alpha}^p$ (which is a unit) we may assume that $u = 1 + p^{p\alpha}y$ with $y \in A$. If $y - z^p$ is divisible by p, $x = 1 + p^\alpha z$ works for β. Repeating this induction step we reach any exponent $< 1/(p-1)$. If $v = u^{1/p} \in B$, it follows that $v_\alpha = (v - x_\alpha)/p^\alpha$ is an element of B. It generates a subring whose discriminant over A is $p^{p-(p-1)\alpha}$. Hence this p-power makes $e_{B/A}$ integral, and by letting α approach $1/(p-1)$ we reach the conclusion. Let us also note that B satisfies the same assumptions as A, that is, $B = B^p + pB$. Calculating $\bmod p$ we see that for α close to $1/(p-1)$ the subring $A[{v_\alpha}^p]$ has discriminant $p^{p(p-(p-1)\alpha)}$. It follows (by expanding in an A-basis of $A[{v_\alpha}^p]$) that after multiplication with this power of p any element of B becomes a pth power $\bmod pB$. But for small $\varepsilon > 0$, $b^p \in p^{p\varepsilon}B$ implies $b \in p^\varepsilon B$. It follows that any element of B is a p-power modulo $p^{1-p\varepsilon}$ and then also modulo p.

Hence we can make induction arguments. For example, if B is the normalization of A in any extension of K which is generated by p-power roots of units in A, then B is almost étale over A.

3. Good reduction.

(a) Our basic setup is as follows. V is a discrete valuation ring whose fraction field K has characteristic 0 and whose residue field k is perfect of positive characteristic. R is a smooth V-algebra or a localization of such or a henselization. We consider normal R-algebras S which are étale over R in characteristic 0, that is, $S[1/p]$ is étale over $R[1/p]$. If R is integral they are classified by the fundamental group $\pi_1(\mathrm{Spec}(R[1/p]))$, which is the Galois group of the largest algebraic extension of the quotient field of R for which the normalization of $R[1/p]$ in it is unramified. Examples of S are obtained by taking p-power roots out of units of R, by étale coverings of R, and by rings of the form $S = R \otimes_V V_1$, where V_1 is the normalization of V in a finite extension of K. We want to show that these are the typical examples.

We choose an increasing sequence of extensions V_n of V which kills ramification (Theorem 1.2). Furthermore we assume that we are given units $u_1, \ldots, u_d \in R^*$ such that the corresponding mapping $V[T_1^{\pm 1}, T_2^{\pm 1}, \ldots, T_d^{\pm 1}] \to R$ is étale. We let R_n denote the normalization of $R \otimes_V V_n$ in the extension generated by the p^nth roots of the u_i. It is clear that

$$R_n \cong (R \bigotimes_V V_n)[X_1, \ldots, X_d]/(X_i^{p^n} - u_i).$$

Let R_∞ denote the union of the R_n.

(b) Our main result is the following.

3.1. **Theorem.** *Suppose S is a normal finite R-algebra such that $S[1/p]$ is étale over $R[1/p]$. Let S_∞ denote the normalization of $S \otimes_R R_\infty$. Then S_∞ is an almost étale covering of R_∞.*

Proof. We reduce to the case that R is the henselization of a smooth V-algebra. We use induction over $\dim(R)$. If this dimension is 1, R is a discrete valuation ring whose residue field is a finite extension of k of transcendence degree d. As $\Omega_{R_n/R} \cong \Omega_{V_n/V} \otimes_{V_n} R_n \oplus (R_n/p^n R_n)^d$, the sequence R_n kills ramification (complete and use part 1, Theorem 1.2), and the assertion follows.

If the dimension is 2, we denote by S_n the normalization of $S \otimes_R R_n$. S_n is a finite reflexive R_n-module, and as R_n is regular S_n is a free R_n-module. Denote the prime divisors of p in R by $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. This is a finite set of height one prime ideals. If $\mathfrak{p}$ is one of them, we may apply induction to the localization in $\mathfrak{p}$. It follows that there exists a sequence $\delta_n \to 0$ such that $p^{\delta_n} \cdot e_{S_n/R_n}$ is in $(S_n \otimes_{R_n} S_n)_\mathfrak{p}$. We may choose this sequence uniform for all $\mathfrak{p}$, and as $(S_n \otimes_{R_n} S_n) = \bigcap_\mathfrak{p} (S_n \otimes_{R_n} S_n)_\mathfrak{p} \cap (S_n \otimes_{R_n} S_n)[1/p]$, it follows that $p^{\delta_n} \cdot e_{S_n/R_n}$ is actually in $(S_n \otimes_{R_n} S_n)$.

(c) So let us assume that $\dim(R) \geq 3$. The residue field of R is a separable extension of k, and some subset of the u_i, say $\{u_1, \dots, u_s\}$, forms a separating transcendence base for this extension. Let $W = V(T_1, \dots, T_s)$ (the localization of the polynomial ring in the extension of the maximal ideal of V). W maps to R (sending T_i to u_i), and R is also étale over W. Let $\tilde{R}$, respectively $\widetilde{W}$, denote the extension obtained by adjoining p-power roots of $\{u_1, \dots, u_s\}$. Then $\widetilde{W}$ is a discrete valuation ring with perfect residue field, and we may replace the pair (R, V) by $(\tilde{R}, \widetilde{W})$. So we may assume that the residue field of R is finite over k. As R was supposed to be henselian, we now replace V by an unramified extension, so that R has residue field k, and R is isomorphic to the henselization of $\mathscr{G}_m{}^d$ in a k-rational point, the u_i being given by the canonical units on $\mathscr{G}_m{}^d$. We may also suppose that k is algebraically closed (use an unramified extension of V). It follows that $R = V\{t_1, \dots, t_d\}$ (henselization of the polynomial-ring), with $u_i = v_i + t_i$, $v_i \in V^*$. Now S is the normalization of R in a finite extension of its fraction field, unramified in characteristic 0, and we denote by S_n the normalization of $S \otimes_R R_n$, and S_∞ is the union of the S_n. There exists a sequence $\delta_n \to 0$, such that for each prime divisor $\mathfrak{p} \subset R$ of p the idempotent e_{S_n/R_n} becomes integral in the $\mathfrak{p}$-localization after multiplication with p^{δ_n}. It follows that the dual of S_n (for the trace form tr_{S_n/R_n}) is contained in $p^{-\delta_n} \cdot S_n$, and one derives that p^{δ_n} annihilates the cokernel of $S_n \otimes_{R_n} R_\infty \to S_\infty$.

Let $\mathfrak{n}$ denote the maximal ideal of R, and consider local cohomology $H_\mathfrak{n}{}^*(\cdot)$. As $\dim(R) \geq 3$, $H_\mathfrak{n}{}^2(S_n)$ is a finite R-module, and we also know that $H_\mathfrak{n}{}^0(S_n)$ and $H_\mathfrak{n}{}^1(S_n)$ vanish. It follows that some power of $\mathfrak{n}$ (depending on n) annihilates $p^{\delta_n} \cdot H_\mathfrak{n}{}^2(S_\infty)$.

(d) We now apply induction to $\widetilde{R} = R/t_dR$ and the sequence of units $u_1, \dots, u_{d-1}$ in it. Note that the surjection $R \to \widetilde{R}$ has a section (send t_d to 0), so we can view $\widetilde{R}$ as a subring of R. Abbreviate $t = t_d$. If $\widetilde{S}$ denotes the normalization of S/tS, we derive that the corresponding sequence $\widetilde{S}_n$ converges to an almost étale covering $\widetilde{S}_\infty$ of $\widetilde{R}_\infty$. Let $A_n = R \otimes_{\widetilde{R}} \widetilde{R}_n$ denote the extension obtained by adjoining p^nth roots of $u_1, \dots, u_{d-1}$, $B_n = R \otimes_{\widetilde{R}} \widetilde{S}_n$, and A_∞, respectively B_∞, are the union of the A_n, respectively B_n. So B_∞ is almost étale over A_∞. Finally let $C_n = \widetilde{R}_\infty \otimes_{\widetilde{R}_n} R_n$. (This is normal. It is obtained from A_∞ by adjoining the p^nth root of u_d.) D_n = normalization of $S \otimes_R C_n$. That is, $A_n \subset B_n$ is obtained from $\widetilde{R}_n \subset \widetilde{S}_n$ via base extension $\widetilde{R} \subset R$, and (up to normalization) $C_n \subset D_n$ from $R_n \subset S_n$ via $\widetilde{R}_n \subset \widetilde{R}_\infty$. Also $C_\infty = R_\infty$ and $D_\infty = S_\infty$. We claim that the map $S \otimes_R C_n \to \widetilde{S} \otimes_R C_n \to (B_\infty \otimes_{A_\infty} C_n)/t(B_\infty \otimes_{A_\infty} C_n)$ extends to D_n. This is clear after inverting p. On the other hand the trace form $\mathrm{tr}_{B_\infty/A_\infty}$ induces an isomorphism of B_∞ with its dual (as A_∞-module). It follows that an element of $(B_\infty \otimes_{A_\infty} C_n)[1/p]$ is integral if the trace relative C_n of its product with any element of $B_\infty \otimes_{A_\infty} C_n$ is integral. The same holds after division by t. So we come down to checking that the trace relative C_n takes integral values on $B_\infty \otimes_{A_\infty} D_n$. But this is obvious, as C_n is integrally closed.

Letting $n \to \infty$ we obtain a regular map

$$S_\infty/tS_\infty \to B_\infty \bigotimes_{A_\infty} R_\infty/t(B_\infty \bigotimes_{A_\infty} R_\infty).$$

If we localize in a prime $\mathfrak{q} \neq \mathfrak{n}$ of R, both S_∞ and $B_\infty \otimes_{A_\infty} R_\infty$ are almost unramified over R_∞. Hence the morphism above is an almost isomorphism and extends over $R_\infty/t^m R_\infty$ for any $m > 0$, provided we replace S_∞ by $R_\infty + \mathfrak{m}S_\infty$ ($\mathfrak{m}$ = maximal ideal in V_∞ = union of the V_n). As $H^1_{\mathfrak{n}}(R_\infty \otimes_{A_\infty} B_\infty/t^m(R_\infty \otimes_{A_\infty} B_\infty)) = (0)$, we obtain a compatible system of liftings $R_\infty + \mathfrak{m}S_\infty \to B_\infty \otimes_{A_\infty} R_\infty/t^m(B_\infty \otimes_{A_\infty} R_\infty)$, whose cokernels consist of $\mathfrak{n}$-torsion up to terms annihilated by $\mathfrak{m}$. If we neglect such terms their cokernels are isomorphic to the kernels of multiplication with t^m on $H^2{}_{\mathfrak{n}}(S_\infty)$ with the transition maps (for $m' > m$) given by multiplication by $t^{m'-m}$. As we know, up to terms annihilated by p^{δ_n}, $H^2{}_{\mathfrak{n}}(S_\infty)$ is equal to $S_\infty \otimes_{R_n} H^2{}_{\mathfrak{n}}(S_n)$ on which t-multiplication is nilpotent; we derive that all the cokernels are annihilated by any p^{δ_n}, hence by $\mathfrak{m}$.

(e) As $R_\infty \otimes_{A_\infty} B_\infty$ is almost flat over R_∞, we have that for each m $S_\infty/t^m S_\infty$ is almost flat over $R_\infty/t^m R_\infty$. As p^{δ_n} annihilates the cokernel of the injection $S_n \otimes_{R_n} R_\infty \to S_\infty$, we derive that $\mathfrak{m} \cdot p^{\delta_n}$ annihilates all $\mathrm{Tor}_i^{R_\infty}(S_n \otimes_{R_n} R_\infty, M)$, $i > 0$ and M any R_∞-module annihilated by some power of t. Since R_∞ is faithfully flat over R_n, p^{δ_n} also annihilates all $\mathrm{Tor}_i^{R_n}(S_n, M)$, $i > 0$ and M any R_n-module annihilated by a power of

t. As R_n is local and noetherian we derive that this holds in fact for any R_n-module M. Letting $n \to \infty$ we derive that S_∞ is almost flat over R_∞.

Finally, by induction we already know that for any $\varepsilon > 0$ $p^\varepsilon \cdot e_{S_\infty/R_\infty}$ is integral in $S_\infty \otimes_{R_\infty} S_\infty$ up to $\mathfrak{m}$-torsion. Almost flatness implies that essentially there is no such $\mathfrak{m}$-torsion, hence the theorem follows. □

(f) There is a version for complements of divisors with normal crossings. In this case we start with a smooth V-algebra R as before together with a set of $u_i \in R$ such that the induced morphism $V[T_1, \dots, T_d] \to R$ is étale. The u_i do not have to be units. We consider normal R-algebras S such that $S[1/(pu_1 \cdots u_d)]$ is étale over $R[1/(pu_1 \cdots u_d)]$. As standard examples we use the extensions R_n generated by V_n and the roots of the u_i of order $n!$ (there are many other choices for the sequence of exponents). Their union R_∞ is the tensor product (over $V[T_1, \dots, T_d]$) of R with $V_\infty[T_i^{\mathbf{Q}+}]$ (all positive rational exponents are allowed for the T_i). It follows that modulo p any element of R_∞ is a p-power.

The reason why we do not restrict to p-power roots is Abhyankhar's lemma (see SGA 1). If S is a finite normal R-algebra, étale after inverting $pu_1 \cdots u_d$, the normalization S_n of $S \otimes_R R_n$ is étale over R_n after inverting only p, provided n is big enough.

We derive the following.

3.2. **Theorem.** *Suppose S is a finite normal torsion free R-algebra such that $S[1/(pu_1 \cdots u_d)]$ is étale over $R[1/(pu_1 \cdots u_d)]$. If S_n denotes the normalization of $S \otimes_R R_n$ and S_∞ the union of all S_n, then S_∞ is almost étale over R_∞.*

Proof. We may assume that R is local and henselian and also that S is already étale after inverting p (replace R by some R_n). We know that S becomes almost étale if we adjoin sufficiently many p-power roots of units in R. It follows that for any $\varepsilon > 0$ we can find an extension A of R, obtained by adjoining finitely many p-power roots of units such that for the normalization B of $S \otimes_R A$ the element $p^\varepsilon \cdot e_{B/A}$ is integral. Let A_n, B_n denote the normalizations of $A \otimes_R R_n$ and $B \otimes_R R_n$, and A_∞ and B_∞ their union. Then A_∞ is almost étale over R_∞, and R_∞ is almost a direct summand in it (the image of the trace form contains $\mathfrak{m}R_\infty$). It follows that B_∞ is almost isomorphic to $S_\infty \otimes_{R_\infty} A_\infty$ as this is almost étale over S_∞. So enlarging ε a little we see that $p^\varepsilon \cdot e_{S_\infty/R_\infty}$ becomes integral after $\otimes_{R_\infty} A_\infty$. As R_∞ is almost a direct summand in A_∞, this already holds before tensoring. The assertion now follows as ε can be arbitrarily small. □

4. Differentials and cohomology.

(a) Let R, V be as before. We especially assume that R has units u_i with the properties specified above. Furthermore, we assume that V is integrally closed in R. Denote by $\overline{V}$ the integral closure of V in an algebraic closure

$\overline{K}$ of K. Then $R\otimes_V \overline{V}$ is a normal domain. We denote by $\overline{R}$ the integral closure of R in the maximal algebraic extension unramified in characteristic 0 of the fraction field of this domain. We have seen that $\overline{R}$ is an inductive limit of almost étale coverings of R_∞. This allows us to transfer properties from R_∞ to $\overline{R}$. Note that $\overline{R}$ does not depend on the choice of the u_i. Let Γ denote the Galois group of $\overline{R}$ over R and $\Delta \subset \Gamma$ the kernel of the natural surjection $\Gamma \to \mathrm{Gal}(\overline{K}/K)$. If R is only integral but V is not integrally closed in R, its integral closure V_1 in R is a finite étale extension of V. V_1 is the normalization of V in the extension K_1 of K determined by the image of $\Gamma \to \mathrm{Gal}(\overline{K}/K)$. This image operates on all objects, and we can form the induced $\mathrm{Gal}(\overline{K}/K)$-module. This procedure allows us to weaken the general hypotheses "geometrically irreducible" to "irreducible," in the rest of this chapter. We leave the details to the reader.

(b) We first consider differentials. It is known (see [Fo] or use part 1 above) that $\Omega_{\overline{V}/V} \cong \overline{K}/\rho^{-1}\overline{V}(1)$, the isomorphism being induced from dlog: $\mu_{p\infty} \to \Omega_{\overline{V}/V}$. Here (1) denotes Tate twist by the cyclotomic character $\mathrm{Gal}(\overline{K}/K) \to \mathbf{Z}_p^*$, and ρ is an element of $\overline{V}$ whose p-valuation can be determined. For unramified rings it is $1/(p-1)$. Otherwise one has to add the different of V over the Witt vectors.

Now consider the exact squences

$$\Omega_{\overline{V}/V} \bigotimes_{\overline{V}} \overline{R} \to \Omega_{\overline{R}/R} \to \Omega_{\overline{R}/R\otimes_V \overline{V}} \to 0$$

and

$$\Omega_{R/V} \bigotimes_{R} \overline{R} \to \Omega_{\overline{R}/\overline{V}} \to \Omega_{\overline{R}/R\otimes_V \overline{V}} \to 0.$$

Up to $\mathfrak{m}$-torsion they are isomorphic to the tensor product over R_∞ with $\overline{R}$ of the corresponding sequences where we replace $\overline{V}$ by V_∞ and $\overline{R}$ by R_∞. These read

$$R_\infty[1/p]/\rho^{-1}R_\infty(1) \to R_\infty[1/p]/\rho^{-1}R_\infty(1) \oplus (R_\infty[1/p]/R_\infty)^d \\ \to (R_\infty[1/p]/R_\infty)^d \to 0$$

and

$$R_\infty{}^d \to R_\infty[1/p]^d \to (R_\infty[1/p]/R_\infty)^d \to 0.$$

The direct sums $R_\infty{}^d, R_\infty[1/p]^d$, etc. are free modules with the $\mathrm{dlog}(u_i)$ as basis. We derive the following result, formulated in a canonical way, without reference to the u_i.

4.1. **Theorem.** (i) *The map $\Omega_{R/V} \to \Omega_{\overline{R}/\overline{V}}$ induces almost isomorphisms*

$$\Omega_{R/V} \bigotimes_{R} \overline{R}[1/p] \cong \Omega_{\overline{R}/\overline{V}} \quad \textit{and} \quad \Omega_{R/V} \bigotimes_{R} (\overline{R}[1/p]/\overline{R}) \cong \Omega_{\overline{R}/R\otimes_V \overline{V}}.$$

(ii) *The sequence*

$$0 \to \Omega_{\overline{V}/V} \bigotimes_{\overline{V}} \overline{R} \to \Omega_{\overline{R}/R} \to \Omega_{\overline{R}/R\otimes_V \overline{V}} \to 0$$

is exact.

Proof. We only have to show that the first map of the exact sequence in (ii) is injective. We already know that its kernel is annihilated by $\mathfrak{m}$. But $\Omega_{\overline{V}/V} \otimes_{\overline{V}} \overline{R}$ has no $\mathfrak{m}$-torsion. □

By using the second isomorphism in (i) to pull back the exact sequence in (ii) we obtain a canonical extension

$$0 \to (\overline{R}[1/p]/\rho^{-1}\overline{R})(1) \to ? \to \Omega_{R/V} \bigotimes_{R} (\overline{R}[1/p]/\overline{R}) \to 0.$$

The Galois group Γ operates on this extension, and it is functorial with respect to morphisms between different R's. It splits as extensions of $R \otimes_V \overline{V}$-modules, but the splitting is not $\mathrm{Gal}(\overline{K}/K)$-equivariant. Forming Tate modules ($= \mathrm{Hom}(\mathbf{Q}_p/\mathbf{Z}_p, .)$) and Tate twists we obtain a canonical extension ("$\widehat{\ }$" = p-adic completion)

$$0 \to \rho^{-1}\overline{R}^{\wedge} \to E_\rho \to \Omega_{R/V} \bigotimes_{R} \overline{R}^{\wedge}(-1) \to 0.$$

(c) We can also compute the Galois cohomology of Δ with values in $\overline{R}/p^l\overline{R}$ or with values in the p-adic completion $\overline{R}^{\wedge}$. In the latter case we use topological cohomology defined via continuous cochains. We could avoid this by dealing exclusively with the projective system $\{\overline{R}/p^l\overline{R} \mid l \geq 0\}$. As $\overline{R}$ is an inductive limit of almost étale covers of $R_\infty \otimes_V \overline{V}$, the cohomology groups are almost isomorphic to those of $R_\infty \otimes_V \overline{V}$, where Δ has to be replaced by $\mathbf{Z}_p(1)^d = \mathrm{Gal}(R_\infty/R \otimes_V \overline{V})$. But the p-adic completion of $R_\infty \otimes_V \overline{V}$ is the topological direct sum of Δ-eigenspaces $(R \otimes_V \overline{V})^{\wedge} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_d^{\alpha_d}$, where the exponents α_i are in $\mathbf{Z}[1/p]$ and satisfy $0 \leq \alpha_i < 1$. If all α_i vanish the cohomology of the corresponding eigenspace is the exterior algebra over $(R \otimes_V \overline{V})^{\wedge}(-1)^d$, and otherwise we obtain a finite direct sum of terms $(R \otimes_V \overline{V})/p^{\varepsilon}(R \otimes_V \overline{V})$, where $\varepsilon = p^{-e}/(1 - 1/p)$, p^e the maximal p-power occurring in the denominators of the α_i. ε is the p-valuation of a primitive p^eth root of unity. Hence

$$H^i(\mathbf{Z}_p(1)^d, (R_\infty \bigotimes_V \overline{V})^{\wedge}) \cong \Lambda^i((R \bigotimes_V \overline{V})^{\wedge}(-1)^d) \oplus \bigoplus (R \bigotimes_V \overline{V})/p^{\varepsilon}(R \bigotimes_V \overline{V}).$$

The sum in the second term is direct, the exponents ε are $\leq 1/(p-1)$, and they converge to 0.

We now obtain the following result. In (iii) (Künneth) and (vi) (duality) we view $H^*(\Delta, \overline{R}^{\wedge})$ as an element of the derived category of $(R \otimes_V \overline{V})^{\wedge}$-modules, represented by the canonical complex which computes it.

4.2. **Theorem.** (i) $H^i(\Delta, \overline{R}\hat{}) \cong \Lambda^i((R \otimes_V \overline{V})\hat{}(-1)^d) \oplus (\text{rest})$, *where "rest" is annihilated by* $p^{1/(p-1)}$. *The isomorphism is* $\text{Gal}(\overline{K}/K)$ *equivariant.*

(ii) *If* $R_1 \to R_2$ *is an étale morphism, the induced morphisms*

$$H^i(\Delta, \overline{R}_1\hat{}) \bigotimes_{(R_1 \otimes_V \overline{V})\hat{}} (R_2 \bigotimes_V \overline{V})\hat{} \to H^i(\Delta, \overline{R}_2\hat{})$$

are almost isomorphisms

(iii) *If* $R = R_1 \otimes_V R_2$, *the morphisms* (*Künneth*)

$$\bigoplus_{a+b=i} H^a(\Delta, \overline{R}_1\hat{}) \bigotimes^{\mathbf{L}}{}_{\overline{V}\hat{}} H^b(\Delta, \overline{R}_2\hat{}) \to H^i(\Delta, \overline{R}\hat{})$$

are almost isomorphisms ($\otimes^{\mathbf{L}}$ = *derived tensor product*).

(iv) *The morphism* $\Omega_{R/V} \to H^1(\Delta, p^{-1}\overline{R}\hat{}(+1))$ *given by the extension* E_p *is induced by a functorial isomorphism*

$$H^1(\Delta, \overline{R}\hat{})/(p\text{-}torsion) \cong \Omega_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-1),$$

via $H^1(\Delta, \overline{R}\hat{}(-1)) \to H^1(\Delta, p^{-1}\overline{R}\hat{}(-1))$ (*which kills torsion*).

(v) *The cup products* $\Lambda^i(H^1(\Delta, \overline{R}\hat{})) \to H^i(\Delta, \overline{R}\hat{})$ *induce isomorphisms*

$$H^i(\Delta, \overline{R}\hat{})/(p\text{-}torsion) \cong \Omega^i_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-i).$$

(vi) *The cup products*

$$H^i(\Delta, \overline{R}\hat{}) \times H^{d-i}(\Delta, \overline{R}\hat{}) \to H^d(\Delta, \overline{R}\hat{})/(torsion) \cong \Omega^d{}_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-d)$$

induce almost isomorphisms

$$H^i(\Delta, \overline{R}\hat{}) \to \text{Rhom}_{(R \otimes_V \overline{V})\hat{}}(H^{d-i}(\Delta, \overline{R}\hat{}), \Omega^d{}_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-d)).$$

(*The right side means homomorphisms in the derived category. If we work with coefficients* R/p^lR, *this would be the same as ordinary homomorphisms, by the known* R*-module structure of* $H^i(\Delta, \overline{R}\hat{})$.) *All these morphisms are functorial in* R.

(d) *Proof*. For (i) to (iii) we already know that they hold up to $\mathfrak{m}$-torsion by reducing to $R_\infty \otimes_V \overline{V}$ and checking there. For (i), it follows that multiplication by $p^{1/(p-1)}$ sends p-torsion in $H^i(\Delta, \overline{R}\hat{})$ into $\mathfrak{m}$-torsion in $H^i(\mathbf{Z}_p(1)^d, (R_\infty \otimes_V \overline{V})\hat{})$. As this module has no $\mathfrak{m}$-torsion, it follows that $p^{1/(p-1)}$ annihilates the p-torsion. Now $H^i(\Delta, \overline{R}\hat{})/(p\text{-torsion})$ contains $\Lambda^i((R \otimes_V \overline{V})\hat{}(-1)^d)$, with the quotient annihilated by $\mathfrak{m}$. Since any map from $\mathfrak{m}$ into $\Lambda^i((R \otimes_V \overline{V})\hat{}(-1)^d)$ extends to $\overline{V}$, (i) follows. (ii) and (iii) are now obvious. For (iv) use that extension given by E_p splits as extension of $\overline{R}\hat{}$-modules. A splitting can be defined by sending the ith basis element $\text{dlog}(u_i)$ of the third term, which represents the sequence $\mathrm{dlog}(u_i^{p^{-n}})$ in $\Omega_{\overline{R}/(R\otimes_V\overline{V})}$, into the same sequence in $\Omega_{\overline{R}/\overline{V}}$. This splitting is not equivariant under the Galois group $\mathbf{Z}_p(1)^d$ as an element of this group multiplies $u_i^{p^{-n}}$ by a root of unity ζ, hence $\mathrm{dlog}(u_i^{p^{-n}})$ gets a summand $\mathrm{dlog}(\zeta) \in \Omega_{\overline{V}/V}$. This association defines the class of the extension E_ρ in Galois cohomology, and now (iv) reduces to a simple check of the various isomorphisms. (v) follows the same way. For (vi) we reduce (by Künneth) to the case $d = 1$, $R = V[T, T^{-1}]$. It also suffices to consider $R_\infty = \bigoplus_{\alpha\in\mathbf{Z}[1/p]} V\cdot T^\alpha$. Choose a generator σ of the relevant Galois group $\mathbf{Z}_p(1)$. Then $H^*(\mathbf{Z}_p(1), (R_\infty \otimes_V \overline{V})\hat{})$ is given by the kernel, respectively co-kernel, of $(\sigma - 1)$ on $(R_\infty \otimes_V \overline{V})\hat{}$, which is the topological direct sum (= p-adic completion of algebraic direct sum) of $\overline{V}\hat{} \cdot T^\alpha$, $\alpha \in \mathbf{Z}[1/p]$. The dual of this complex is given by the product of these $\mathbf{Z}_p(1)$-eigenspaces, and the duality map (vi) is induced by the inclusion from the topological direct sum into the direct product. As for any $\varepsilon > 0$ p^ε annihilates kernel and cokernel of $(\sigma - 1)$ on almost all summands, the assertion follows. □

(e) We also have shown that the canonical extension E_ρ can be obtained via pushout by $\overline{R}\hat{} \to \rho^{-1}\overline{R}\hat{}$ from a Γ-equivariant extension

$$0 \to \overline{R}\hat{} \to E \to \Omega_{R/V} \bigotimes_R \overline{R}\hat{}(-1) \to 0.$$

However, this extension is not functorial, but depends on the choice of the u_i. If $\sigma \in \overline{V}$ is an element of p-valuation $1/(p-1)$, σ annihilates the p-torsion in $H^1(\Delta, \overline{R}\hat{})$, so the induced extension

$$0 \to \sigma^{-1}\overline{R}\hat{} \to E_\sigma \to \Omega_{R/V} \bigotimes_R \overline{R}\hat{}(-1) \to 0$$

is, up to Δ-isomorphisms, independent of choices. However, these isomorphisms are in general not Γ-linear.

(f) Now let us comment on the open case, that is, where some of the u_i may be nonunits, and we consider extensions which are étale outside the locus of $pu_1 \cdots u_d$. We can proceed as before by denoting by $\overline{R}$, respectively Γ and Δ, the maximal extension of this type, respectively its Galois groups over V or $\overline{V}$. There are, however, two relevant theories reflecting the distinction between cohomology and cohomology with compact support. One of them is $H^*(\Delta, \overline{R}\hat{})$ and the other $H^*(\Delta, \overline{J}\hat{})$. Here $\overline{J} \subset \overline{R}$ is the ideal defining the locus of $u_1 \cdots u_d$, that is, $\overline{J}$ is the kernel of $\overline{R} \to (\overline{R}/u_1 \cdots u_d\overline{R})]1/p]/(\text{nilradical})$. For the differentials we use differentials with logarithmic poles. Define $\Omega_{\overline{R}}(\mathrm{dlog}\,\infty)$ as the submodule of $\Omega_{\overline{R}} \otimes_R R[1/u_1 \cdots u_d]$ generated by $\Omega_{\overline{R}}$ and elements du/u, $u \in \overline{R}$, a p-power root of some u_i. We check that this is independent of the choice of the u_i. Replace V by an unramified extension so that all irreducible components of $\mathrm{Spec}(R/u_iR)$ are geometrically irreducible over K. As all these components are disjoint, we may localize and assume that each

u_i is either a unit or a prime element. Then any unit f in $R[1/u_1 \cdots u_d]$ is a product of a unit in R and a monomial in the u_i. It follows that for any p-power root g of $f\,\mathrm{dlog}(g)$ is contained in $\Omega_{\overline{R}}(\mathrm{dlog}\,\infty)$. Similarly we define differentials with logarithmic poles relative to some subring of $\overline{R}$.

The results of 3 above allow us to reduce computations to the case of the polynomial ring $R = V[T_1, \dots, T_d]$, and by using Künneth we end up with the case $R = V[T]$. Here

$$R_\infty \bigotimes_{V_\infty} \overline{V} = \bigoplus \overline{V} \cdot T^\alpha \qquad (\alpha \in \mathbf{Q}, \alpha \geq 0),$$

$$J_\infty \bigotimes_{V_\infty} \overline{V} = \bigoplus \overline{V} \cdot T^\alpha \qquad (\alpha \in \mathbf{Q}, \alpha > 0),$$

$$\Omega_{R_\infty \otimes \overline{V}/\overline{V}}(\mathrm{dlog}\,\infty) = \bigoplus \overline{V}[1/p] \cdot T^\alpha dT/T \qquad (\alpha \in \mathbf{Q}, \alpha \geq 0),$$

and the relevant Galois group is $\mathbf{Z}_p(1)$. We derive

4.3. **Theorem.** *There exists a Γ-equivariant functorial extension*

$$0 \to p^{-1}\overline{R}\hat{\ } \to E_p \to \Omega_{R/V}(\mathrm{dlog}\,\infty) \bigotimes_R \overline{R}\hat{\ }(-1) \to 0.$$

This extension is obtained via pushout by $\overline{R}\hat{\ } \to p^{-1}\overline{R}\hat{\ }$ from an extension

$$0 \to \overline{R}\hat{\ } \to E \to \Omega_{R/V}(\mathrm{dlog}\,\infty) \bigotimes_R \overline{R}\hat{\ }(-1) \to 0.$$

Γ operates on this extension but not functorially.

4.4. **Theorem.** (i) $H^i(\Delta, \overline{R}\hat{\ }) \cong \Lambda^i((R \otimes_V \overline{V})\hat{\ }(-1)^d) \oplus (rest)$, *where "rest" is annihilated by $p^{1/(p-1)}$. The isomorphism is* $\mathrm{Gal}(\overline{K}/K)$ *equivariant. Similarly for J,*

$$H^i(\Delta, \overline{J}\hat{\ }) \cong \Lambda^i((R \bigotimes_V \overline{V})\hat{\ }(-1)^d) \oplus (rest).$$

(ii) *If $R_1 \to R_2$ is an étale morphism, the induced morphisms*

$$H^i(\Delta, \overline{R}_1\hat{\ }) \bigotimes_{(R_1 \otimes_V \overline{V})\hat{\ }} (R_2 \bigotimes_V \overline{V})\hat{\ } \to H^i(\Delta, \overline{R}_2\hat{\ })$$

and

$$H^i(\Delta, \overline{J}_1\hat{\ }) \bigotimes_{(R_1 \otimes_V \overline{V})\hat{\ }} (R_2 \bigotimes_V \overline{V})\hat{\ } \to H^i(\Delta, \overline{J}_2\hat{\ })$$

are almost isomorphisms.

(iii) *If $R = R_1 \otimes_V R_2$, the morphisms* (*Künneth*)

$$\bigoplus_{a+b=i} H^a(\Delta, \overline{R}_1\hat{\ }) \bigotimes^{\mathbf{L}}_{\overline{V}\hat{\ }} H^b(\Delta, \overline{R}_2\hat{\ }) \to H^i(\Delta, \overline{R}\hat{\ })$$

and

$$\bigoplus_{a+b=i} H^a(\Delta, \overline{J}_1\hat{\ }) \bigotimes^{\mathbf{L}}_{\overline{V}\hat{\ }} H^b(\Delta, \overline{J}_2\hat{\ }) \to H^i(\Delta, \overline{J}\hat{\ })$$

are almost isomorphisms.

(iv) *The morphism* $\Omega_{R/V} \to H^1(\Delta, p^{-1}\overline{R}^\wedge(1))$ *given by the extension* E_p *is induced by a functorial isomorphism*

$$H^1(\Delta, \overline{R}^\wedge)/(p\text{-}torsion) \cong \Omega_{R/V}(\mathrm{dlog}\infty) \bigotimes_R (R \bigotimes_V \overline{V})^\wedge(-1),$$

via $H^1(\Delta, \overline{R}^\wedge(-1)) \to H^1(\Delta, p^{-1}\overline{R}^\wedge(-1))$ (*which kills* p-*torsion*).

(v) *The cup products* $\Lambda^i(H^1(\Delta, \overline{R}^\wedge)) \to H^i(\Delta, \overline{R}^\wedge)$ *induce isomorphism*

$$H^i(\Delta, \overline{R}^\wedge)/(p\text{-}torsion) \cong \Omega^i_{R/V}(\mathrm{dlog}\infty) \bigotimes_R (R \bigotimes_V \overline{V})^\wedge(-i).$$

(vi) *If* $J \subset R$ *denotes the ideal generated by* $u_1 \cdots u_d$, *the cup products*

$$H^0(\Delta, \overline{J}^\wedge) \times H^i(\Delta, \overline{R}^\wedge) \to H^i(\Delta, \overline{J}^\wedge)$$

induce isomorphisms

$$H^i(\Delta, \overline{J}^\wedge)/(p\text{-}torsion) \cong J\Omega^i_{R/V}(\mathrm{dlog}\infty) \bigotimes_R (R \bigotimes_V \overline{V})^\wedge(-i).$$

(vii) *The cup product*

$$H^i(\Delta, \overline{R}^\wedge) \times H^{d-i}(\Delta, \overline{J}^\wedge) \to H^d(\Delta, \overline{J}^\wedge)/(torsion) \cong \Omega^d{}_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})^\wedge(-d)$$

induces almost isomorphisms (*in the derived category*)

$$H^i(\Delta, \overline{R}^\wedge) \to \mathrm{Rhom}_{(R\otimes_V \overline{V})^\wedge}(H^{d-i}(\Delta, \overline{J}^\wedge), \Omega^d{}_{R/V} \bigotimes_R (R \otimes_V \overline{V})^\wedge(-d))$$

and

$$H^i(\Delta, \overline{J}^\wedge) \to \mathrm{Rhom}_{(R\otimes_V \overline{V})^\wedge}(H^{d-i}(\Delta, \overline{R}^\wedge), \Omega^d{}_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})^\wedge(-d)).$$

(*The right side means homomorphisms in the derived category. If we work with coefficients* $R/p^l R$, *those would be the same as ordinary homomorphisms*).

All these morphisms are functorial in R.

(f) Finally we have to investigate the relation with the previous theory. That is, we assume that $(R, \{u_i\})$ satisfies the same hypotheses as before but that in addition R has enough units, so that we can study $\widetilde{R}$, the maximal extension unramified in characteristic 0. We also assume that for each subset $M \subset \{1, \ldots, d\}$ the quotient ring $R_M = R/(\{u_i \mid i \in M\})$ is geometrically integral if it does not vanish and that the image of R^* gives us enough units in it. Finally there should be a maximal M for which $R_M \neq 0$. Let us denote by Δ the Galois group of $\overline{R}$ over $R \bigotimes_V \overline{V}$ and by $\widetilde{\Delta}$ its quotient corresponding to extensions unramified in characteristic 0. By $I \subset \Delta$ we denote the kernel of $\Delta \to \widetilde{\Delta}$.

For any $M \subset \{1, \ldots, d\}$, choose a prime ideal $\mathfrak{q}_M \subset \overline{R}$ lying over $(\{u_i \mid i \in M\}) \subset R$. If $(\{u_i \mid i \in M\}) = R$, let $\mathfrak{q}_M = \overline{R}$. We can do this in such a way that for $M \subset N$ it is also true that $\mathfrak{q}_M \subset \mathfrak{q}_N$ (start with a maximal M and decrease). In the following let M denote only subsets for which $R_M \neq 0$. $I_M \subset \Delta_M \subset \Delta$ denotes the inertia, respectively decomposition, group of $\mathfrak{q}_M$, so that for $M \subset N$ $I_M \subset I_N \subset I \cap \Delta_N \subset \Delta_N \subset \Delta_M$. The extension of $R \otimes_V \overline{V}$ defined by $u_i^{1/n}$ defines a map $\Delta \to \mathbf{Z}\hat{}(1)^d$ ($\mathbf{Z}\hat{}$ = integral finite adeles, also equal to the product of $\mathbf{Z}_l$, l running through all primes). This map sends I_M into the product of those factors $\mathbf{Z}\hat{}$ which are indexed by M (so I_M is abelian), and induces an isomorphism of the p-part of I_M with $\mathbf{Z}_p(1)^M$. The quotient Δ_M/I_M is a quotient of the group $\Delta(R_M)$ (the analogue of Δ for R_M) which corresponds to the extension $\overline{R}/\mathfrak{q}_M$ of R_M. The subgroup $(I \cap \Delta_M)/I_M$ is the quotient of the corresponding inertia $I(R_M)$.

(g) If M has m elements, we obtain Δ_M-linear morphisms

$$\begin{aligned} H^m(I, \overline{R}\hat{}) &\to H^m(\Delta_M \cap I, (\overline{R}/\mathfrak{q}_M)\hat{}) \\ &\to H^0((\Delta_M \cap I)/I_M, H^m(I_M, (\overline{R}/\mathfrak{q}_M)\hat{})) \\ &= H^0((\Delta_M \cap I)/I_M, (\overline{R}/\mathfrak{q}_M)\hat{})(-m) \to \widetilde{R}_M(-m), \end{aligned}$$

hence also

$$H^*(\widetilde{\Delta}, H^m(I, \overline{R}\hat{})) \to H^*(\Delta_M/(\Delta_M \cap I),$$

$$H^0((\Delta_M \cap I)/I_M, (\overline{R}/\mathfrak{q}_M)\hat{})(-m)) \to H^*(\widetilde{\Delta}(R_M), \widetilde{R}_M\hat{})(-m).$$

We use them in the following result, the proof of which will be given only up to a certain point which will be established later. Let us remark that $\Omega^i_{R/V}(\operatorname{dlog}\infty) = \Lambda^i\Omega_{R/V}(\operatorname{dlog}\infty)$ has a filtration (the weight filtration) obtained by exterior power from $\Omega_{R/V} \subset \Omega_{R/V}(\operatorname{dlog}\infty)$. The successive quotients in this filtration are $\bigoplus_M \Omega^a{}_{R_M/V}$, where the direct sum is over all M of cardinality b and $a = i - b$.

4.5. **Theorem.** (i) *The spectral sequence*

$$E_2^{a,b} = H^a(\Delta, H^b(I, \overline{R}\hat{})) \Rightarrow H^{a+b}(\Delta, \overline{R}\hat{})$$

degenerates up to $\mathfrak{m}$*-torsion.*

(ii) *The mappings* (*the direct sum* $\bigoplus_M$ *is over all* M *of size* b)

$$\begin{aligned} E_2^{a,b} &\to \bigoplus_M H^a(\Delta_M/(\Delta_M \cap I), H^0((\Delta_M \cap I)/I_M, (\overline{R}/\mathfrak{q}_M)\hat{})(-b)) \\ &\leftarrow \bigoplus_M H^a(\widetilde{\Delta}(R_M), \widetilde{R}_M\hat{})(-b) \end{aligned}$$

are almost isomorphisms.

(iii) *Under the isomorphism*

$$H^i(\Delta, \overline{R}\hat{})/(p\text{-torsion}) \cong \Omega^i_{R/V}(\operatorname{dlog}\infty) \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-i),$$

the filtration on $H^i(\Delta, \overline{R}\hat{\ })$ *given by the spectral sequence corresponds to the canonical filtration on* $\Omega^i_{R/V}(\operatorname{dlog}\infty)$ *such that the induced morphisms*

$$E_\infty{}^{a,b} \to \bigoplus_M \Omega^a{}_{R_M/V} \bigotimes_R (R_M \bigotimes_V \overline{V})\hat{\ }(-a-b)$$

correspond to the direct sum of the canonical mappings

$$H^a(\tilde{\Delta}(R_M), \tilde{R}_M\hat{\ }) \to \Omega^a{}_{R_M/V} \bigotimes_R (R_M \bigotimes_V \overline{V})\hat{\ }(-a).$$

A corresponding result (*with the obvious modification of* (iii)) *holds for cohomology with coefficients* $\overline{R}/p^l\overline{R}$.

Proof. Our standard argument reduces everything (together with some uninspiring bookeeping) to the case where R is a localization of $V[T]$ and $u_1 = T$. In this case the groups Δ, $\tilde{\Delta}$, I have cohomological dimension 1, as they are fundamental groups of affine curves in characteristic 0. Hence (i) follows by a degree argument. Also only $E_2{}^{0,0}$, $E_2{}^{1,0}$ and $E_2{}^{0,1}$ are nontrivial (up to $\mathfrak{m}$-torsion). The term $E_2{}^{0,0} = H^0(\Delta, \overline{R}\hat{\ })$ has been computed, and it is what it is supposed to be. Furthermore, for $M = \{1\}$ we have $I_M = \Delta_M \cap I = \Delta_M$, and so $E_2{}^{0,1}$ maps to $(R \bigotimes_V \overline{V})\hat{\ }(-1)$. This map is surjective as up to p-torsion $H^1(\Delta, \overline{R}\hat{\ })$ is generated over $(R \otimes_V \overline{V})\hat{\ }$ by the class in cohomology corresponding to the homomorphism $\Delta \to \mathbf{Z}_p(1)$, and as this class maps to a generator of $(R \bigotimes_V \overline{V})\hat{\ }$. Moreover, $H^1(\tilde{\Delta}, \tilde{R}\hat{\ }) \to E_2{}^{1,0} \to H^1(\Delta, \overline{R}\hat{\ }) \to \Omega_{R/V}(\operatorname{dlog}\infty) \otimes (R \bigotimes_V \overline{V})\hat{\ }(-1)$ has image $\Omega_{R/V} \otimes (R \bigotimes_V \overline{V})\hat{\ }(-1)$ which implies (iii) and (ii) up to possible p-torsion. Moreover we can already assert that the map $H^1(\tilde{\Delta}, \tilde{R}\hat{\ }) \to E_2{}^{1,0}$ is almost injective as it is after inverting T and as $H^1(\tilde{\Delta}, \tilde{R}\hat{\ })$ has no T-torsion (as follows from its explicit computation).

All in all we have proved (i), (iii), and (ii) up to the unknown middle cohomology of the complex (in case $d = 1$)

$$0 \to H^1(\tilde{\Delta}, \tilde{R}\hat{\ }) \to H^1(\Delta, \overline{R}\hat{\ }) \to (R \bigotimes_V \overline{V})\hat{\ }(-1) \to 0.$$

We already know that it is almost exact at the other positions, that this middle cohomology is p-torsion, and that multiplication by $T = u_1$ is locally nilpotent on it. We shall see that it is in fact $\mathfrak{m}$-torsion. □

II

1. Construction of $\mathscr{H}^*$.

(a) Suppose V is a complete discrete valuation ring with fraction field K of characteristic 0 and residue field k perfect of positive characteristic $p > 0$. An affine V-scheme U is called small if there exists an étale V-morphism $U \to \mathscr{G}_m{}^d$, $\mathscr{G}_m$ the multiplicative group over V. Suppose X is a smooth V-scheme such that the geometric fiber X_K is geometrically irreducible over K. For any open nonempty subset $U \subset X$ the geometric fundamental group $\Delta(U) = \pi_1(U \otimes_K \overline{K})$ is a quotient of the absolute Galois group $\mathrm{Gal}(\overline{K(X)}/\overline{K}(X))$ of $\overline{K}(X)$, and for $U_1 \subset U_2$ an inclusion there is a surjection $\Delta(U_1) \to \Delta(U_2)$. If U is affine we denote by $R(U)$ its affine ring and by $\overline{R}(U)$ the maximal extension of $R(U)$ which is unramified in characteristic 0. Again for $U_1 \subset U_2$ there is a map $\overline{R}(U_2) \to \overline{R}(U_1)$, so $\overline{R}(U)$ defines a presheaf of rings on the site defined by the affine open subsets of X. The cohomology $H^*(\Delta(U), \overline{R}(U)\hat{\ })$ can be computed by the conical complex $C^*(\Delta(U), \overline{R}(U)\hat{\ })$ with $C^n(\Delta(U), \overline{R}(U)\hat{\ }) =$ continuous maps $\Delta(U)^n \to \overline{R}(U)\hat{\ }$. It can be constructed from the canonical simplicial models for $B(\Delta(U))$, is a presheaf of complexes, and moreover, $C^*(\Delta(U), \overline{R}(U)\hat{\ })$ is equal to the projective limit of $C^*(\Delta(U), \overline{R}(U)/p^l\overline{R}(U))$, $l > 0$. Further properties follow. The absolute Galois group of $K(X)$ acts on $C^*(\Delta(U), \overline{R}(U)\hat{\ })$, and the restriction of this action to the absolute Galois group of $\overline{K}(X)$ is homotopic to the trivial action by a nice compatible system of homotopies. We also have a functorial cup product $C^*(\Delta(U), \overline{R}(U)\hat{\ }) \otimes_{\overline{V}\hat{\ }} C^*(\Delta(U), \overline{R}(U)\hat{\ }) \to C^*(\Delta(U), \overline{R}(U)\hat{\ })$, associative up to canonical homotopy.

So far $C^*(\Delta(U), \overline{R}(U)\hat{\ })$ has been defined as a presheaf of complexes with values in p-adically complete $(\mathscr{O}_X \otimes_V \overline{V})$-modules. This abelian category has the full subcategory of $\mathfrak{m}$-torsion modules, and from now on we view $C^*(\Delta(U), \overline{R}(U)\hat{\ })$ as a presheaf with values in the quotient category. That means that we worry about exactness only up to $\mathfrak{m}$-torsion and that maps $M \to N$ in this quotient category are represented by "real" maps $\mathfrak{m}M/\mathfrak{m}$-torsion $\to \mathfrak{m}N/\mathfrak{m}$-torsion. The quotient category has arbitrary inverse and direct (small) limits.

(b) Let us now restrict to small U's. Then the cohomology $H^*(\Delta(U), \overline{R}(U)\hat{\ }) = H^*(C^*(\Delta(U), \overline{R}(U)\hat{\ }))$ has been computed in §I, Theorem 4.2, and it follows that it is a sheaf (this means up to $\mathfrak{m}$-torsion, to remind the reader of our conventions). It follows that for $U_. \to W$ a hypercovering in the category of small affine open subsets of X, the induced map $C^*(\Delta(W), \overline{R}(W)\hat{\ }) \to \mathrm{Tot}(C^*(\Delta(U_.), \overline{R}(U_.)\hat{\ }))$ is a quasi-isomorphism. In turn this implies that for $U_. \to X$ any hypercovering with all U_n affine and disjoint unions of small pieces, the total complex $\mathrm{Tot}(C^*(\Delta(U_.), \overline{R}(U_.)\hat{\ }))$ is, up to quasi-isomorphism, independent of the choice of $U_.$. We define $\mathscr{H}^*(X)$ as the cohomology of this complex, which takes values in $\overline{V}\hat{\ }$-modules/$\mathfrak{m}$-torsion and has a $\mathrm{Gal}(\overline{K}/K)$-action. However, sometimes we use derived categories, and then $\mathscr{H}^*(X)$ is represented by the complex above, which has an action of the absolute Galois group of $K(X)$, null-homotopic on that of $\overline{K}(X)$.

For any number a we define the subcomplex

$$_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ }) \subset C^*(\Delta(U_.), \overline{R}(U_.)\hat{\ })$$

by the rule

$$_{\leq a}C^i(\Delta(U), \overline{R}(U)\hat{\ }) = C^i(\Delta(U), \overline{R}(U)\hat{\ }), \quad \text{if } i < a,$$

$$\mathrm{Kernel}(C^i(\Delta(U), \overline{R}(U)\hat{\ }) \to C^{i+1}(\Delta(U), \overline{R}(U)\hat{\ })), \quad \text{if } i = a$$
$$\to 0, \text{ if } i > a.$$

It follows that the inclusion ${}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ }) \subset C^*(\Delta(U), \overline{R}(U)\hat{\ })$ induces an isomorphism on cohomology in degree $\leq a$ and that ${}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ })$ is acyclic in degree $> a$.

The subcomplex ${}_{<a}C^*(\Delta(U), \overline{R}(U)\hat{\ })$ of ${}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ })$ coincides with ${}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ })$ in degrees $\neq a$, while in degree a ${}_{<a}C^a(\Delta(U), \overline{R}(U)\hat{\ })$ is the preimage of the p-torsion in $H^a({}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ }))$ under ${}_{\leq a}C^a(\Delta(U), \overline{R}(U)\hat{\ }) \to H^a({}_{\leq a}C^*(\Delta(U), \overline{R}(U)\hat{\ }))$. All these constructions are functorial in U, and the construction above gives cohomology groups ${}_{<a}\mathscr{H}^*(X)$ and ${}_{\leq a}\mathscr{H}^*(X)$ with the obvious mappings between them. Everything can be defined on the level of derived categories associated to complexes with a suitable filtration. We can also work mod p^l by tensoring. However, then ${}_{<a}\mathscr{H}^*(X)$ cannot be so easily described.

The cup product respects filtrations, i.e. it maps ${}_{\leq a}\mathscr{H}^*(X) \times {}_{\leq b}\mathscr{H}^*(X)$ into ${}_{\leq a+b}\mathscr{H}^*(X)$ and ${}_{\leq a}\mathscr{H}^*(X) \times {}_{<b}\mathscr{H}^*(X)$ into ${}_{<a+b}\mathscr{H}^*(X)$, also in the sense of derived categories.

(c) If $f: X \to Z$ is a V-morphism, we define $f^*: \mathscr{H}^*(Z) \to \mathscr{H}^*(X)$, as follows. There is a morphism f_* from $\mathrm{Gal}(\overline{K(X)}/K(X))$ to the quotient of $\mathrm{Gal}(\overline{K(Z)}/K(Z))$ which classifies extensions unramified in the image of the generic point of X. It is well defined up to conjugacy, and for small affines $U \subset X$ and $W \subset Z$ with $f(U) \subset W$ this induces compatible maps $\Delta(U) \to \Delta(W)$, $\overline{R}(W) \to \overline{R}(U)$. If we choose hypercoverings $U_. \to X$ and $W_. \to Z$ such that f extends to a map of simplicial schemes $U_. \to W_.$, we obtain an induced map $f^*: \mathrm{Tot}(C^*(\Delta(W_.), \overline{R}(W_.)\hat{\ })) \to \mathrm{Tot}(C^*(\Delta(U_.), \overline{R}(U_.)\hat{\ }))$. Up to homotopy it is independent of all choices and represents f^*. We have corresponding results for ${}_{\leq a}\mathscr{H}^*$ and ${}_{<a}\mathscr{H}^*$. Also cup products are functorial. Finally, for smooth V-schemes X which are not geometrically irreducible, we can define $\mathscr{H}^*(X)$ by the following procedure. If X is irreducible, denote by V_1 the integral closure of V in $\mathscr{O}_X$ so that X is geometrically irreducible over V_1. V_1 is a finite étale extension of V, and $X \otimes_V \overline{V}$ is the disjoint union of $[K_1 : K]$ copies of $X \otimes_{V_1} \overline{V}$ permuted by $\mathrm{Gal}(\overline{K}/K)$. Define $\mathscr{H}^*(X)$ as the induced (via $\mathrm{Gal}(\overline{K}/K_1) \subset \mathrm{Gal}(\overline{K}/K)$) module associated to the cohomology of X relative to V_1, or equivalently as the direct sum of $[K_1 : K]$ copies of this cohomology, with the corresponding extension of the $\mathrm{Gal}(\overline{K}/K_1)$-action to $\mathrm{Gal}(\overline{K}/K)$. For nonirreducible X use the direct sum of the cohomologies of the connected components. These constructions also apply to ${}_{\leq a}\mathscr{H}^*$ and ${}_{<a}\mathscr{H}^*$.

Finally all these extend to simplicial schemes.

(d) We now exhibit some properties of $\mathscr{H}^*$. Let us first remark that

$$ {}_{\le a}C^*(\Delta(U), \overline{R}(U)\hat{})/{}_{<a}C^*(\Delta(U), \overline{R}(U)\hat{}) \cong \Omega^a{}_{X/V} \bigotimes_{R(U)} (R(U) \bigotimes_V \overline{V})\hat{}(-a)[-a], $$

and that ${}_{\le d}C^*(\Delta(U), \overline{R}(U)\hat{}) \to C^*(\Delta(U), \overline{R}(U)\hat{})$ is a quasi-isomorphism if X has relative dimension d over V. Also we denote by $\mathscr{X}$ the formal scheme associated to X.

1.1. **Theorem.** (i) *There exists a spectral sequence*

$$ E_2{}^{a,b} = H^a(\mathscr{X} \bigotimes_V \overline{V}, H^b(\Delta(U), \overline{R}(U)\hat{})) \Rightarrow \mathscr{H}^{a+b}(X). $$

$E_2{}^{a,b}$ *vanishes unless* $0 \le a, b \le d$. *Furthermore, up to p-torsion*

$$ E_2{}^{a,b} = H^a(\mathscr{X}, \Omega^b{}_{X/V} \bigotimes_{\mathscr{O}_X} \mathscr{O}_{\mathscr{X}}) \bigotimes_{V\hat{}} \overline{V}\hat{}(-b). $$

(ii) $\mathscr{H}^i(X) = 0$ *unless* $0 \le i \le 2d$.
(iii) *If* $X = X_1 \times_V X_2$ *is a product, the map* ($\bigotimes^{\mathbf{L}}$ = *derived tensor product*)

$$ \mathscr{H}^*(X_1) \bigotimes^{\mathbf{L}}{}_{\overline{V}\hat{}} \mathscr{H}^*(X_2) \to \mathscr{H}^*(X) $$

is an isomorphism (*Künneth formula*).

(iv) *Suppose* X *is proper over* V *of pure relative dimension* d. *The trace map defined by*

$$ \mathscr{H}^{2d}(X) = {}_{\le d}\mathscr{H}^{2d}(X) \to H^d(\mathscr{X}, \Omega^d{}_{X/V} \bigotimes_{\mathscr{O}_X} \mathscr{O}_{\mathscr{X}}) \bigotimes_{V\hat{}} \overline{V}\hat{}(-d) \to \overline{V}\hat{}(-d) $$

defines an isomorphism (*Poincaré duality*)

$$ \mathscr{H}^i(X) \cong \mathrm{Rhom}_{\overline{V}\hat{}}(\mathscr{H}^{2d-i}(X), \overline{V}\hat{}(-d)). $$

It also induces isomorphisms

$$ {}_{\le a}\mathscr{H}^i(X) \cong \mathrm{Rhom}_{\overline{V}\hat{}}((\mathscr{H}/{}_{<d-a}\mathscr{H})^{2d-i}(X), \overline{V}\hat{}(-d)) $$

and

$$ {}_{<a}\mathscr{H}^i(X) \cong \mathrm{Rhom}_{\overline{V}\hat{}}((\mathscr{H}/{}_{\le d-a}\mathscr{H})^{2d-i}(X), \overline{V}\hat{}(-d)). $$

These isomorphisms are compatible with the Künneth formula.

(v) *If* $f: X \to Z$ *is a morphism between proper smooth* V*-schemes,* $f^*: \mathscr{H}^*(Z) \to \mathscr{H}^*(X)$ *has an adjoint* (*for Poincaré duality*) $f_*: \mathscr{H}^*(X) \to \mathscr{H}^*(Z)$. *Especially, if* $X \subset Z$ *is a smooth closed subscheme of pure codimension* t, *it has a characteristic class* $c(X) \in \mathscr{H}^{2t}(Z)(t)$. *This class lifts canonically to* $c(X) \in_{\le t} \mathscr{H}^{2t}(Z)(t)$ *and is invariant under* $\mathrm{Gal}(\overline{K}/K)$.

Proof. The spectral sequence in (i) is well known, and the computation of the E_2-term follows from §I. (ii) is now obvious, and (iii) follows from §I if we use as hypercovering of X the product of hypercoverings of X_1 and X_2. Finally, the duality (iv) reduces to the duality in §I, Theorem 4.2 and Serre duality on $\mathscr{X}$, and (v) is a formal consequence: $c(X)$ is the image of $1 \in {}_{\leq 0}\mathscr{H}^0(X)$ under $f_*: {}_{\leq 0}\mathscr{H}^0(X) \to {}_{\leq t}\mathscr{H}^{2t}(Z)(t)$. □

Remark. There are also versions with coefficients $\overline{R}/p^l\overline{R}$. Note that duality becomes simpler in this context as $\overline{V}/p^l\overline{V}$ is (almost) injective over itself.

(e) As an example let us compute the cohomology of $X = \mathbf{P}^1{}_V$. It is clear that $\mathscr{H}^0(X) = {}_{<1}\mathscr{H}^0(X) = \overline{V}^\wedge$, and we shall see (by injecting into the cohomology of $\mathbf{P}^1 - \{0, \infty\}$, see (f) below) that the analogous result also holds $\bmod p^l$. From the exact cohomology sequence to $0 \to \overline{R}^\wedge \to \overline{R}^\wedge \to \overline{R}/p^l\overline{R} \to 0$ we get that $\mathscr{H}^1(X)$ has no p-torsion. Finally, from the spectral sequence above (which degenerates) we obtain that $\mathscr{H}^1(X) = {}_{<1}\mathscr{H}^1(X)$ is p-torsion, hence vanishes, and that ${}_{<1}\mathscr{H}^2(X) =$ torsion in $\mathscr{H}^2(X)$, with quotient $\overline{V}^\wedge(-1)$. By duality ${}_{<1}\mathscr{H}^2(X) = \mathrm{Ext}^1_{\overline{V}^\wedge}({}_{<1}\mathscr{H}^1(X), \overline{V}^\wedge) = 0$. As a result we see that $\mathscr{H}^*(\mathbf{P}^1{}_V)$ is precisely what it should be in any reasonable cohomology theory. As this coincides with the cohomology of $\mathscr{O}_X$, respectively $\Omega_{X/V}$, it follows from the spectral sequence (i) above that the sheaf "p-torsion in $H^1(\Delta, \overline{R}^\wedge)$" has trivial cohomology on $\mathscr{X}$.

(f) Now let us explain how to deal with the open case. Suppose X is smooth over V, as before, and $D \subset X$ is a divisor with normal crossings, relative to V, with smooth components $D_1, \dots, D_r$. We assume that X as well as all irreducible components of finite intersections of D_i's are geometrically irreducible. This is not really necessary as we can deal with these problems by standard methods, but it simplifies the exposition. An open affine subset $U \subset X$ is called small if the intersection of U with any finite intersection of D_i's is irreducible and if there exists an étale morphism $U \to \mathbf{A}^d{}_Y$ into an affine space such that each $U \cap D_i$ is pullback of a standard hyperplane in $\mathbf{A}^d{}_V$. Furthermore, there is a maximal subset of the D_i whose intersection meets U. For each such U we can form $\overline{R}(U)$, the maximal extension of U which ramifies only along D or in characteristic p, and we also define the ideal $\overline{J}(U) \subset \overline{R}(U)$. After this we proceed as before, and obtain two cohomology theories $\mathscr{H}^*(X - D)$ and $\mathscr{H}_c^*(X - D)$(= cohomology with compact support), or even ${}_{<a}\mathscr{H}^*(X - D)$, ${}_{\leq a}\mathscr{H}^*(X - D)$, ${}_{<a}\mathscr{H}_c^*(X - D)$, and ${}_{\leq a}\mathscr{H}_c^*(X - D)$. The notation is not quite precise as they depend on the pair (X, D) and not just on the complement $X - D$. Before we list their properties let us compute an example, which will also allow us to finish the proof of Theorem 4.5 in §I.

Consider $D = \{0, \infty\} \subset X = \mathbf{P}^1{}_V$. We obtain a small open covering of X by $X = U_0 \cup U_\infty = (X - \{\infty\}) \cup (X - \{0\})$. The corresponding rings $\overline{R}_0$, $\overline{R}_\infty$, and $\overline{R} = \overline{R}(U_0 \cap U_\infty)$ are free $\overline{V}$-modules whose bases are given by T^α where $\alpha \in \mathbf{Q}$ and either $\alpha \geq 0$, $\alpha \leq 0$, or α is arbitrary. The relevant Galois group is $\Delta = \mathbf{Z}^\frown(1)$, operating on the complex $\overline{R}^\frown{}_0 \oplus \overline{R}^\frown{}_\infty \to \overline{R}^\frown$, and we have to compute hypercohomology. As the complex is quasi-isomorphic to $V^\frown$, this is trivial. $\mathscr{H}^0(X-D) = \overline{V}^\frown$, $\mathscr{H}^1(X-D) = \overline{V}^\frown(-1)$, and all other groups vanish. Furthermore, this coincides with the cohomology of $\mathscr{O}_x$, respectively $\Omega_{x/v}(\operatorname{dlog}\infty)$, so that from the spectral sequence Theorem 1.2(i) below (which is the analogue of the spectral sequence Theorem 1.1(i) above) we obtain that the sheaf "p-torsion in $H^1(\Delta, \overline{R}^\frown)$" has trivial cohomology on $\mathscr{X}$.

If we refine our covering above to a hypercovering by small open subsets (in the previous sense) we can compare this with $\mathscr{H}^*(X)$, which we already computed. First the map is induced from the maps $H^i(\widetilde{\Delta}, \widetilde{R}^\frown) \to H^i(\Delta, \overline{R}^\frown)$ of §I, which were isomorphisms for $i = 0$ and injective for $i = 1$ with the cokernel concentrated at 0 and ∞, and an extension of $V^\frown(-1)$ by a p-torsion group. This p-torsion group is the cokernel of the injection "p-torsion in $H^1(\widetilde{\Delta}, \widetilde{R}^\frown)$" $\to$ "p-torsion in $H^1(\Delta, \overline{R}^\frown)$," and we wanted to show that is vanishes (that is, in our previous terminology "up to $\mathfrak{m}$-torsion"). But both sheaves above have trivial cohomology on X, so this also holds for the cokernel. As the cokernel is supported only at 0 and ∞, it must vanish. □

(g) We now can exhibit the main properties of $\mathscr{H}^*(X-D)$ and $\mathscr{H}^*{}_c(X-D)$. Choose hypercoverings of X whose pieces U are very small, that is, they are small and their intersections with any irreducible component of an intersection of D_i's is small in the previous sense. The spectral sequence associated to the surjection $\Delta(U) \to \widetilde{\Delta}(U)$ defines a filtration on $\mathscr{H}^*(X-D)$, the weight filtration $W_\cdot$. The sucessive quotients in this filtration are $\mathscr{H}^*$ applied to finite intersections of D_i's, by §I, Theorem 4.5. Finally, the filtrations by $< \alpha$ and $\leq \alpha$ are transversal to the weight filtration and induce the corresponding filtrations on successive quotients. In fact all this should (and can) be done in the derived category, but we leave this to the reader interested in such matters. We get

1.2. **Theorem.** (i) *There exist spectral sequences*

$$E_2^{a,b} = H^a(\mathscr{X} \bigotimes_V \overline{V}, H^b(\Delta(U), \overline{R}(U)^\frown)) \Rightarrow \mathscr{H}^{a+b}(X-D),$$

$$E_2^{a,b} = H^a(\mathscr{X} \bigotimes_V \overline{V}, H^b(\Delta(U), \overline{J}(U)^\frown)) \Rightarrow \mathscr{H}_c^{a+b}(X-D).$$

The $E_2^{a,b}$ *vanish unless* $0 \leq a, b \leq d$. *Furthermore, up to p-torsion*

$$E_2^{a,b} = H^a(\mathscr{X}, \Omega^b{}_{X/V}(\operatorname{dlog}\infty) \bigotimes_{\mathscr{O}_X} \mathscr{O}_{\mathscr{X}}) \bigotimes_{V^\frown} \overline{V}^\frown(-b),$$

respectively

$$E_2^{a,b} = H^a(\mathscr{X}, J\Omega^b{}_{X/V}(\operatorname{dlog}\infty) \bigotimes_{\mathscr{O}_X} \mathscr{O}_{\mathscr{X}}) \bigotimes_{V^\frown} \overline{V}^\frown(-b).$$

(ii) $\mathscr{H}^i(X) = \mathscr{H}_c^i(X) = 0$, *unless* $0 \leq i \leq 2d$.

(iii) *If* $X = X_1 \times_V X_2$ *is a product and* $D = D_1 \times_V X_2 \cup X_1 \times_V D_2$, *the maps* ($\bigotimes^{\mathbf{L}}$ = *derived tensor product*)

$$\mathscr{H}^*(X_1 - D_1) \bigotimes^{\mathbf{L}}{}_{\overline{V}^\wedge} \mathscr{H}^*(X_2 - D_2) \to \mathscr{H}^*(X - D),$$

$$H_c^*(X_1 - D_1) \bigotimes^{\mathbf{L}}{}_{\overline{V}^\wedge} \mathscr{H}_c^*(X_2 - D_2) \to \mathscr{H}_c^*(X - D)$$

are isomorphisms (Künneth formula).

(iv) *Suppose* X *is proper over* V, *of pure relative dimension* d. *The trace map defined by*

$$\mathscr{H}_c^{2d}(X-D) = {}_{\leq d}\mathscr{H}_c^{2d}(X-D) \to H^d(\mathscr{X}, \Omega^d{}_{X/V} \bigotimes_{\mathscr{O}_X} \mathscr{O}_X) \bigotimes_{V^\wedge} \overline{V}^\wedge(-d) \to \overline{V}^\wedge(-d)$$

defines isomorphisms (Poincaré duality)

$$\mathscr{H}^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}(\mathscr{H}_c^{2d-i}(X - D), \overline{V}^\wedge(-d))$$

and

$$\mathscr{H}_c^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}(\mathscr{H}^{2d-i}(X - D), \overline{V}^\wedge(-d)).$$

It also induces isomorphisms

$${}_{\leq a}\mathscr{H}^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}((\mathscr{H}_c/{}_{<d-a}\mathscr{H}_c)^{2d-i}(X - D), \overline{V}^\wedge(-d)),$$

$${}_{<a}\mathscr{H}^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}((\mathscr{H}_c/{}_{\leq d-a}\mathscr{H}_c)^{2d-i}(X - D), \overline{V}^\wedge(-d)),$$

$${}_{\leq a}\mathscr{H}_c^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}((\mathscr{H}/{}_{<d-a}\mathscr{H})^{2d-i}(X - D), \overline{V}^\wedge(-d)),$$

$${}_{<a}\mathscr{H}_c^i(X - D) \cong \mathrm{Rhom}_{\overline{V}^\wedge}((\mathscr{H}/{}_{\leq d-a}\mathscr{H})^{2d-i}(X - D), \overline{V}^\wedge(-d)).$$

These isomorphisms are compatible with the Künneth formula.

(v) *For any subset* $M \subset \{1, \dots, r\}$, *denote by* $X(M) \subset X$ *the intersection of the* D_i, $i \in M$. *Then the associated grading of the weight filtration* $W_\bullet$ *is given (in the usual numbering) by*

$$W_{i+s}(\mathscr{H}^i(X - D))/W_{i+s-1}(\mathscr{H}^i(X - D)) \cong \bigoplus_{|M|=s} \mathscr{H}^{i-s}(X(M))(-s),$$

$$W_{i+s}({}_{<a}\mathscr{H}^i(X - D))/W_{i+s-1}({}_{<a}\mathscr{H}^i(X - D)) \cong \bigoplus_{|M|=s} {}_{<a-s}\mathscr{H}^{i-s}(X(M))(-s),$$

$$W_{i+s}({}_{\leq a}\mathscr{H}^i(X - D))/W_{i+s-1}({}_{\leq a}\mathscr{H}^i(X - D)) \cong \bigoplus_{|M|=s} {}_{\leq a-s}\mathscr{H}^{i-s}(X(M))(-s).$$

These equations should be read in the derived category. The isomorphisms

$${}_{\leq a}\mathscr{H}^* / {}_{<a}\mathscr{H}^*(X - D) \to H^{*-a}(\mathscr{X}, \Omega^a{}_{X/V}(\mathrm{dlog}\,\infty) \bigotimes_{\mathscr{O}_X} (\mathscr{O}_{\mathscr{X}} \bigotimes_V \overline{V})(-a))$$

transform the $W_\bullet$*-filtration on the left into the* $W_\bullet$*-filtration on the right. For* (i) *to* (v) *there are corresponding results with coefficients* $\overline{R}/p^l\overline{R}$.

Finally, a remark about functoriality for $\mathscr{H}^*(X - D)$. It holds for maps $f: X \to Z$ for which $D \supset f^{-1}(E)$, $D \subset X$, and $E \subset Z$ the divisors at infinity.

2. The isomorphism with étale cohomology

(a) Let us recall the definition of a $K(\pi, 1)$-space in étale topology. Suppose X is a smooth irreducible scheme over an algebraically closed field K, of characteristic 0, and π its fundamental group. If F is a finite abelian group with continuous π-action corresponding to a locally constant étale sheaf $\mathbf{F}$ on X, there exists a natural transformation $H^*(\pi, F) \to H^*(X, \mathbf{F})$ (étale topology). We say that X is a $K(\pi, 1)$ if this is an isomorphism for all $\mathbf{F}$ as above. This is equivalent to the assertion that for any $\mathbf{F}$ and any class $\phi \in H^i(X, \mathbf{F})$, $i > 0$, there exists a finite étale covering $Y \to X$ such that the pullback of ϕ to Y vanishes. It is known (SGA 4) that the open subsets $U \subset X$ which are $K(\pi, 1)$'s form a base for the topology of X, and we derive that for each point $x \in X$ the spectrum of the local ring $\mathscr{O}_{X,x}$ is a filtering projective limit of $K(\pi, 1)$'s, hence itself a $K(\pi, 1)$.

We need a generalization of this statement. Let V denote a discrete valuation ring with the usual hypotheses and K its field of fractions. Suppose now X is a smooth V-scheme. An open subset $U \subset X$ is called a $K(\pi, 1)$ if $U \otimes_V \overline{K}$ is so.

2.1. **Lemma.** *Any $x \in X$ is contained in an open $U \subset X$ which is a $K(\pi, 1)$.*

Proof (Compare SGA 4, XI.3.3 or [Fr, Theorem11.5/11.6]). We may assume that the residue field k of V is algebraically closed, that x is a k-rational point in the closed fiber, and that X is affine and irreducible of relative dimension d. Embed X into some projective space $P = \mathbf{P}^n{}_V$, with closure $\overline{X}$, $X = \overline{X} - Y$, Y reduced of relative dimension $d-1$. On the dense set $Y^0 \subset Y$ of points where Y is smooth over V $\Omega_{Y/V}$ is a quotient of the restriction of $\Omega_{P/V}$ to Y, and we obtain a map from Y^0 into the (relative to P) Grassmannian of rank-$(d-1)$ quotients of $\Omega_{P/V}$. Let Z denote the closure of the graph of this map, so that Z is a proper modification of Y on which $\Omega_{Y/V}$ extends to a quotient bundle $\mathscr{E}$ of $\Omega_{P/V}$. A linear subspace $L \subset P$ of codimension $d-1$ is called good if the intersection of L with X is smooth, if L meets the generic fiber Y_K in $Y^0{}_K$, and if for any $z \in Z$ with projection $y \in Y$ lying on L the map $\Omega_{P/V}(y) \to \mathscr{E}(z) \oplus \Omega_{L/V}(y)$ is an isomorphism. A dimension count shows that good L's through x form a dense open subset in the parameter space of all such L's, provided one modifies the embedding $X \subset P = \mathbf{P}^n{}_V$ via some $\mathbf{P}^n{}_V \subset \mathbf{P}^N{}_V$. Note also that for a good L the intersection $(Y \cap L)_K$ is reduced.

We find open subsets $x \in U \subset X$, $W \subset \mathbf{P}_V{}^{d-1}$, and a projection $U \to W$ such that all fibers are intersections $X \cap L$, L good. On the generic fibers this induces an elementary fibration, that is, the complement of a finite étale divisor in a family of complete smooth curves. The projection thus is a fibration in étale homotopy, and we conclude by induction. □

2.2. **Corollary.** *For X as above, $x \in X$,* $\operatorname{Spec}(\mathscr{O}_{X,x} \otimes_V \overline{K})$ *is a $K(\pi, 1)$.*

Let us also note the "open" version.

2.3. **Lemma.** *Suppose X is smooth over V, $D \subset X$ a divisor with normal crossings (relative to V). If $x \in X$ and $f \in \mathcal{O}_{X,x}$ is a local equation for D near x,* $\operatorname{Spec}(\mathcal{O}_{X,x}[1/f] \otimes_V \overline{K})$ *is a* $K(\pi, 1)$.

Proof. We may assume that $f = f_1 f_2 \cdots f_t$, where the f_i are elements of the maximal ideal of $\mathcal{O}_{X,x}$ which extend to a regular system of parameters. We have to show that for any finite π-module M and any $\phi \in H^i_{\text{ét}}(\mathbf{M})$, $i > 0$, the class ϕ dies in a finite étale extension of $\operatorname{Spec}(\mathcal{O}_{X,x}[1/f] \otimes_V \overline{K})$. In the course of the proof we may always pass over to the extension generated by taking nth roots of all f_i, since the normalization of $\mathcal{O}_{X,x}$ in this extension is local and satisfies the same hypotheses as $\mathcal{O}_{X,x}$ itself. From Abhyankhar's lemma it follows that we may assume that M is unramified, that is, M is a module for the fundamental group of $\operatorname{Spec}(\mathcal{O}_{X,x} \otimes_V \overline{K})$. If j denotes the inclusion $\operatorname{Spec}(\mathcal{O}_{X,x}[1/f] \otimes_V \overline{K}] \subset \operatorname{Spec}(\mathcal{O}_{X,x} \otimes_V \overline{K})$, there is a spectral sequence

$$E_2^{a,b} = H^a(\operatorname{Spec}(\mathcal{O}_{X,x} \otimes_V \overline{K}), \mathbf{R}j_*^{\,b}(\mathbf{M})) \Rightarrow H^{a+b}(\operatorname{Spec}(\mathcal{O}_{X,x}[1/f] \otimes_V \overline{K}), \mathbf{M}).$$

Taking nth roots induces multiplication by n^b on $E_2^{a,b}$, composed with some other map. As M is finite we may use this procedure several times and reduce to the case that ϕ is restriction of some cohomology class on $\operatorname{Spec}(\mathcal{O}_{X,x} \otimes_V \overline{K})$. But now we only have to apply the corollary above. □

(b) These technicalities out of the way we make the following construction. Suppose X is smooth over V, $l > 0$. By associating to any affine open subset $U \subset X$ the group cohomology $H^*(\Delta(U), \mathbf{Z}/p^l\mathbf{Z})$ or the étale cohomology $H^*(U \bigotimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z})$ we obtain presheaves on X with a map from the first to the second which induces an isomorphism on associated sheaves. The stalks in $x \in X$ are

$$H^*(\pi_1(\operatorname{Spec}(\mathcal{O}_{X,x} \bigotimes_V \overline{K})), \mathbf{Z}/p^l\mathbf{Z}),$$

respectively

$$H^*(\operatorname{Spec}(\mathcal{O}_{X,x} \bigotimes_V \overline{K}), \mathbf{Z}/p^l\mathbf{Z}),$$

which coincide by the technicalities above. It is clear that the associated sheaves are the higher direct images of $\mathbf{Z}/p^l\mathbf{Z}$ under the map (étale topos of $X \bigotimes_V \overline{K}$) $\to$ (Zariski topos of X). If we view everything as objects in a suitable derived category, the hypercohomology of these sheaves is $H^*(X \bigotimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z})$.

The hypercohomology can be computed on the level of presheaves as the direct limit of the associated complexes of sections on $U_.$ over all hypercoverings $U_. \to X$. If $U_.$ consists of small open affines the natural transformation $H^*(\Delta(U), \mathbf{Z}/p^l\mathbf{Z}) \to H^*(\Delta(U), \overline{R}(U)/p^l\overline{R}(U))$ now defines a natural transformation $H^*(X \bigotimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \to \mathcal{H}^*(X, \overline{R}/p^l\overline{R})$. Of course this transformation should be seen in the derived category as a map between complexes. If X is proper over V the theory $\mathcal{H}^*(X)$ satisfies the necessary finiteness conditions so that we can pass to the limit and obtain, at least on the level of cohomology groups, a natural transformation $H^*(X \otimes_Y \overline{K}, \mathbf{Z}_p) \to \mathscr{H}^*(X)$ or $H^*(X \otimes_V \overline{K}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \overline{V}\hat{} \to \mathscr{H}^*(X)$. We shall see that this is an almost isomorphism.

(c) All this can be extended to the open case. If X is smooth over V, $D \subset X$ a divisor with normal crossings (relative to V), we might redo things above using fundamental groups or étale cohomology of $U \cap (X - D)$ and arrive at natural transformations $H^*((X - D) \otimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \to \mathscr{H}^*(X - D, \overline{R}/p^l\overline{R})$ (in the derived category), and for proper X $H^*((X - D) \otimes_V \overline{K}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \overline{V}\hat{} \to \mathscr{H}^*(X - D)$. A check of the definitions reveals that these respect weight filtrations and that on the associated graded pieces we obtain the previous transformations. Hence in this more general case we also obtain almost isomorphisms.

(d) From now on we mainly work with cohomology $\bmod p^l$, as Poincaré duality is simpler in this context. We do not have to worry about Ext groups. The transformations $H^*(X \otimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \otimes \overline{V}/p^l\overline{V} \to \mathscr{H}^*(X, \overline{R}/p^l\overline{R})$ preserve cup products. Let us show that this also holds for characteristic classes of subvarieties. So assume that X is proper and smooth over V and that $Z \subset X$ is a smooth closed subscheme of pure relative dimension t. We claim that the characteristic class $c(Z) \in H^{2t}(X \otimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z})(t)$ maps to $c(Z) \in \mathscr{H}^{2t}(X, \overline{R}/p^l\overline{R})(t)$. This holds if $t = 1$, that is, if Z is a smooth divisor: We may assume that X has pure relative dimension d over V. The weight filtration on the complex $C^*(X - Z, \overline{R}/p^l\overline{R})$ (which computes $\mathscr{H}^*(X - Z, \overline{R}/p^l\overline{R})$) gives rise to an extension

$$0 \to C^*(X, \overline{R}/p^l\overline{R}) \to C^*(X - Z, \overline{R}/p^l\overline{R}) \to C^*(Z, \overline{R}/p^l\overline{R})(-1)[-1] \to 0.$$

The corresponding extension in étale topology represents $c(Z)$ there, and it maps into this extension. So we have to show that it represents $c(Z)$ in $\mathscr{H}^*$, that is, we have to show that for any class $\phi \in \mathscr{H}^{2(d-1)}(Z, \overline{R}/p^l\overline{R})(d - 1)$, the Z-trace of ϕ is equal to the X-trace of the image of ϕ under the connecting morphism $\mathscr{H}^{2(d-1)}(Z, \overline{R}/p^l\overline{R})(d - 1) \to \mathscr{H}^{2d}(Z, \overline{R}/p^l\overline{R})(d)$ associated to the exact sequence above. But trace maps are defined via the transformation from $\mathscr{H}^*$ into differentials, which maps the exact sequence above into

$$0 \to H^*(X, \Omega^d{}_{X/V}) \bigotimes_V \overline{V}\hat{} \to H^*(X, \Omega^d{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \overline{V}\hat{} \to H^*(Z, \Omega^{d-1}{}_{Z/V}) \bigotimes_V \overline{V}\hat{} \to 0,$$

associated to $0 \to \Omega^d{}_{X/V} \to \Omega^d{}_{X/V}(\mathrm{dlog}\,\infty) \to \Omega^{d-1}{}_{Z/V} \to 0$. However, this sequence represents the characteristic class of Z in Hodge cohomology, and so the assertion follows.

For $t > 1$ we denote by $\widetilde{X}$ the blow-up of X along Z. The preimage of Z in $\widetilde{X}$ is a smooth divisor E which is a $\mathbf{P}^{t-1}$-bundle over Z. If

$\xi \in {}_{\leq 1}\mathscr{H}^2(\tilde{X}, \overline{R}/p^l\overline{R})(1)$ denotes the characteristic class of E, we claim that for any $\phi \in \mathscr{H}^{2(d-t)}(Z, \overline{R}/p^l\overline{R})(d-t)$, the Z-trace of ϕ is equal to the E-trace of the product with ξ^{t-1} of the pullback of ϕ. If we show that the image of ξ under ${}_{\leq 1}\mathscr{H}^2(\tilde{X}, \overline{R}/p^l\overline{R})(1) \to H^1(\tilde{X}, \Omega_{\tilde{X}/V}) \otimes_V \overline{V}/p^l\overline{V}$ is the characteristic class of E in Hodge cohomology, everything follows from the corresponding formula in this theory, and from compatibility with cup product of the projections ${}_{\leq a}\mathscr{H}^*(\tilde{X}, \overline{R}/p^l\overline{R})(a) \to H^{*-a}(X, \Omega^a_{\tilde{X}/V}) \otimes_V \overline{V}/p^l\overline{V}$. However, the remaining assertion follows also by those methods, mapping

$$0 \to {}_{\leq 1}C^*(\tilde{X}, \overline{R}\hat{\ }) \to {}_{\leq 1}C^*(\tilde{X} - E, \overline{R}\hat{\ }) \to {}_{\leq 0}C^*(E, \overline{R}\hat{\ })(-1)[-1] \to 0$$

to differentials.

We also derive that the transformation

$$H^{2d}(X \bigotimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \otimes \overline{V}/p^l\overline{V}(d) \to \mathscr{H}^{2d}(X, \overline{R}/p^l\overline{R})(d)$$

is compatible with trace maps by evaluating at characteristic classes of points.

(e) Finally we can show the main result.

2.4. **Theorem.** *The transformations* (X *proper and smooth,* $D \subset X$ *a divisor with normal crossings*)

$$H^*((X - D) \bigotimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \otimes \overline{V}/p^l\overline{V} \to \mathscr{H}^*(X - D, \overline{R}/p^l\overline{R}),$$

$$H^*((X - D) \bigotimes_V \overline{K}, \mathbf{Z}_p) \otimes \overline{V}\hat{\ } \to \mathscr{H}^*(X - D)$$

are almost isomorphisms. They respect weight filtrations.

Proof. It suffices to consider the first mappings, and we may assume that $D = \varnothing$. We compute module $\mathfrak{m}$-torsion. By Poincaré duality and compatibility of trace maps the transformation $H^*(X \otimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \otimes \overline{V}/p^l\overline{V} \to \mathscr{H}^*(X, \overline{R}/p^l\overline{R})$ has a left inverse. If we show that this left inverse preserves cup product, the same reasoning gives a left inverse for it, and so all maps are isomorphisms. As everything is already compatible with Künneth products, it suffices if the left inverse commutes with δ^*, $\delta: X \subset X \times_V X$ the diagonal embedding. This is equivalent to $H^*(X \otimes_V \overline{K}, \mathbf{Z}/p^l\mathbf{Z}) \otimes \overline{V}/p^l\overline{V} \to \mathscr{H}^*(X, \overline{R}/p^l\overline{R})$ commuting with δ_*, which follows easily from the fact that it respects the characteristic class of the diagonal, and that δ^* is surjective. □

(f) Some examples.

(1) Let $X = \mathbf{P}^n{}_V$. We want to compute ${}_{\leq a}\mathscr{H}^*(X)$ and ${}_{<a}\mathscr{H}^*(X)$. If $\xi \in {}_{\leq 1}\mathscr{H}^2(X)$ is the characterisitc class of a hyperplane, we claim that ${}_{<a}\mathscr{H}^*(X) = {}_{\leq a-1}\mathscr{H}^*(X) = \overline{V}\hat{\ }[T]/(T^a)$, with T corresponding to ξ. This is the result we expect in any decent theory. It is easily shown by induction over n. In the induction step we consider the divisor $D \subset \mathbf{P}^n{}_V$, where D is the union

of the $n+1$ standard hyperplanes. The computation of ${}_{<a}\mathscr{H}^*(X-D)$ and ${}_{\leq a-1}\mathscr{H}^*(X-D)$ proceeds as in the case $n=1$, using the standard open covering of X, and the result is what we expect from any reasonable cohomology theory. As for the intersections of components D_i the result is known by induction; the assertion follows.

(ii) If $T \cong \mathscr{G}_m{}^d$ is a torus over V and $T \subset X$ a smooth torus embedding, where X is proper over V, and $D = X - T$ is a divisor with normal crossings, the cohomology $\mathscr{H}^*(X-D)$ can be computed using the small covering defined by T-invariant open affines. We derive that $\mathscr{H}^*(X-D)$ is independent of the embedding and an exterior algebra in d generators of degree 1. These generators lie in ${}_{\leq 1}\mathscr{H}^1(X-D)$, and ${}_{<a}\mathscr{H}^*(X-D) = {}_{\leq a-1}\mathscr{H}^*(X-D)$ has as basis monomials of degree $< a$ in these generators.

(iii) Let $B\mathscr{G}_m$ denote the classifying simplicial scheme for $\mathscr{G}_m$, so $(B\mathscr{G}_m)_n = \mathscr{G}_m{}^n$. We would like to compactify to a proper simplicial scheme. We cannot quite do that, but at least we can find torus embeddings $(B\mathscr{G}_m)_n \subset X_n$, with X_n smooth and proper over V, and $D_n = X_n - (B\mathscr{G}_m)_n$ a divisor with normal crossings, such that the degeneracies $\partial_i : (B\mathscr{G}_m)_n \to (B\mathscr{G}_m)_{n-1}$ extend to $X_n \to X_{n-1}$. These suffice to define $\mathscr{H}^*(X_. - D_.)$ (only the $\partial_i{}^*$ appear in the relevant double complex), and by the usual spectral sequences and the computation (ii) above we see that $\mathscr{H}^*(X_. - D_.)$ is independent of all choices, and given by the polynomial algebra in one generator ξ in degree two. ξ lies in ${}_{\leq 1}\mathscr{H}^2(X_. - D_.)(1)$, is $\mathrm{Gal}(\overline{K}/K)$-invariant, and the groups ${}_{\leq a-1}\mathscr{H}^*(X_. - D_.) = {}_{<a}\mathscr{H}^*(X_. - D_.)$ are generated by monomials of degree $< a$ in ξ. ξ is the first Chern class of the universal line bundle over $B\mathscr{G}_m$. Of course, we could also use the cohomology of $B\mathscr{G}_m$ itself, to which $\mathscr{H}^*(X-D)$ maps. However, there we encounter difficulties with p-torsion, which makes it more difficult to check identities.

If $\mathscr{L}$ is a line bundle over some X, there exists a hypercovering $U_. \to X$ and a map $U_. \to B\mathscr{G}_m$ such that the pullback of $\mathscr{L}$ to $U_.$ is isomorphic to the pullback of the universal line bundle on $B\mathscr{G}_m$. It follows that we can define the first Chern class of $\mathscr{L}$ in ${}_{\leq 1}\mathscr{H}^2(X)(1) = {}_{\leq 1}\mathscr{H}^2(U_.)(1)$ as the pullback of ξ. Its image in $\mathscr{H}^2(X)(1)$ is the image of the first Chern class in étale topology.

We can also check that for a smooth divisor $Z \subset X$ (X smooth and proper) the Chern class of $\mathscr{O}_X(Z)$ coincides with the characteristic class of Z. The action of $\mathscr{G}_m$ on $\mathbf{P}^1$ defines a $\mathbf{P}^1$-bundle over $B\mathscr{G}_m$ with a smooth divisor given by $\{0, \infty\} \subset \mathbf{P}^1$. There exists a hypercoving $U_. \to X$ such that the pullback of Z to $U_.$ is induced from this universal example. Hence it suffices to treat this universal case. Here the method of torus embeddings implies that ${}_{\leq 1}\mathscr{H}^2$ is isomorphic to $\mathscr{H}^2$, and the assertion follows. We also derive that the image of the first Chern class in $({}_{\leq 1}\mathscr{H}/{}_{<1}\mathscr{H})^2(X)(1) = H^1(X, \Omega_{X/V}) \otimes_V \overline{V}^\wedge$ is equal to the Chern class there, as this holds in the universal example.

(iv) Let $f\colon Y = \mathbf{P}_X(\mathscr{E}) \to X$ denote a projective bundle associated to a rank-e vector bundle $\mathscr{E}$ on X. By $\xi \in {}_{\leq 1}\mathscr{H}^2(Y)$ we denote the first Chern class of $\mathscr{O}(1)$. Then

$$ {}_{\leq a}\mathscr{H}^i(Y) \cong \bigoplus_{0\leq j<e} {}_{\leq a-j}\mathscr{H}^{i-j}(X)\xi^j , $$

and

$$ {}_{<a}\mathscr{H}^i(Y) \cong \bigoplus_{0\leq j<e} {}_{<a-j}\mathscr{H}^{i-j}(X)\xi^j . $$

Proof. The assertion holds for $({}_{\leq a}\mathscr{H}/{}_{<a}\mathscr{H})^*(Y)$, which is given by Hodge cohomology. It thus suffices to treat $({}_{<a}\mathscr{H}/{}_{\leq a-1}\mathscr{H})^*(Y)$. Choose a hypercovering $U_. \to X$ which trivializes $\mathscr{E}$, so that the pullback of Y to $U_.$ is isomorphic to $\mathbf{P}^{e-1} \times U$. The cohomology $({}_{<a}\mathscr{H}/{}_{\leq a-1}\mathscr{H})^*$ of this is abutment of a spectral sequence, which starts with the cohomology of the pieces $\mathbf{P}^{e-1} \times U_n$. As these are known (by Künneth), we derive that the spectral sequence degenerates and that the required result holds. □

It follows that we can define Chern classes $c_i(\mathscr{E}) \in {}_{\leq i}\mathscr{H}^{2i}(X)(i)$. For a smooth subscheme $Z \subset X$ of pure codimension t this leads to a new definition of the characteristic class $c(Z) \in {}_{\leq t}\mathscr{H}^{2t}(X)(t)$, which again coincides with the previous one. Blow up Z and reduce to the case of a divisor. Finally, we can use the splitting principle to reduce assertions about these Chern classes to assertions about line bundles, which usually can be easily shown for the universal case $B\mathscr{G}_m$.

(v) Just as in the case of $B\mathscr{G}_m$ we also can define classifying simplicial schemes $B\mathscr{G}\mathscr{L}(e)$ for vector bundles of rank e. However, it is more difficult to compactify them. We restrict ourselves to note that one can define canonical Chern classes (of the universal bundle) in their cohomology which for example allows to define regulator mappings from K-theory into $\mathscr{H}^*$ (compare [B]).

3. Relations to Hodge cohomology.

(a) The Hodge cohomology of a pair (X, D) (as usual) with coefficients $\overline{V}^\wedge$ is defined by

$$ H^i{}_h(X - D) = \bigoplus_{a+b=i} H^a(X, \Omega^b{}_{X/V}(\operatorname{dlog}\infty)) \bigotimes_V \overline{V}^\wedge(-b) . $$

It can be represented by a complex in the derived category. There are variants with coefficients $\overline{V}/p^l\overline{V}$. $H^*{}_h$ has the usual properties of a cohomology theory, like cup product, Künneth formula or Poincaré duality. We intend to define mappings from $H^*{}_h$ to $\mathscr{H}^*$. This is very easy if we invert p. The spectral sequence

$$ E_2{}^{a,b} = H^a(X, \Omega^b{}_{X/V}(\operatorname{dlog}\infty)) \bigotimes_V \overline{V}^\wedge[1/p](-b) \Rightarrow \mathscr{H}^{a+b}(X - D)[1/p] $$

degenerates, as it is $\mathrm{Gal}(\overline{K}/K)$-equivariant, and there is no nonzero map between $\overline{V}\hat{}[1/p](-b)$'s of different weights b [T]. The same holds for extensions, and we get a natural isomorphism

$$\mathscr{H}^n(X-D)[1/p] = \bigoplus_{a+b=n} H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \overline{V}\hat{}[1/p](-b).$$

It preserves all relevant structures, as products, duality, Chern classes (use the classifying scheme $B\mathscr{G}_m$). In the following we try to control the powers of p in the denominators above. Let us remark immediately that all our transformations will respect Chern classes of vector bundles (and hence also characteristic classes of subvarieties) by considering $B\mathscr{G}_m$, whose cohomology has no torsion.

(b) Let us recall the functorial extensions

$$0 \to \rho^{-1}\overline{R}\hat{} \to E_\rho \to \Omega_{R/V}(\mathrm{dlog}\,\infty) \bigotimes_R \overline{R}\hat{}(-1) \to 0$$

where $R = R(U)$ for some small open $U \subset X$. Their exterior powers give rise to exact sequences

$$0 \to \rho^{-1}\Omega^{i-1}{}_{R/V}(\mathrm{dlog}\,\infty) \bigotimes_R \overline{R}\hat{}(1-i)$$
$$\to \Lambda^i E_\rho \to \Omega^i{}_{R/V}(\mathrm{dlog}\,\infty) \otimes \overline{R}\hat{}(-i) \to 0.$$

Apply $C^*(\Delta(U), \dots)$ and take hypercohomology to get natural transformations

$$H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^b \overline{V}\hat{}(-b) \to \mathscr{H}^{a+b}(X-D),$$

which, as usual, can be defined on the level of derived categories. They are compatible with products. Unfortunately, they are not quite what one expects from Theorem 1.2 (for example) as there appears an annoying factor $b!$. For example if $a = b = d$ we get the inclusion $\rho^d\overline{V}\hat{} \subset \overline{V}\hat{}$, multiplied with $d!$. Similarly for cohomology with compact support. Tensor with the ideal J defining D to obtain

$$H^a(X, J\Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^b \overline{V}\hat{}(-b) \to \mathscr{H}_c^{a+b}(X-D).$$

For $a = b = d$ we get the $d!$ times the inclusion $\rho^d\overline{V}\hat{} \subset \overline{V}\hat{}$. Poincaré duality now gives maps the other way

$$\mathscr{H}^{a+b}(X) \to H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^{b-d}\overline{V}\hat{}(-b),$$

such that the composition of both maps is $d!$ times (the reader might like to reduce this factor to $b! \cdot (d-b)!$) the inclusion $\rho^b\overline{V}\hat{} \subset \rho^{b-d}\overline{V}\hat{}$, tensored with Hodge cohomology. This also holds in the derived category. We similarly claim the reverse composition

$$\mathscr{H}^n(X-D) \to \bigoplus_{a+b=n} H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^{b-d}\overline{V}\hat{}(-b) \to \rho^{-d}\mathscr{H}^n(X-D)$$

is equal to $d!$ times the canonical inclusion, at least if $D = \varnothing$. It follows from the fact that our mappings preserve the characteristic class of the diagonal $X \subset X \times X$.

(c) Let us also recall the extensions

$$0 \to \overline{R}^\wedge \to E \to \Omega_{R/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(-1) \to 0,$$

for $R = R(U)$ the affine ring of a small affine $U \subset X$. Choose sections $\phi_R : \Omega_{R/V}(\operatorname{dlog}\infty) \otimes_R \overline{R}^\wedge(-1) \to E$ for the projections. They are neither functorial in R nor Δ-linear. But if we pass over to extensions

$$0 \to \rho^{-1}\overline{R}^\wedge \to E_\rho \to \Omega_{R/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(-1) \to 0,$$

the E_ρ's glue together to a sheaf $E_\rho(U)$, and for two open subsets U_1 and U_2 with intersection $U = U_1 \cap U_2$ the difference between the two sections ϕ_R of $E_\rho(U)$ induced by the E's is measured by a map

$$H(U_1, U_2) : \Omega_{R/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(-1) \to \rho^{-1}\overline{R}^\wedge.$$

$H(U_1, U_2)$ satisfies the cocycle condition, and it is $\Delta(U)$-linear modulo $\overline{R}^\wedge$. $Id + H(U_1, U_2)$ defines a unipotent automorphism of $E_\rho(U)$, which relates the two E's.

For any i and U the extensions above define exact sequences

$$0 \to \Omega^{i-1}{}_{R/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(1-i) \to \Lambda^i E \to \Omega^i{}_{R/V}(\operatorname{dlog}\infty) \otimes \overline{R}^\wedge(-i) \to 0.$$

Unfortunately these sequences are not functorial in U but become so if we invert p.

Suppose $U_. \to X$ is a hypercovering by small open affine subsets. We construct a double complex $D^*(\Lambda^i E)$ with

$$D^{a,b}(\Lambda^i E) = C^b(\Delta(U_a), \Lambda^i E(U_a))[1/p],$$

where $C^b(\Delta(U_a), \dots)$ stands for the product of the corresponding complexes over all connected components of U_a. The a-differential is given by the alternating sum of the degeneracies ∂_j^*, associated to $\partial_j : U_{a+1} \to U_a$, $0 \le j \le a+1$. For the b-differential we use the differential on C^b.

We can perform the analogous construction with Ω^i. We then obtain an exact sequence of double complexes

$$\begin{aligned} 0 \to D^{a,b}(\Omega^{i-1}{}_{X/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(1-i)) \\ \to D^{a,b}(\Lambda^i(E)) \to D^{a,b}(\Omega^i{}_{X/V}(\operatorname{dlog}\infty) \bigotimes_R \overline{R}^\wedge(-i)) \to 0. \end{aligned}$$

It is functorial in the hypercovering $U_{.}$. The corresponding sequence of connecting homomorphisms defines our transformation from Hodge cohomology into $\mathscr{H}^*$. Now each $D^{a,b}$ has an integral structure as

$$\Omega^i{}_{X/V}(\mathrm{dlog}\,\infty) \subset \Omega^i{}_{X/V}(\mathrm{dlog}\,\infty)[1/p]$$

and $\Lambda^i(E) \subset \Lambda^i(E)[1/p]$. These integral structures are preserved by the b-differentials while the a-differentials on $D^{a,b}(\Lambda^i(E))$ have a factor ρ^{-1}. The connecting homomorphisms are defined by lifting a cycle in

$$\bigoplus_{a+b=n} D^{a,b}(\Omega^i{}_{X/V}(\mathrm{dlog}\,\infty) \bigotimes_R \overline{R}\hat{}\,(-i))$$

to $\bigoplus_{a+b=n} D^{a,b}(\Lambda^i(E))$ and applying $(d_a + d_b)$, which gives something in $\bigoplus_{a+b=n+1} D^{a,b}(\Omega^{i-1}{}_{X/V}(\mathrm{dlog}\,\infty) \otimes_R \overline{R}\hat{}\,(1-i))$. As only the d_a-differential induces a denominator, we see that the sequence of i connecting homomorphisms applied to a cycle in $D^{0,q}(\Omega^i{}_{X/V}(\mathrm{dlog}\,\infty) \otimes_R \overline{R}\hat{}\,(-i))$ gives a cycle in $\bigoplus_{a+b=i} \rho^{-a} D^{a+q,b}(\overline{R}\hat{}\,)$.

Changing by a boundary or using different choices for the liftings leads to an indeterminancy in $(d_a + d_b)\{\bigoplus_{a+b=i-1} \rho^{-a} D^{a+q,b}(\overline{R}\hat{}\,)\}$. The maps $C^*(\Delta(U), \overline{R}\hat{}\,(U)) \to C^*(\Delta(U), \overline{R}\hat{}\,(U))/_{<c}C^*(\Delta(U), \overline{R}\hat{}\,(U))$ are functorial and both complexes are torsion free. So we also get the corresponding projection for the D^{**}-complexes. If we compose the i-fold connecting homomorphism with them, we obtain transformations

$$H^q(X, \Omega^i{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \overline{V}\hat{}\,(-i) \to \rho^{c-i}\mathscr{H}^{q+i}/_{<c}\mathscr{H}^{q+i}(X-D),$$

and also the corresponding $\bmod p^l$ (divide each complex by p^l). The same holds for cohomology with compact support, and by duality ($q = d-a$, $i = d-b$, $c = d-n$) this corresponds to

$$_{\le n}\mathscr{H}^m(X-D) \to \bigoplus_{a+b=m} H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^{b-n}\overline{V}\hat{}\,(-b),$$

i.e., first defined modulo p^l, for all l, and the letting $l \to \infty$. Again the compositions one way are $d!$ times the natural inclusions, while the other way this holds at least if $D = \varnothing$.

(d) As a final result we have shown the following

3.1. **Theorem.** *Suppose X is proper and smooth over V, $D \subset X$ a divisor with normal crossings (relative V).*

(i) *There exist natural transformations*

$$\bigoplus_{0\le b\le a} H^{n-b}(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^b\overline{V}\hat{}\,(-b) \to {}_{\le a}\mathscr{H}^n(X-D).$$

They are isomorphisms up to p-torsion and respect weight filtrations, and, up to some factors, also products, Chern classes, and characteristic classes. They can be defined on the level of derived categories.

(ii) *If* $\dim(X) = d$, *these maps have left inverses* (*up to a factor* $d! \cdot \rho^d$)

$$_{\leq a}\mathscr{H}^n(X - D) \to \bigoplus_{0 \leq b \leq a} H^{n-b}(X, {\Omega^b}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^{b-d}\overline{V}^\wedge(-b).$$

If $D = \varnothing$ *they are also right inverses* (*up to* $d! \cdot \rho^d$), *and in any case this is true up to* p*-torsion, or on the associated graded of the weight filtration. These assertions also hold in the derived category.*

(iii) *There also exist left inverses* (*up to* $d! \cdot \rho^a$)

$$_{\leq a}\mathscr{H}^n(X - D) \to \bigoplus_{0 \leq b \leq a} H^{n-b}(X, {\Omega^b}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_V \rho^{b-a}\overline{V}^\wedge(-b).$$

They are right inverses under the same conditions as in (ii), *but are not defined in the derived category.* □

(e) Some applications. We derive an "algebraic" proof of the degeneration of the Hodge-de Rham spectral sequence for compact algebraic manifolds over the complex numbers. It suffices to show that the total dimension of singular cohomology is equal to that of Hodge cohomology. For this, use the fact that the variety is defined over some **Z**-algebra of finite type and make base change to a V satisfying our usual hypotheses. There the assertion holds by (i) above.

Another application is the Kodaira vanishing theorem.

3.2. **Theorem.** *Suppose* X *is proper and smooth over* $V = W(k)$ (*Witt vectors*) *of pure dimension* $d < p$. *If* $\mathscr{L}$ *is an ample line bundle on* X, *then* $H^a(X, \mathscr{L} \otimes {\Omega^b}_X)$ *vanishes for* $a + b > d$.

Proof. Suppose first that there exists a smooth hypersurface $Y \subset X$ with $\mathscr{O}_X(Y) \cong \mathscr{L}$. As $X - Y$ is affine, its étale cohomology with compact support vanishes in degrees $< d$. The same holds for $\mathscr{H}_c(X - Y)$ up to $\mathfrak{m}$-torsion. By (ii) above there are transformations $H^a(X, \mathscr{L} \otimes {\Omega^d}_X) \to \mathscr{H}^{a+d}(X - Y)$ whose kernel is annihilated by ρ^a. Thus $\rho^a\mathfrak{m}$ annihilates $H^a(X, \mathscr{L} \otimes {\Omega^d}_X) \bigotimes_V \overline{V}^\wedge$, and as the annihilator of a nontrivial element in this group contains no p-power of exponent less than 1, the assertion follows for $b = d$. For other values of b we use the exact sequence

$$0 \to {\Omega^b}_X(\mathrm{dlog}\, Y) \to \mathscr{L} \otimes {\Omega^b}_X \to \mathscr{L} \otimes {\Omega^{b-1}}_Y \to 0$$

and descending induction over b.

In general some power $\mathscr{L}^n$, n prime to p, has a global section f, which defines a smooth hypersurface. Replace X by a finite covering $X_1 \to X$, ramified only along Y. X_1 can be constructed via the algebra $\bigoplus_{0 \leq i < m} \mathscr{L}^{-i}$ with multiplication defined by $\mathscr{L}^{-m} \subset \mathscr{O}_X$. It is smooth over V, and the pullback of $\mathscr{L}$ to it satisfies the previous hypotheses. The assertion follows. □

III

1. Commutative algebra.

(a) In the case of bad reduction we start with V as before, but now R is just a V-algebra essentially of finite type over V and such that $R \otimes_V K$ is smooth over K. As before we can define $\overline{R}$ as the integral closure of V in the maximal étale covering of $R \otimes_V K$. To study it we now use the concept of bounded ramification instead of almost étale. It means that assertions hold up to some fixed power p^e instead of p^ε for any $\varepsilon > 0$. Let us assume that in R there exist units $u_1, \ldots, u_d$ such that the $\operatorname{dlog}(u_i)$ form a basis of $\Omega_{R/V}[1/p]$ over $R[1/p]$, which is equivalent to the fact that the maps $V[T_i^{\pm 1}] \to R$ defined by the u_i are étale in characteristic 0. Our strategy is to compare the big extension $\overline{R}$ of R to the well-known union R_∞ of $R_n = R \otimes_{V[T_i^{\pm 1}]} V_n[T_i^{\pm 1}, T_i^{p^{-n}}]$ (V_n a sequence of extensions of V which kills ramification as before). Our first task is to compare R_n to its normalization.

1.1. **Lemma.** *There exists an e independent of n such that the normalization of R_n is contained in $p^{-e} R_n$.*

Proof. We may assume that R is normal and that the residue field of V is algebraically closed. Let $S = V[T_i^{\pm 1}]$, $S_n = V_n[T_i^{\pm 1}, T_i^{p^{-n}}]$, and denote by $\widetilde{R}_n$ the normalization of $R_n = R \otimes_S S_n$. As R_n is a free R-module, it satisfies the S_2-condition, and hence it suffices to prove the assertion after localization in a height-one prime $\mathfrak{p}$ of R containing p (for all other height-one primes the localizations of R_n and $\widetilde{R}_n$ are equal already). So we may assume that R is a discrete valuation ring.

As V_n is a totally ramified extension of V, after a finite number of steps the semilocal rings W_n = normalization of $R \otimes_V V_n$ all have the same number of maximal ideals and form a sequence of totally ramified extensions. Renumbering we may assume that all W_n are local and totally ramified extensions of R. If Π_n denotes uniformizing elements for W_n and π_n uniformizing elements for V_n, Π_n divides π_m for some $m > n$. It follows that for some n the element $d\pi_n \in \Omega_{W_n/R}$ does not vanish. Otherwise, $d\Pi_n$ would map to 0 in $\Omega_{W_m/R}$, hence it would vanish as $\Omega_{W_n/R} \to \Omega_{W_m/R}$ is injective. But $d\Pi_n$ generates $\Omega_{W_n/R}$ which is nonzero.

Now choose an n with $d\pi_n \neq 0$ in $\Omega_{W_n/R}$. If for $m > n$ the different of V_m/V_n is p^δ, it follows that up to units $d\pi_n = p^\delta d\pi_m \neq 0$ in $\Omega_{W_m/R}$, hence W_m has different at least p^δ over R. Looking at traces we derive that there exists a fixed f such that $W_n \subset p^{-f}(R \otimes_V V_n)$ for all n. Similarly we can look at Ω_{R_n/W_n}. First consider $\Omega_{R_n/W_n}/(p\text{-torsion}) = \Omega_{R_n/V_n}/(p\text{-torsion}) \subset \Omega_{R_n/V_n}[1/p]$ = free $R_n[1/p]$-module with basis $\operatorname{dlog}(u_i)$. It contains the free R_n-submodule with basis $p^{-n}\operatorname{dlog}(u_i)$. For n big the intersection of this free module with the submodule generated by $\Omega_{R/V}$ is contained in $p^{n-c} \cdot$ (this free module), c fixed and independent of n. As $W_n = R \otimes_V V_n$ up to bounded

p-power, it follows that we may enlarge c such that Ω_{R_n/W_n} contains a subquotient $(R_n/p^{n-c}R_n)^d$, and by the exact sequence

$$0 \to \Omega_{W_n/R} \bigotimes_{W_n} R_n \to \Omega_{R_n/R} \to \Omega_{R_n/W_n} \to 0$$

we see that the different of R_n/R differs only by a uniformly bounded p-power from that of S_n/S. Looking at traces it follows that $R_n = R \otimes_S S_n$ up to uniformly bounded p-power. The lemma has been shown. We also have derived that for $\dim(R) = 1$ the sequence of extensions R_n kills ramification (§I, Theorem 1.2). □

(b) Let us call a system of units $\{u_1, \dots, u_d\} \in R^*$ good if their logarithmic derivatives $\mathrm{dlog}(u_i)$ form a basis of $\Omega_{X/V}[1/p]$ and if there exists a fixed power p^e such that $p^e \cdot e_{B/R}$ is integral for any normal extension (of finite degree) $B \supset A = R_\infty$ (defined as before) which is étale in characteristic 0. If such a system of units exists, it follows that $\overline{R}$ is flat over R up to a bounded p-power, that is, some fixed power p^e annihilates all $\mathrm{Tor}_i^R(\overline{R}, M)$ for $i > 0$ and any R-module M. This property is transitive for extensions, and it holds for $R \subset R_\infty$ and $R_\infty \subset \overline{R}$. We can also define a more restricted notion of a good system of units by replacing $\overline{R}$ by a smaller (infinite) Galois extension C of R_∞ and restricting to subextensions of this smaller extension. For example, if R is smooth over V, any system of units u_i is good if their logarithmic derivatives form a base of $\Omega_{R/V}[1/p]$. We intend to show that once we have one good system of units, we have many.

1.2. **Proposition.** *Suppose R has one system of good units for some (infinite) Galois extension C of R unramified in characteristic 0. Suppose the units $u_1, \dots, u_d \in R^*$ have the property that their* dlog*'s form a basis of $\Omega_{R/V}[1/p]$, and that C contains their p-power roots. Then $\{u_1, \dots, u_d\}$ is a good system of units for C.*

Proof. Let $S = V[T_i^{\pm 1}]$, $S_n = V_n[T_i^{\pm 1}, T_i^{p^{-n}}]$, $\widetilde{R}_m =$ normalization of R in the extension generated by V_m and the p^m-th roots of the good system of units $\{v_1, \dots, v_d\}$ (whose existence is claimed in the hypotheses), $R_{m,n}$ (for $m \geq n$) the normalization of $\widetilde{R}_m \otimes_{V_n \otimes S} S_n$. There exists a p-power p^e (independent of n) such that for each n $R_{m,n}$ is flat up to p^e over R provided m is big enough. This is true for all extensions $R \subset \widetilde{R}_m$ and, for m big, also for $\widetilde{R}_m \subset R_{m,n}$. We claim that this assertion also holds for $R_{m,n}$ over R_n or, equivalently, for $R_{m,n}$ over $R \otimes_S S_n$.

We want to show that some fixed p-power annihilates $\mathrm{Tor}_i^{R\otimes_S S_n}(R_{m,n}, M)$, any $i > 0$ and any $R \otimes_S S_n$-module M. This holds if M is of the form $M = N \otimes_s S_n$, N an R-module (this is equivalent to the assertion relative to R). As any M has an exact resolution $0 \to M \to N_0 \otimes_S S_n \to N_1 \otimes_S S_n \to \cdots$, we can restrict to indices $i > d + 1 = \dim(S)$. Also $R[1/p]$ is étale over $S[1/p]$;

some fixed p-power makes $e_{R/S}$ integral. It follows that R is a projective $R \otimes_S R$-module up to this p-power. Also some fixed p-power annihilates the finitely generated $R \otimes_S R$-modules $\mathrm{Tor}_i^S(R, R)$, $i > 0$. It follows that the same conclusion holds for higher Tor's $\mathrm{Tor}_i^{S_n}(R_{m,n}, R \otimes_S S_n)$ with a p-power independent of m and n. Now the functor which sends an $R \otimes_S S_n$-module M to $R_{m,n} \otimes_{R\otimes_S S_n} M$ is the composition of the functor $M \to R_{m,n} \otimes_{S_n} M$ and of the tensor product with R over $R \otimes_S R$. The left derivatives of the first functor are, up to bounded p-power, equal to the $\mathrm{Tor}_i^{S_n}(R_{m,n}, M)$ as those form an exact δ-functor and vanish (up to bounded p-power) for $M = R \otimes_S S_n$. Hence these left derivatives vanish for $i > d+1$. The second functor is exact, again up to bounded p-power. The assertion now follows easily. □

(c) We claim that $R_{m,n}$, m big, is even faithfully flat over R_n up to a p-power independent of n. This means that if a map between R_n-modules M induces 0 on $R_{m,n} \otimes_{R_n} M$, then it is annihilated by this p-power. It suffices if R_n is a direct summand in $R_{m,n}$ up to bounded p-power. To see this we remark that the differents over R of $R_{m,n}$ (m big) and R_n in different height-one primes of R differ only by bounded p-powers, so we find δ (depending on m) and e (independent of m and n) such that for m big $p^\delta e_{R_{m,n}/R_n}$ is integral after localizing in height-one primes and such that $p^{e-\delta} \mathrm{tr}_{R_{m,n}/R_n} : R_{m,n} \to R_n$ is integral. As $R_{m,n}$ is flat over R_n (up to bounded p-power) and as R_n satisfies the S_2-condition, we may assume that $p^\delta e_{R_{m,n}/R_n} = \sum x_i \otimes y_i$ is integral even before localization. Then $r \to p^{e-\delta} \mathrm{tr}_{R_{m,n}/R_n}(\sum x_i y_i r)$ defines a left inverse of the inclusion $R_n \subset R_{m,n}$ up to p^e. We derive the assertion and also that C is faithfully flat over each R_n up to a bounded p-power independent of n. Use transitivity. $R_n \subset R_{\infty,n} =$ union of all $R_{m,n} \subset C$. In the limit C is also faithfully flat over R_∞ up to a bounded p-power. Finally, if $A = R_\infty \subset B \subset C$ is a normal extension of finite degree, unramified in characteristic 0, B is almost unramified over R_∞ after localization in primes of height 1. Hence $p^\varepsilon \cdot e_{B/A}$ is integral after localization for any positive ε. As $C \otimes_A B$ is flat over B (up to a bounded p-power), $p^e \cdot e_{B/A}$ lies in $C \otimes_A B$ for some exponent e independent of B. As C is Galois over A, $C \otimes_A B[1/p] \cong C[1/p]^r$ decomposes and this induces $C \otimes_A B \to C^r$ with kernel and cokernel annihilated by p^e. So $C \otimes_A B$ is étale over C up to p^e and by faithfully flat descent this also holds for B over A.

(d) Let us now give some applications.

1.3. **Theorem.** *Suppose that R is a V-algebra of finite type, geometrically irreducible, such that R has a good system of units $u_1, \dots, u_d$, for some Galois extension $C \supset R \otimes_V \overline{V}$ with group Δ. The following assertions hold up to some p-power p^e.*

(i) *The mappings* $\Omega_{R/V} \to \Omega_{C/\overline{V}} \to \Omega_{C/R\otimes_V \overline{V}}$ *induce isomorphisms.*

$$\Omega_{R/V} \bigotimes_R C[1/p] \cong \Omega_{C/\overline{V}}, \qquad \Omega_{R/V} \bigotimes_R (C[1/p]/C) \cong \Omega_{C/R\otimes_V \overline{V}}.$$

(ii) *The sequence*

$$0 \to \Omega_{\overline{V}/V} \bigotimes_V C \to \Omega_{C/R} \to \Omega_{C/R\otimes_V \overline{V}} \to 0$$

is exact and splits as a sequence of C-modules. It defines a functorial Δ-linear extension

$$0 \to p^{-1}C\hat{} \to E \to \Omega_{R/V} \bigotimes_R C\hat{}(-1) \to 0.$$

(iii) *The induced morphisms*

$$\Omega^i{}_{R/V} \bigotimes_R (R \bigotimes_V \overline{V})\hat{}(-i) \to H^i(\Delta, C\hat{})$$

are isomorphisms.

Proof. Replace C and Δ by $(R \otimes_V \overline{V})[u_i^{p^{-\infty}}]$ and $\mathbf{Z}_p(1)^d$ and compute. □

2. Global methods.

(a) Stable punctured curves.

2.1. **Definition.** A stable punctured curve of type (g, t) over an algebraically closed field k consists of C, $\{x_1, \dots, x_t\}$, where

—C is a proper connected algebraic curve of arithmetic genus g over k, with only simple double points as singularities (so C is reduced).

—$\{x_1, \dots, x_t\}$ is an ordered t-tuple of different k-rational points in the smooth locus of X.

Thus tuple has to satisfy the following conditions.

(i) If an irreducible component of C is a rational curve, then at least three of its k-points are either equal to an x_i, $1 \le i \le t$, or rational double points of C.

(ii) If an irreducible component of C is an elliptic curve, then at least one of its k-points is either an x_i, $1 \le i \le t$, or one of the rational double points of C.

A family of stable punctured curves over a base B consists of a flat morphism of finite presentation $f\colon C \to B$ together with t sections such that each geometric fiber is a stable punctured curve of type (g, t). If C, $\{x_1, \dots, x_t\}$ is a stable punctured curve, the line bundle $\omega_C(\infty)$ (ω_C = dualizing bundle = bidual of Ω_C, "∞" = simple poles in $\{x_1, \dots, x_t\}$) is ample on C, and its third power is very ample. It has degree $t + 2g - 2 > 0$. It follows that each family $f\colon C \to B$ is projective. Also the deformation theory of a stable punctured curve is controlled by the dual $\omega_C(\infty)^*$. As $H^0(C, \omega_C(\infty)^*) = 0$, C has no infinitesimal automorphisms, and its automorphism group is finite.

Also versal deformations exist with tangent space $H^1(C, \omega_C(\infty)^*)$ of dimension $2t + 3g - 3$. Finally $H^2(C, \omega_C(\infty)^*)$ vanishes, so deformations are unobstructed, and the versal deformation has a smooth base.

2.2. **Lemma.** *If V is a complete discrete valuation ring with fraction field K, a stable punctured curve C_K over K can be extended to V after replacing V by its normalization in a finite extension of K. This extension is unique up to isomorphism.*

Proof. First extend C_K to a regular scheme over V such that its special fiber is a reduced divisor with normal crossings. This is possible (after extending V) by the stable reduction theorem if $g > 1$, by the theory of minimal models for elliptic curves if $g = 1$, and by hand if $g = 0$. The sections x_i extend to V and specialize to smooth points in the special fiber. If two such points are equal, we blow up this point. After finitely many steps we are in a situation where this does not occur. After that we contract all components $\mathbf{P}^1$ of the special fiber which do not satisfy condition (i) above. The elliptic components satisfy (ii) (as $t + 2g - 2 > 0$), and we have obtained a stable extension of C_K. It is unique as all steps can be reversed. □

(b) It follows that we can construct a moduli stack $\mathcal{M}_{g,t}$ for stable punctured curves which is proper and smooth over $\operatorname{Spec}(\mathbf{Z})$ or any base we prefer to use. The details consist in following the arguments of Deligne and Mumford [DM]. For example, if $t = 1$ and $g > 0$, $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is the universal stable curve. One also shows that in general the total space the universal curve C over $\mathcal{M}_{g,t}$ is $\mathcal{M}_{g,t+1}$ (at first glance this might seem surprising), hence it is smooth over the base.

Let us spell out what the notion "stack" means in this context. We have given a smooth scheme S and a stable punctured curve $C \to S$ of type (g, t) such that S is versal everywhere or, equivalently, that the tangent space of S is isomorphic to the tangent space of the deformation problem. We further assume that each stable punctured curve of type (g, t) over an algebraically closed field can be obtained by base change from this family. Over $S \times_{\mathbf{Z}} S$ the groupoid $R = \operatorname{Isom}(\operatorname{pr}_1^*(C), \operatorname{pr}_2^*(C))$ is representable by a finite unramified morphism $R \to S \times_{\mathbf{Z}} S$ such that the projections of R onto S are étale. Also, if V is a complete discrete valuation ring with fraction field K, any K-rational point lifts (after enlarging K and V) along pr_1 to a K-rational point of R whose second projection extends to a V-rational point of S.

3. Rigid coverings.

(a) From now on V is a complete discrete valuation ring as usual, with fraction field K of characteristic 0 and residue field k perfect of characteristic $p > 0$. Let us consider flat V-schemes of finite type.

3.1. **Definition.** A V-map $f: X \to Y$ is called a rigid étale covering if

(i) The induced map $f_K: X_K \to Y_K$ on generic fibers is étale.

(ii) If W is the normalization of V in a finite extension L of K, any W-valued point of Y lifts to X after replacing W by its normalization in a finite extension of L.

Examples of rigid étale coverings are proper modifications along the special fiber or étale coverings in the usual sense (it even suffices if they cover the special fiber). If we restrict to coverings which induce open immersions on the generic fiber and if we assume that Y is normal, we can exhibit a cofinal system: Replace Y by a proper modification along the special fiber, and denote by X the disjoint union of open subsets of Y which cover the special fiber.

We may assume that X is affine. Factor $X \to Y$ as $X \to \mathbf{P}^n \times_V Y \to Y$ such that X is open in its closure in $\mathbf{P}^n \times_V Y$. After proper modification of Y we may assume that this closure is a disjoint union of connected components of Y so that X is a direct sum of open subsets of Y. These open subsets cover the special fiber by condition (ii) above.

It is also true that any Zariski open covering of Y_K can be refined to a rigid covering. We may assume that $Y = \mathrm{Spec}(R)$ is affine. The open covering of Y_K is given by complements of loci of ideals $I_j \subset R$ such that their sum I contains a power of p. Blowing up I we may assume that I is generated by one element t. Then the complements of the loci of I_j/t define an open covering of Y which induces the given covering in the generic fiber.

(b) Rigid étale coverings satisfy the axioms of a Grothendieck topology. They also satisfy cohomological descent for formal cohomology with p inverted as follows. Denote the cohomology of formal V-sheaves by $H^*(X^\wedge, \mathcal{F}^\wedge)$, etc. They can be viewed either as topological V-modules or (which usually is the better definition) as projective systems of $V/p^l V$-modules. The assertion below will be true in both contexts.

3.2. **Theorem.** *Suppose $f\colon U_\cdot \to Y$ is a rigid étale hypercovering, $\mathcal{F}$ a coherent sheaf on X. Then the natural map*

$$H^*(Y^\wedge, f^\wedge)[1/p] \to H^*(U_\cdot^\wedge, f^*(\mathcal{F})^\wedge)[1/p]$$

is an isomorphism.

Proof. It suffices to treat Cech coverings $X \to Y$. The assertion holds for open coverings of the special fiber or for proper modifications (by EGA III). In general we use noetherian induction. We may assume that Y is irreducible and normal. Replacing it by its normalization in a finite extension of its fraction field we may assume that X_K is a disjoint union of open subsets of Y_K. For such coverings the assertion holds (we know their structure by (a) above) and everything follows easily. □

(c) Rigid étale coverings occur in the following context. Suppose $C_K \to Y_K$ is a stable punctured curve of type (g, t) and that Y is normal. We intend to construct a rigid étale covering $X \to Y$ and a regular mapping $X \to S$ (S the scheme occurring in the definition of $\mathcal{M}_{g,t}$ as S/R) such that the pullback of

C_K to X_K is isomorphic to the pullback of the universal curve over S. We proceed as follows.

Let $X_K \to Y_K \times_K S_K$ denote the scheme which classifies isomorphisms between the stable punctured curves on both factors. Locally in the étale topology (on Y_K) it is the pullback of R_K, hence it is finite over $Y_K \times_K S_K$, and its projection onto Y_K is étale. Let X denote the normalization of $Y \times_V S$ in X_K. We have to show the lifting condition for discrete valuation rings $W \supset V$. A W-valued point of Y defines a stable punctured curve over the fraction field L of W. After extending W and L we may assume that this curve is isomorphic to the pullback of the universal curve on S via some W-rational point of S. This isomorphism defines an L-rational point of X_K which lifts to a W-rational point.

(d) We now can show the main result.

3.3. **Theorem.** *Suppose R is a V-algebra of finite type, smooth in characteristic 0, which contains a system of units $u_1, \dots, u_d$ whose logarithmic derivatives form a basis of $\Omega_{R/V}[1/p]$. Then there exists a rigid étale covering of* Spec(R) *by affines* Spec(R_i) *such that for each i the units $u_1, \dots, u_d \in R_i$ form a good system for the normalization of R_i in the quotient field of the maximal extension $\overline{R}$ of R which is étale in characteristic 0.*

Proof. Before giving the cumbersome details let us explain the idea. Locally Spec($R[1/p]$) is an elementary fibration [Fr, Theorem 11.5/11.6] $U_K \subset \overline{U}_K \to W_K$. Its geometric fundamental group is an extension of that of W_K by the fundamental group of the geometric fibers of the fibration. The fundamental group of W_K is handled by induction, and so we are reduced to considering the fundamental group of the geometric fibers. For this we use that the elementary fibration is pullback of a universal fibration $C - \{\infty\} \to S$ and that in the universal case everything is smooth (and so is covered by what we already know). Now the formalities. We use induction over d, $d = 0$ being the trivial case. There exists a Zariski open covering of Spec($R[1/p]$) by elementary fibrations $U_K \subset \overline{U}_K \to W_K$ such that $\overline{U}_K$ is a smooth proper curve over W_K with geometrically connected fibers in which U_K is the complement of a divisor of degree $t \geq 3$ which is finite étale over W_K. Also W_K is affine, and its geometric fiber is a $K(\pi, 1)$ in the étale topology. Refine the covering given by these U_K's to a rigid covering of Spec(R) by affines Spec(R_i). We may choose models W over V for the W_K's such that the maps Spec(R_i) $\to W$ are regular. After replacing W by its normalization $\widetilde{W}$ in a finite étale covering $\widetilde{W}_K$ of W_K we may assume that $U_K \subset \overline{U}_K$ is given by a stable punctured curve over $\widetilde{W}_K$. Over a rigid étale covering of $\widetilde{W}$ this curve is pullback of a universal curve over some smooth V-scheme S. Hence refining the rigid étale covering $\{$Spec(R_i)$\}$ we may assume that each Spec(R_i) maps to the total space U of a stable punctured curve $U \subset \overline{U} \to \widetilde{W}$, which is pullback under $\widetilde{W} \to S$ of the universal $C - \{\infty\} \to S$, and such that in the generic fiber we obtain a pullback of the original elementary fibration $U_K \subset \overline{U}_K \to W_K$.

As $C - \{\infty\}$ is smooth over V, there exists an open affine covering of it such that each affine ring has a good system of units, where in addition all these units except one are functions on the base S, that is, they are constant along the fibers of $C \to S$. Refining the rigid étale covering $\{\mathrm{Spec}(R_i)\}$ we may assume that it maps to this open covering of $C - \{\infty\}$. By pullback we obtain normal extensions of $\mathrm{Spec}(R_i)$ which are étale in characteristic 0, and it follows that they are étale up to some p^e over the extension of R_i generated by $\overline{V}$ and p-power roots of the pullbacks of the units constructed above. As all of them except one come from units on W, we may use induction on W and obtain (after refinements) for each R_i a Galois extension, étale in characteristic 0, which is étale up to p^e over the extension generated by $\overline{V}$ and the p-power roots of a system of d units in R_i. We have to convince ourselves that this extension is big enough. It contains the maximal (étale in characteristic 0) extension of W_K, as well as the pullbacks of the maximal extensions of the open covering of $C - \{\infty\}$. We may assume that this covering refines a covering of S_K by $K(\pi, 1)$'s, and it follows that we also obtain at least the full fundamental group of the geometric fibers of U_K over W_K. By the homotopy sequence for $U_K \to W_K$ we see that our Galois extension contains at least the maximal étale covering of U_K, hence also that of $\mathrm{Spec}(R_K)$. Finally, we can replace the d units in R_i by the chosen units in R, and everything follows. □

4. Intermediate cohomology.

(a) Suppose X is a proper flat V-scheme with smooth and geometrically irreducible generic fiber X_K. There exists a rigid covering of X by affines $\mathrm{Spec}(R_i)$ such that each R_i contains d-units $u_1, \dots, u_d$ whose logarithmic derivatives generate $\Omega_{X/V} \otimes R_i[1/p]$. First find an open covering such that $\Omega_{X/V}$ is generated by logarithmic derivatives of units and then refine it. As in the case of good reduction we define presheaves on the rigid étale topos by sending $U \to X$ to $C^*(\Delta(U), \mathbf{Z}/p^l\mathbf{Z})_{l\geq 0}$, respectively $C^*(\Delta(U), \overline{R}(U)/p^l\overline{R}(U))_{l\geq 0}$. They have coefficients in the abelian category of projective systems of $\mathbf{Z}/p^l\mathbf{Z}$-modules. However, we work modulo bounded p-torsion, that is, we divide by the subcategory of projective systems which are annihilated by some power of p. In the quotient category infinite direct or inverse limits do not exist in general. However, if they do exist they are unique up to isomorphism. The associated sheaves are p-adic étale cohomology (same proof as before), respectively Hodge cohomology. If we fix U, we have for any sufficiently small rigid étale cover W of it the canonical extension $E(W)$, which defines (in the derived category) a natural transformation

$$\bigoplus \Omega^i{}_{X/V} \bigotimes_{\mathcal{O}_X} (\mathcal{O}_X \bigotimes_V \overline{V})^\wedge(-i)[-i] \to C^*(\Delta(U), \overline{R}/p^l\overline{R})_{l\geq 0}.$$

This is a quasi-isomorphism.

All in all we have obtained natural transformations

$$H^n{}_{et}(X \bigotimes_V \overline{K}, \mathbf{Q}_p) \to \bigoplus_{a+b=n} H^a(X, \Omega^b{}_{X/V}) \bigotimes_V \overline{V}\hat{}(-a),$$

which preserve cup products, $\mathrm{Gal}(\overline{K}/K)$-action, characteristic classes of cycles, and Chern classes of vector bundles (this is true in the universal case). By now familiar arguments (characteristic class of the diagonal) they induce isomorphisms

$$H^n{}_{et}(X \bigotimes_V \overline{K}, \mathbf{Q}_p) \bigotimes_{\mathbf{Q}_p} \overline{K}\hat{} \cong \bigoplus_{a+b=n} H^a(X, \Omega^b{}_{X/V}) \bigotimes_V \overline{K}\hat{}(-a).$$

The result extends to X which are not geometrically irreducible. Note also that the transformation is even compatible with transformations $f: X_K \to Y_K$ which do not extend to a regular map $X \to Y$.

(b) We can extend to the open case by a method different from the previous one. Suppose X is again proper over V, $D \subset X$ a divisor, such that over K we obtain the familiar situation (X smooth, D normal crossings). We first assume that all irreducible components D_i of D, as well as of finite intersections of D_i's, are geometrically irreducible over K. If X_n, $n \geq 0$, denotes the disjoint union of intersections of n different D_i, we obtain natural maps $X_{n+1} \to X_n$ which satisfy the rules for the degeneracies of a simplicial object. If for each n we choose a rigid étale hypercovering $U_n. \to X_n$ such that the transformations above extend, we get double complexes $C^*(\Delta(U_n.), \mathbf{Z}/p^l\mathbf{Z})_{l\geq 0}$, respectively $C^*(\Delta(U_n.), \overline{R}(U_n.)/p^l\overline{R}(U_n.))_{l\geq 0}$, and a natural map from the first to the second. Passing over to the limit over all systems of hypercoverings $U_n.$ we obtain étale cohomology of $(X - D) \otimes_V \overline{K}$ with compact support, respectively Hodge cohomology with compact support. The corresponding transformation is an isomorphism as this is true for all X_n, and we have a familiar (weight-) spectral-sequence. By duality we also obtain an isomorphism for ordinary cohomology. These are natural transformations for maps $f: X_1 \to X_2$ with $D_1 \supset f^{-1}(D_2)$. Factor into $X_1 \subset X_1 \times X_2 \to X_2$. The projection from the product is easy to treat, so we can concentrate on the injection j of the graph. Here everything follows as the transformations respect the characteristic class of this graph in the cohomology of $X_1 \times X_2$. The kernel of j^* consists of elements which have cup product 0 with this characteristic class, and the cohomology of $X_1 \times X_2$ is the direct sum of this kernel and the image of $\mathrm{pr}_1{}^*$ ($\mathrm{pr}_1 =$ first projection).

(c) Let us collect all information. We remark that by a theorem of Nagata any proper X over K extends to V.

4.1. **Theorem.** *Suppose X is a smooth proper K-scheme, $D \subset X$ a divisor with normal crossings. Then there are* $\mathrm{Gal}(\overline{K}/K)$*-lineaar isomorphisms*

$$H^n{}_{et}((X - D) \bigotimes_K \overline{K}, \mathbf{Q}_p) \bigotimes_{\mathbf{Q}_p} \overline{K}\hat{} \cong \bigoplus_{a+b=n} H^a(X, \Omega^b{}_{X/V}(\mathrm{dlog}\,\infty)) \bigotimes_K K\hat{}(-a).$$

They are natural transformations.

BIBLIOGRAPHY

[B] A. Beilinson, *Higher regulators and values of L-functions*, Modern Problems in Mathematics, VINITI series **24** (1984), 181–238.

[BK] S. Bloch and K. Kato, *P-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. **69** (1986), 107–152.

[DM] P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

[D] P. Deligne, *Théorie de Hodge* II, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 5–58.

[Fa] G. Faltings, *Hodge-Tate structures and modular forms*, Math. Ann. (to appear).

[Fo] J. M. Fontaine, *Formes différentielles et modules de Tate des variétés abeliennes sur les corps locaux*, Invent. Math. **65** (1982), 379–409.

[Fr] E. Friedlander, *Etale homotopy of simplicial schemes*, Princeton Univ. Press, Princeton, N. J., 1982.

[H] O. Hyodo, *On the Hodge-Tate decomposition in the imperfect residue field case*, Crelles J. **365** (1986), 97–113.

[M] H. Matsumura, *Commutative algebra*, Benjamin, New York, 1979.

[T] J. Tate, *P-divisible groups*, in Driebergen Conference on Local Fields, 1966 (T. A. Springer, ed.), Springer-Verlag, Berlin, 1967, pp. 158–183.

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