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ABOUT THE COVER: ALFRED CLEBSCH ON CRYSTALLOGRAPHY

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Alfred Clebsch was born in Königsberg (now Kaliningrad) on January 19, 1833. His high school was the "Altstädtisches Gymnasium" in Königsberg.[1] One of his school friends was Carl Neumann (1832–1925), the son of Franz Neumann (1798–1895), professor of physics at the University of Königsberg and a very prominent figure in 19th-century science, one of the "grandfathers" of theoretical physics in Germany. Franz Neumann's career started in 1820 with studies in crystallography, in particular, in electromagnetic properties of crystals.[2] After graduating from high school, Alfred Clebsch and Carl Neumann enrolled in the University of Königsberg. Clebsch's *Doktorvater* was Franz Neumann.

In 1854 Clebsch defended his thesis on hydrodynamics and subsequently became a high school teacher in Berlin.[3] Clebsch's 1858 "Habilitation" was also in mathematical physics. He became "Privatdozent" at the University of Berlin, delivered a single lecture, and left for Karlsruhe immediatly after to join the faculty of the "Polytechnikum", an engineering school. Here he wrote his first book, on elasticity theory, and thus liberated, immersed himself in algebraic geometry and invariant theory, the hot topics of the time. Throughout the 1860's Clebsch wrote an enormous amount on invariants and algebraic geometry (and their relations), most of the papers appearing in *Crelle's Journal*.

In 1863 Clebsch moved to Giessen, where he met P. Gordan. Their collaboration culminated in Clebsch-Gordan coefficients and a book on abelian functions giving a rather algebraic treatment of the subject [CG66]. Around the same time, Carl Neumann published a book on the same subject, but with an emphasis on function theory and Riemann surfaces [Neu84]. In 1868 Clebsch and Carl Neumann finally combined their efforts and founded the journal *Mathematische Annalen*. The same year Clebsch moved to Göttingen to succeed B. Riemann on the chair of Gauss and assembled around him a remarkable group of young mathematicians, including A. Brill, M. Noether and F. Klein. Unfortunately, his tenure in Göttingen was very short; he died suddenly on November 7, 1872, of diphtheria.

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[1]Some thirty years later, David Hilbert and Hermann Minkowski graduated from the same school – a good place for mathematics and physics (the school library subscribed to *Crelle's Journal*).

[2]One hundred years later Carl Neumann, who worked mainly in analysis, potential theory and mathematical physics, reedited the early papers of his father [Neu17].

[3]A postdoctoral position not uncommon in the 19th century, e.g., K. Weierstrass and H. Grassmann.

Crystallography was rather far from Clebsch's early works in hydrodynamics and elasticity. A science at the interface of mineralogy, physics and mathematics, it attracted a lot of interest in the 19th century. The concept of atoms and molecules, though widely accepted, was still only a working hypothesis. It was hoped that by studying properties of crystals one could indirectly grasp the atomic structure of matter — this was later confirmed with X-ray experiments of Bragg and von Laue. Mathematically, there were also many interesting things to consider. One major problem was to classify crystals appearing in nature. It was observed early on that there are many "rational proportions" in such a crystal, i.e., that after fixing four independent planes of a crystal, all other planes ideally are planes in the underlying *rational* vector space inside $\mathbb{R}^3$, determined by these four planes.[4] The symmetries of crystals are reminiscent of geometries with symmetries and are connected to the theory of groups. This is already in A. Bravais [Bra51] and in C. Jordan's influential book [Jor70]. The Erlangen Program of Klein was on the horizon.

The cover of this issue shows a page from lecture notes of a course by Alfred Clebsch on crystallography, courtesy of the library of the Göttingen Mathematical Institute (also see Figure 1). These notes are not signed, as was customary. It is very likely — comparing the handwriting — that they were written by Clebsch himself. It is hard to say who learned what and from whom — but it is apparent that many pictures in these notes were already in Franz Neumann's original work in 1823.

Let us add that Gauss observed a connection between ternary quadratic forms and crystallography (cf. [Gau63a]). In fact, for a short time he got very interested in crystallographic questions. In 1831 he made quite accurate measurements of crystals with a theodolite. In this connection he also found a nice (and easy to prove) criterion for rationality: given five planes contained in (euclidean) space, there is an underlying rational structure containing these five planes iff the following condition for the angles occurring in the various spherical triangles formed by unit normal vectors to the five planes is fulfilled:

> Das Grundgesetz der Krystallisation lässt sich am kürzesten so aussprechen: Zwischen je fünf Ebenen, welche dabei vorkommen, gibt es folgende Relation: Sind ihre Normalen auf der Kugeloberfläche (0), (1), (2), (3), (4), so sind allezeit die Producte $\sin 102 \cdot \sin 304$, $\sin 103 \cdot \sin 204$, $\sin 203 \cdot \sin 104$ in einem rationalen Verhältniss. Ist dies wie $\alpha : \beta : \gamma$, so ist $\beta = \alpha + \gamma$.[5]

Cf. [Gau63b, pp. 308–310].

Of course, checking this condition experimentally makes sense only if the numerators and denominators of the rational proportions can be assumed to be small natural numbers.

[4] A first systematic treatment of such concepts (i.e., $\mathbb{Q}^3$ or $\mathbb{Z}^3$) can be traced back to crystallographic studies by J. G. Grassmann, the father of H. Grassmann, to whom we owe the "Lineale Ausdehnungslehre" — the modern theory of vector spaces (cf. [Sch89]).

[5] The fundamental law of crystallization is best expressed as follows: Between each 5 occurring planes there is the relation: if (0), (1), (2), (3), (4) are the normals on the sphere, then $\sin 102 \cdot \sin 304$, $\sin 103 \cdot \sin 204$, $\sin 203 \cdot \sin 104$ are in a rational relation. If these are $\alpha : \beta : \gamma$, then $\beta = \alpha + \gamma$.

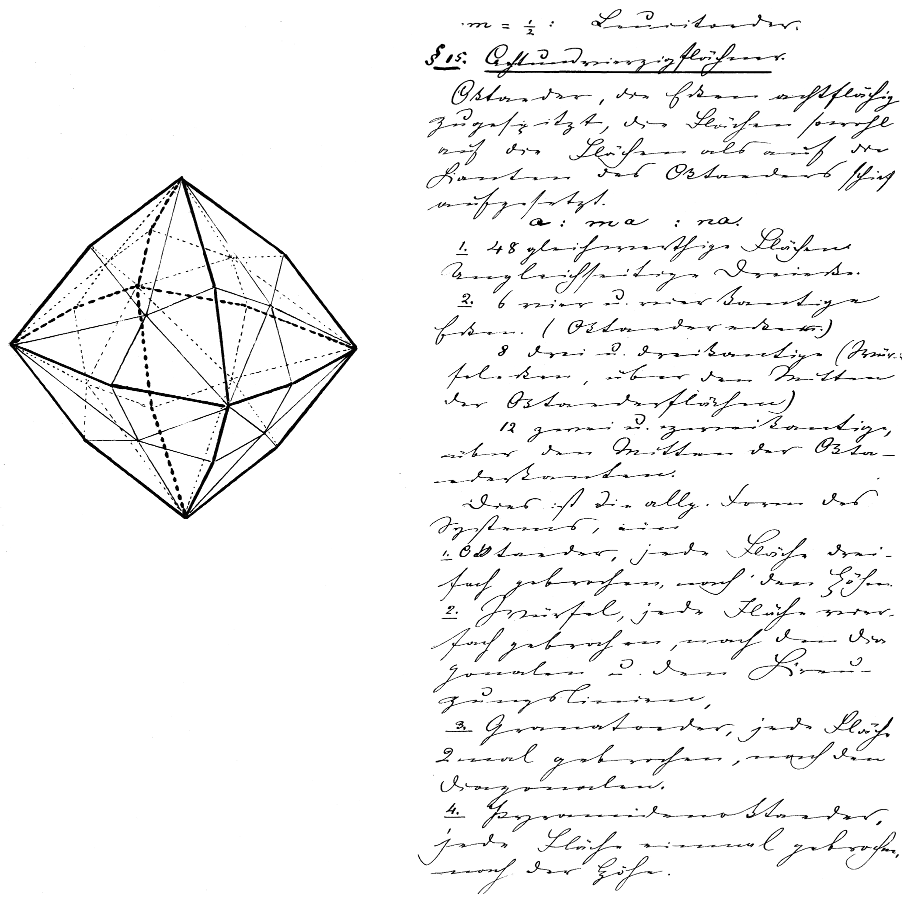

FIGURE 1. Page from lecture notes on crystallography, courtesy of the library of the Göttingen Mathematical Institute.

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