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ON THE RADICAL OF A LIE ALGEBRA

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Let $\mathfrak{L}$ be a Lie algebra over a field K of characteristic zero. For any $X\in\mathfrak{L}$ we denote, as usual, the linear mapping $Y\rightarrow[X, Y]$ of $\mathfrak{L}$ into itself by ad X. Let Γ be the radical of $\mathfrak{L}$. Consider the set $\mathfrak{N}$ consisting of all $N\in\Gamma$ such that ad N is nilpotent. It was shown in a recent paper[1] that $\mathfrak{N}$ is the unique maximal nilpotent ideal[2] of $\mathfrak{L}$. Further if D is a derivation of Γ then $D\Gamma\subset\mathfrak{N}$.

For any $X, Y, Z\in\mathfrak{L}$ put $B(X, Y)=sp(\text{ad } X \text{ ad } Y)$ and $T(X, Y, Z) =sp(\text{ad } [X, Y] \text{ ad } Z)$. Then $B(X, Y)$ is a symmetric bilinear form on $\mathfrak{L}$ while $T(X, Y, Z)$ is a skewsymmetric trilinear form. It is easily verified that they are both invariant under all derivations of $\mathfrak{L}$, that is,

$$B(DX, Y) + B(X, DY) = 0,$$
$$T(DX, Y, Z) + T(X, DY, Z) + T(X, Y, DZ) = 0$$

for any derivation D and $X, Y, Z\in\mathfrak{L}$.

An ideal $\mathfrak{M}$ in $\mathfrak{L}$ is called characteristic if $D\mathfrak{M}\subset\mathfrak{M}$ for every derivation D of $\mathfrak{L}$. Our first theorem may now be stated as follows:

THEOREM 1. *An element X of $\mathfrak{L}$ belongs to the radical Γ if and only if $T(X, Y, Z)=0$ for all $Y, Z\in\mathfrak{L}$.*[3]

As an immediate corollary we get the following:

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[1] Ann. of Math. vol. 50 (1949) p. 68.

[2] My attention has been drawn to a paper by Malcev (Bull. Acad. Sci. URSS. vol. 9 (1945) pp. 329–356) where it is shown that $\mathfrak{N}$ is an ideal.

[3] Since $T(X, Y, Z)=-T(Z, Y, X)$ this condition is clearly equivalent to $B(X, Y) =0$ for all $Y\in\mathfrak{L}'=[\mathfrak{L}, \mathfrak{L}]$. Professor Jacobson has kindly brought it to my notice that this theorem is contained in Cartan's thesis p. 109.

COROLLARY 1. *Both* Γ *and* $\mathfrak{N}$ *are characteristic ideals in* $\mathfrak{L}$.

For every $N\in\mathfrak{N}$, ad N is a nilpotent derivation of $\mathfrak{L}$. Hence $\sigma_N=\exp$ (ad N) is defined and is an automorphism of $\mathfrak{L}$. Let $\mathfrak{M}$ be any nilpotent ideal in $\mathfrak{L}$. Then $\mathfrak{M}\subset\mathfrak{N}$. By $G_{\mathfrak{M}}$ we denote the group of all automorphisms of $\mathfrak{L}$ of the form $\sigma_{M_1}\sigma_{M_2}\cdots\sigma_{M_r}$ where $M_1,\cdots,M_r\in\mathfrak{M}$ and $r\geqq 1$. Clearly every ideal is invariant under $G_{\mathfrak{M}}$. Our second theorem now runs as follows:

THEOREM 2. *Let* $\mathfrak{S}$ *be a semisimple subalgebra of* $\mathfrak{L}$ *such that* $\mathfrak{L}=\mathfrak{S}+\Gamma$. *Then, given any semisimple subalgebra* $\mathfrak{M}$ *of* $\mathfrak{L}$, *there exists a* $\sigma\in G_{\mathfrak{N}}$ *such that* $\mathfrak{M}\subset\sigma\mathfrak{S}$.

The following two corollaries follow immediately from this theorem.

COROLLARY 2. *Any maximal semisimple subalgebra of* $\mathfrak{L}$ *is isomorphic to* $\mathfrak{L}/\Gamma$.

COROLLARY 3. *Given any two maximal semisimple subalgebras* $\mathfrak{S}_1$, $\mathfrak{S}_2$ *of* $\mathfrak{L}$, *there exists a* $\tau\in G_{\mathfrak{N}}$ *such that* $\tau\mathfrak{S}_1=\mathfrak{S}_2$.

Corollary 3 is a sharper form of a result due to Malcev.[4]

PROOF OF THEOREM 1. First we shall prove that for any $N\in\mathfrak{N}$, $B(N, Z)=0$ for all $Z\in\mathfrak{L}$. For any $s\geqq 1$ define $\mathfrak{N}_{(s)}$ by induction as follows. $\mathfrak{N}_{(1)}=\mathfrak{N}$, $\mathfrak{N}_{(s+1)}=[\mathfrak{N}, \mathfrak{N}_{(s)}]$. Then $\mathfrak{N}_{(s)}$ is an ideal in $\mathfrak{L}$ and therefore (ad N ad $Z)\mathfrak{L}\subset\mathfrak{N}$ and (ad N ad $Z)\mathfrak{N}_{(s)}\subset\mathfrak{N}_{(s+1)}$. Hence (ad N ad $Z)^{s+1}\mathfrak{L}=\mathfrak{N}_{(s)}$. But $\mathfrak{N}$ is nilpotent and therefore $\mathfrak{N}_{(s)}=\{0\}$ for some s. Hence (ad N ad $Z)^{s+1}\mathfrak{L}=\{0\}$ or (ad N ad $Z)^{s+1}=0$. Therefore ad N ad Z is nilpotent and sp(ad N ad $Z)=B(N, Z)=0$.

Let $\mathfrak{M}$ be the set of all $X\in\mathfrak{L}$ such that $T(X, Y, Z)=0$ for all $Y, Z\in\mathfrak{L}$. Since T is invariant under all derivations of $\mathfrak{L}$, $\mathfrak{M}$ is a characteristic ideal in $\mathfrak{L}$. We have to show that $\mathfrak{M}=\Gamma$. Let $X\in\Gamma$. For any $Y\in\mathfrak{L}$, ad Y is a derivation of $\mathfrak{L}$ and Γ is invariant under ad Y. Hence ad Y induces a derivation of Γ and therefore (ad $Y)\Gamma\subset\mathfrak{N}$. So $[X, Y]=-(\text{ad } Y)X\in\mathfrak{N}$. Hence $B([X, Y], Z)=T(X, Y, Z)=0$ for all $Z\in\mathfrak{L}$. Since this is true for every Y, $X\in\mathfrak{M}$. Hence $\Gamma\subset\mathfrak{M}$. On the other hand let $\mathfrak{M}'=[\mathfrak{M}, \mathfrak{M}]$. Then it is clear that for any $M\in\mathfrak{M}'$, $B(M, Z)=0$ for all $Z\in\mathfrak{L}$. Hence by Cartan's criterion for solvability $\mathfrak{M}'$ is solvable. Hence $\mathfrak{M}$ is solvable and therefore $\mathfrak{M}\subset\Gamma$. So the theorem is proved.

Since $\mathfrak{M}$ is a characteristic ideal the same is true of Γ. Hence if D is a derivation of $\mathfrak{L}$, $D\Gamma\subset\Gamma$ and so D induces a derivation of Γ. Therefore $D\Gamma\subset\mathfrak{N}$ and so $D\mathfrak{N}\subset D\Gamma\subset\mathfrak{N}$. Hence $\mathfrak{N}$ is also a characteristic ideal.

[4] A. Malcev, C. R. Acad. Sci. URSS. vol. 36 (1942) p. 42.

PROOF OF THEOREM 2. Since $\mathfrak{S}$ is semisimple $\mathfrak{S}\cap\Gamma=\{0\}$. Hence every $P\in\mathfrak{M}$ can be written uniquely as

$$P = S(P) + \nu(P)$$

where $S(P)\in\mathfrak{S}$ and $\nu(P)\in\mathfrak{N}$. Hence

$$\begin{aligned}[P, Q] = {} & [S(P), S(Q)] + [S(P), \nu(Q)] \\ & + [\nu(P), S(Q)] + [\nu(P), \nu(Q)].\end{aligned}$$

Therefore

$$S([P, Q]) = [S(P), S(Q)],$$
$$\nu([P, Q]) = [\nu(P), S(Q)] + [S(P), \nu(Q)] + [\nu(P), \nu(Q)].$$

We have already seen that $(\mathrm{ad}\ X)\Gamma\subset\mathfrak{N}$ for any $X\in\mathfrak{L}$. Hence, $\nu([P, Q])\in\mathfrak{N}$. Let ν denote the mapping $P\to\nu(P)$. Then $\nu([\mathfrak{M}, \mathfrak{M}])\subset\mathfrak{N}$. But $\mathfrak{M}$ is semisimple. Therefore $[\mathfrak{M}, \mathfrak{M}]=\mathfrak{M}$ and $\nu(\mathfrak{M})\subset\mathfrak{N}$. Hence $\mathfrak{M}\subset\mathfrak{S}+\mathfrak{N}$.

Put $\mathfrak{L}_0=\mathfrak{S}+\mathfrak{N}$. First suppose that $\mathfrak{N}$ is abelian. The mapping $P\to S(P)$ is a homomorphic mapping of $\mathfrak{M}$ into $\mathfrak{S}$. For any $S\in\mathfrak{S}$ let D_S denote the derivation of $\mathfrak{N}$ given by $D_SN=[S, N]$ $(N\in\mathfrak{N})$. Then the mapping ρ defined by $\rho(P)=D_{S(P)}$ is a representation of $\mathfrak{M}$. Also, since $\mathfrak{N}$ is abelian

$$\begin{aligned}\nu([P, Q]) &= [S(P), \nu(Q)] - [S(Q), \nu(P)] \\ &= \rho(P)\nu(Q) - \rho(Q)\nu(P).\end{aligned}$$

Hence ν is a Whitehead mapping of $\mathfrak{M}$ into $\mathfrak{N}$ with respect to the representation ρ. Therefore by the first Whitehead lemma[5] there exists an element $-N\in\mathfrak{N}$ such that $\nu(P)=-\rho(P)N=[N, S(P)]$. Hence

$$P = S(P) + \nu(P) = S(P) + [N, S(P)] = \exp(\mathrm{ad}\ N)S(P)$$

since $(\mathrm{ad}\ N)^2=0$, $\mathfrak{N}$ being abelian. Therefore $\mathfrak{M}\subset\sigma_N\mathfrak{S}$ and so the theorem is proved in this case.

Now consider the general case. Let $n=\dim\mathfrak{N}$. If $n\leqq 1$, $\mathfrak{N}$ is abelian and so the theorem is true. Hence we can assume $n>1$ and use induction on n. Further we can assume that $\mathfrak{N}$ is not abelian so that $\mathfrak{N}'=[\mathfrak{N}, \mathfrak{N}]\neq\{0\}$. Let $X\to\overline{X}$ denote the natural homomorphism of $\mathfrak{L}_0$ onto $\overline{\mathfrak{L}}_0=\mathfrak{L}_0/\mathfrak{N}'$. The radical of $\overline{\mathfrak{L}}_0$ is $\overline{\mathfrak{N}}=\mathfrak{N}/\mathfrak{N}'$ which is abelian. Let $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{S}}$ be the images of $\mathfrak{M}$ and $\mathfrak{S}$ respectively in $\overline{\mathfrak{L}}_0$. Then they are both semisimple and

[5] See Hochschild, Amer. J. Math. vol. 64 (1942) p. 677.

$$\overline{\mathfrak{M}} \subset \overline{\mathfrak{S}} + \overline{\mathfrak{N}}.$$

Since $\overline{\mathfrak{N}}$ is abelian it follows from the above proof that there exists an $\overline{N} \in \overline{\mathfrak{N}}$ such that

$$\overline{\mathfrak{M}} \subset \sigma_{\overline{N}} \overline{\mathfrak{S}}.$$

Let $N \in \overline{N}$ $(N \in \mathfrak{N})$. The complete inverse image of $\sigma_{\overline{\mathfrak{N}}} \overline{\mathfrak{S}}$ in $\mathfrak{L}_0$ is $\mathfrak{S}_1 + \mathfrak{N}'$ where $\mathfrak{S}_1 = \sigma_N \mathfrak{S}$. Hence

$$\mathfrak{M} \subset \mathfrak{S}_1 + \mathfrak{N}'.$$

Since $\mathfrak{N}$ is nilpotent, dim $\mathfrak{N}' <$ dim $\mathfrak{N} = n$. Hence the induction hypothesis is applicable to $\mathfrak{L}_1 = \mathfrak{S}_1 + \mathfrak{N}'$ and therefore there exists a $\sigma_1 \in G_{\mathfrak{N}'}$ such that $\mathfrak{M} \subset \sigma_1 \mathfrak{S}_1$. But $G_{\mathfrak{N}'} \subset G_{\mathfrak{N}}$ and therefore $\sigma = \sigma_1 \sigma_N \in G_{\mathfrak{N}}$ and $\mathfrak{M} \subset \sigma \mathfrak{S}$.

Now let $\mathfrak{S}^*$ be any maximal semisimple algebra of $\mathfrak{L}$. It follows from the above theorem that $\mathfrak{S}^* \subset \sigma \mathfrak{S}$ for some $\sigma \in G_{\mathfrak{N}}$. Since σ is an automorphism of $\mathfrak{L}$, $\sigma \mathfrak{S}$ is semisimple. Therefore $\mathfrak{S}^* = \sigma \mathfrak{S} \cong \mathfrak{S} \cong \mathfrak{L}/\Gamma$.

If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are two maximal semisimple subalgebras of $\mathfrak{L}$, we can find $\tau_1, \tau_2 \in G_{\mathfrak{N}}$ such that $\mathfrak{S}_1 = \tau_1 \mathfrak{S}$, $\mathfrak{S}_2 = \tau_2 \mathfrak{S}$. Then $\tau \mathfrak{S}_1 = \mathfrak{S}_2$ where $\tau = \tau_2 \tau_1^{-1} \in G_{\mathfrak{N}}$.

Summary. Let $\mathfrak{L}$ be a Lie algebra over a field of characteristic zero and let Γ be its radical. It is proved that any $X \in \mathfrak{L}$ belongs to Γ if and only if $sp(\text{ad }[X, Y] \text{ ad } Z) = 0$ for all $Y, Z \in \mathfrak{L}$. Here $X \rightarrow \text{ad } X$ is the adjoint representation of $\mathfrak{L}$. Further let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{L}$ and let $\mathfrak{S}$ and $\mathfrak{S}^*$ be any two maximal semisimple subalgebras of $\mathfrak{L}$. Then $\mathfrak{S} + \mathfrak{N} = \mathfrak{S}^* + \mathfrak{N}$ and $\mathfrak{S}$ and $\mathfrak{S}^*$ are conjugate in a certain strict sense.

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