PURELY TRANSCENDENTAL SUBFIELDS OF $k(x_1, \cdots, x_n)$

A. EVYATAR AND A. ZAKS

1. Let $k_n = k[x_1, \cdots, x_n]$ be the ring of polynomials in n independent variables over a field k. Let G be an algebraic group acting regularly on k_n and let R be the corresponding subring of invariants. We call a subring R in k_n factorable in k_n if whenever an element of R factors in k_n, then all its factors lie already in R. We will discuss an example of a group G such that R is not a ring of polynomials.

Having this example in mind we consider the following problem:

(A) When is a factorable ring a ring of polynomials?

Our counterexample to an affirmative answer in general suggested the following question:

(B) When is a field of quotients of a factorable ring R a purely transcendental extension of k?

This last problem is related to similar questions discussed by Samuel [4] and Zariski [5].

We obtain the following main results:

(1) A factorable ring of transcendence degree n is a ring of polynomials.

(2) Let R be a factorable subring of k_3. Let K_3, K be the field of quotients of k_3, R respectively. Then, if k is algebraically closed, and K_3 is a separable extension of K, it follows that K is a purely transcendental extension of k.

We are indebted to Professor Max Rosenlicht for mentioning the example of §3 and for several other suggestions.

2. In this section we state some results needed later on and introduce notations and definitions.

Let k be a field of characteristic $p \geqq 0$ and $x_1, \cdots, x_n$ algebraically independent over k. We denote by R a subring of $k_n = k[x_1, \cdots, x_n]$ containing k, by K the field of quotients of R in $K_n = k(x_1, \cdots, x_n)$.

The following is known about subfields of K_n:

(1) *Theorem of Lüroth-Samuel* [4]. Every subfield K of K_n of transcendence degree one over an infinite field k is a purely transcendental extension of k.

(2) *Theorem of Castelnuovo-Zariski* [5]. Let K be a subfield of K_2 of transcendence degree two over an algebraically closed field k, such that K_2 is a separable extension of K. Then K is a purely transcendental extension of k.

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We observe that together those theorems yield the following result: Let K be a subfield of K_2 such that K_2 is a separable extension of K. If k is algebraically closed, then K is a purely transcendental extension of k.

3. Let k be a field of characteristic zero. Let $G=\{f_t\}_{t\in k}$ be the group acting on $S=k[x_1, x_2, x_3, y_1, y_2, y_3]$—the ring of polynomials in six independent variables over k—defined by

$$f_t(x_i) = x_i + ty_i \quad \text{for } i = 1, 2, 3, \qquad f_t(y_j) = y_j \quad \text{for } j = 1, 2, 3.$$

f_t extends naturally to all of S.

Let R be the ring of invariants of S for the group G. R obviously contains y_i, $i=1, 2, 3$, and

$$u_1 = x_2y_3 - x_3y_2, \qquad u_2 = x_3y_1 - x_1y_3, \qquad u_3 = x_1y_2 - x_2y_1.$$

Note that $y_1u_1+y_2u_2+y_3u_3=x_1u_1+x_2u_2+x_3u_3=0$.

Since $S[t]$—the ring of polynomials in one variable over S—is a unique factorization domain it immediately follows that R is a factorable ring. We claim

LEMMA 1. *R is generated as a k-algebra by $y_1, y_2, y_3, u_1, u_2, u_3$.*

LEMMA 2. *R is not a ring of polynomials over k.*

LEMMA 3. *The field of quotients K of R is a purely transcendental extension of k.*

PROOF OF LEMMA 1. Let $s=s(x_1, \cdots, y_3)$ be an invariant polynomial in S. In each monomial of s we may replace x_1y_2 by $u_3+x_2y_1$ whenever x_1y_2 is a factor of this monomial. Next we replace x_1y_3 by $x_3y_1-u_2$ whenever x_1y_3 is a factor. We thus obtain

$$s = s_1 = s_1(x_1, x_2, x_3, y_1, y_2, y_3, u_2, u_3)$$

in which every monomial of which x_1 is a factor is free of y_2 and y_3, viewing u_2 and u_3 as additional independent variables. Since $s_1\in R$ we have $y_1\partial s_1/\partial x_1+y_2\partial s_1/\partial x_2+y_3\partial s_1/\partial x_3=0$. Unless $\partial s_1/\partial x_1=0$ we get that every monomial of s_1 of which x_1 is a factor also contains y_2 or y_3—which is a contradiction. Therefore s_1 is free of x_1.

Now, replacing x_2y_3 by $u_1+x_3y_2$, the same argument implies that the polynomial s_2,

$$s = s_1 = s_2 = s_2(x_2, x_3, y_1, y_2, y_3, u_1, u_2, u_3)$$

is actually free of x_2 and x_3. This completes the proof.

PROOF OF LEMMA 2. Identify R naturally with the residue ring H of $T=k[y_1, y_2, y_3, z_1, z_2, z_3]$—the ring of polynomials in six variables over k—modulo the ideal J generated by $y_1z_1+y_2z_2+y_3z_3$.

Consider the prime ideal I generated by $(y_1, y_2, y_3, z_1, z_2, z_3)$ in T. The localization $H_{I'}$ of H at I'—the image of I—is isomorphic with the residue ring of T_I modulo the ideal JT_I. Since $JT_I \subset I^2T_I$, it follows that $H_{I'}$ is not a regular local ring. Since I' is a prime ideal in H this implies that H is not a ring of polynomials over k.

PROOF OF LEMMA 3. Take for instance $K = k(y_1, y_2, y_3, u_1, u_2)$; then $-u_3 = (y_1u_1 + y_2u_2)/y_3 \in K$.

REMARK. One could take the field k to be an infinite field of characteristic $p \neq 0$. The only place where one need to be careful is in the proof of Lemma 1. One still gets the same results by a slight modification of the argument.

Notice that in this case $K_6 = K(x_1)$; in particular K_6 is a separable extension of K.

The following question arises:

(C) Is K_n separably generated over the field of quotients K of a factorable ring R?

4. We now turn to factorable rings.

THEOREM I. *If R is factorable in k_n and of transcendence degree n over k, then R is all of k_n.*

To prove this property we use the following lemma, interesting in its own right.

LEMMA A. *If R is factorable in k_n, K is algebraically closed in K_n.*

PROOF. Let q/t be in K_n and algebraic over K. We can take $(q, t) = 1$. There exist elements $r_0, \cdots, r_n \in R$ such that

$$r_0 + r_1(q/t) + \cdots + r_n(q/t)^n = 0$$

or $r_0t^n + r_1qt^{n-1} + \cdots + r_nq^n = 0$. By the unique-factorization property we have q/r_0t^n and since $(q, t) = 1$ we have q/r_0.

Since R is factorable, $q \in R$.

REMARK. Similarly every element of k_n that is integral over R, lies in R.

The proof of Theorem I is now obvious.

As a consequence of Lemma A we get that R is all of k_n. In particular $K = K_n$, which yields an affirmative answer to both questions (B) and (C) for this case.

This last property can also be put this way.

If R is factorable in k_n, and tr. deg. $K/k = n$, then R is a polynomial ring.

This result, together with the example discussed in §3, seems to give the best possible answer to question (A).

5. We now turn to the study of the field of quotients of factorable rings.

LEMMA B. *If R is factorable in k_n, then the mapping $f: k_n \to k_{n-1}$, defined by the specialization $x_n \to \alpha_n$, is a k-isomorphism on R for every $\alpha_n \in k$ if and only if $x_n \notin R$.*

PROOF. Obviously if $x_n \in R$, $f(x_n - \alpha_n) = 0$.

Now the other way round: Let $s(x_1 \cdots x_n) \in R$ and $f(s) = 0$. Write s as a polynomial in $(x_n - \alpha_n)$; then

$$s(x_1, \cdots, x_n) = (x_n - \alpha_n) s_1(x_1 \cdots x_n).$$

And by the factorability of R, $x_n - \alpha_n \in R$. Therefore $x_n \in R$.

THEOREM II. *Let k_3 be a ring of polynomials in three independent variables over an algebraically closed field k, and let R be a factorable subring of k_3. If K_3 (the quotient field of k_3) is a separable extension of K (the quotient field of R) and if* tr.deg. $K/k = 2$, *then also K is a purely transcendental extension of k.*

PROOF. Since K_3 is a separable extension of K, we may assume that x_3 is a separating transcendence basis of K_3 over K. Let now u be a primitive element for K_3 over $K(x_3)$; then $K_3 = K(x_3)(u)$. Moreover, there exist elements a_{ij}, c_{ij}^m, d_{ie}^m in R, such that

$$\text{(a)} \qquad h(t) = \sum_i \left(\sum_j a_{ij} x_3^j \right) t^i$$

is a polynomial which satisfies $h(u) = 0$ and $h'(u) \neq 0$.

$$\text{(b)} \qquad x_m = \sum_i \frac{\sum_j c_{ij}^m x_3^j}{\sum_e d_{ie}^m x_3^e} u^i \qquad \text{for } m = 1, 2.$$

Let $u = g_1/g_2$, where g_1, g_2 are polynomials. In the infinite field k we can find an element α_3 such that none of the following vanish:

(i) $g_2(x_1, x_2, \alpha_3)$.

(ii) $\sum_i (\sum_j a_{ij} (x_1, x_2, \alpha_3) \alpha_3^j) i u^{i-1} (x_1, x_2, \alpha_3)$.

(iii) $\sum_e d_{ie}^m (x_1, x_2, \alpha_3) \alpha_3^e$ for $m = 1, 2$, and for every i.

This choice of α_3 assures that the map f from k_3 onto k_2 defined by specializing x_3 into α_3 has the following properties:

(1) It is a k-isomorphism on R by Lemma B. Set $f(R) = R_0$.

(2) As a consequence f naturally extends to a k-isomorphism on K into K_2. Set $f(K) = K_0$.

(3) Being well defined on u, $f(u) = u_0 = u(x_1, x_2, \alpha_3)$, f naturally extends to a map of $R[u]$ into K_2.

By $R[u]$ we denote the R-algebra generated by $(1, u)$ in K_3. If h_0 denotes the natural image under f of the polynomial h then $h_0(u_0)=0$, while $h_0'(u_0)\neq 0$, which implies that u_0 is separable over K_0.

(4) $K_2=K_0(u_0)$ since the expressions for x_1 and x_2 in $K(x_3)(u)$ are carried naturally by f into $K_0(u_0)$.

Therefore, the map f induces an isomorphism of K onto K_0 such that K_2 is a separable extension of K_0. Applying to K_0 the theorem of Castelnuovo-Zariski, K_0 then turns out to be a purely transcendental extension of k, and so does K, being k-isomorphic to K_0.

Summarizing, we obtain

THEOREM III. *Let K be the field of quotients of a factorable ring R in k_3, such that K_3 is a separable extension of K. If k is algebraically closed then K is a purely transcendental extension of k.*

PROOF. The case of tr. deg. $K/k=1$ follows from the Theorem of Lüroth-Samuel. The case of tr. deg. $K/k=2$ follows from Theorem II. The case of tr. deg. $K/k=3$ follows from Theorem I.

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TECHNION, ISRAEL INSTITUTE OF TECHNOLOGY