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ON THE STRUCTURE OF NONSTANDARD MODELS OF ARITHMETIC

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ABSTRACT. In this paper we show that the additive group of each nonstandard model $*Z$ of the integers Z is isomorphic to the group $\langle F\times Z, +\rangle$ where F is a direct sum of α-copies of the rationals Q, α the cardinality of $*Z$, and $+$ is defined by: $(a, x)+(b, y)=(a+b, x+y+g(a, b))$ for certain functions g mapping from $F\times F$ to Z.

Introduction. Let $*Z$ denote any nonstandard model of arithmetic and set

$$*Z/Z = \{\bar{a}: a \in *Z\}, \qquad \bar{a} = a + Z.$$

For each $*Z$ we let $K=K(*Z)$ denote the set of elements of $*Z$ of infinite height in $*Z$. (An element has infinite height if it is divisible by every positive integer.) It is easy to see that $\bar{a}\cap K$ never contains more than one element. Kemeny [2] asked whether there existed a $*Z$ so that $\bar{a}\cap K$ contained exactly one element for every $a \in *Z$. The importance of Kemeny's question lies in the fact that if such a $*Z$ existed, Goldbach's conjecture could be proven false; however, it was shown by Gandy [1] and Mendelson [3] that this question is answered in the negative.

The purpose of this paper is to analyze, in more detail, the structure of the models $*Z$, based on ideas suggested by Kemeny's and Mendelson's work and by the work of MacDowell and Specker [6]. The basic results about nonstandard models used in this paper can be found in Robinson [5]. We begin with Mendelson's solution to Kemeny's problem.

2. The additive group of $*Z$.

THEOREM 1 (MENDELSON). *Any nonzero ring homomorphism of $*Z$ into itself is an order-preserving isomorphism.*

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Now suppose $K\cap\bar{a}$ consists of exactly one element for each a in some $^*Z$. Then, clearly, the additive group of $^*Z$ may be written as $K\oplus Z$. Hence, the map $f(x+n)=n$, where $x\in K$ and $n\in Z$, is not an order-preserving isomorphism, in contradiction to Theorem 1.

That the set K plays an important role in determining the structure of $^*Z$ is seen in the work of MacDowell and Specker. There, the additive group of $^*Z$ is shown to be isomorphic to $K\oplus J$ where J is some subgroup of the product group $\prod_{p=1}^{\infty} Z_p$, Z_p being the integers mod p.

We now present a canonical form (Phillips [4]) for the additive group of each $^*Z$ in which, essentially, only the addition varies from model to model. Given $^*Z$, let $h: {}^*Z/Z \to {}^*Z$ be a choice function; that is, $h(\bar{a})\in\bar{a}$. We "normalize" h so that $h(\bar{0})=0$. Then the map α_h given by

$$a \to (\bar{a},\ a - h(\bar{a}))$$

is a one-to-one map of $^*Z$ onto $^*Z/Z\times Z$.

THEOREM 2. *Let $^*Z/ZhZ$ denote the group whose elements are $^*Z/Z\times Z$ and whose group operation is defined to be*

$$\text{(i)}\qquad (\bar{a}, x) + (\bar{b}, y) = (\overline{a+b},\ x + y + h(\bar{a}) + h(\bar{b}) - h(\overline{a+b})).$$

Then α_h is an additive isomorphism of $^*Z$ onto $^*Z/ZhZ$.

PROOF. We have

$$a + b \to (\overline{a+b},\ a + b - h(\overline{a+b}))$$

and hence we may define addition as

$$(\bar{a}, a - h(\bar{a})) + (\bar{b}, b - h(\bar{b})) = (\overline{a+b},\ a+b - h(\overline{a+b})).$$

It is easy to see that this addition is equivalent to that defined by (i) and the theorem follows.

DEFINITION 1. Let F be some direct sum of infinitely many copies of the rationals Q. By F_0 we mean the set of all functions $g: F\times F\to Z$ such that

1. $g(a, 0)=0$,
2. $g(a, b)=g(b, a)$,
3. $g(a, b)+g(a+b, c)=g(a, b+c)+g(b, c)$.

If $g\in F_0$, we let FgZ denote the set $F\times Z$ together with the additive operation defined as

$$(a, x) + (b, y) = (a + b,\ x + y + g(a, b)).$$

The next theorem is immediate.

THEOREM 3. *For each* $g \in F_0$, FgZ *is an abelian group.*

THEOREM 4. *For each* $^*Z$ *of cardinal* α, $^*Z/Z$ *is isomorphic to a direct sum of* α *copies of the rationals* Q, *which in turn is isomorphic to* $K(^*Z)$.

PROOF. See MacDowell-Specker [6].

THEOREM 5. *For each* $^*Z$ *of cardinal* α *there exists a function* $g \in F_0$, *where* F *is a direct sum of* α *copies of* Q, *so that the additive group of* $^*Z$ *is isomorphic to* FgZ; *the isomorphism mapping* $n \in Z$ *onto* $(0, n)$ *in* FgZ.

PROOF. Let $\pi: F \to {}^*Z/Z$ be the isomorphism given by Theorem 4 and let

$$g(a, b) = h(\pi(a)) + h(\pi(b)) - h(\pi(a) + \pi(b)),$$

where $h: {}^*Z/Z \to {}^*Z$ is a choice function, $h(\bar{0}) = 0$. Then FgZ is isomorphic to $^*Z/ZhZ$ by the map $(a, x) \to (\pi(a), x)$ and now the theorem follows from Theorem 2.

3. **The set** F_1. Let F be an infinite direct sum of α copies of Q and let $F_1 = F_1(\alpha)$ denote the set of all maps $g \in F_0$ such that for some nonstandard model $^*Z$ of cardinal α, the additive group of $^*Z$ is isomorphic to FgZ as in Theorem 5.

THEOREM 6. $g \equiv 0$ *is not in* F_1.

PROOF. Follows from Theorem 1. Hence, there are no linear choice functions from $^*Z/Z \to {}^*Z$.

Since $^*Z/Z$ and F are isomorphic, we now agree to identify corresponding elements in both structures.

THEOREM 7. *There exists a* $g \in F_1$ *such that* $g(a, b) = 0$ *for each* a, $b \in F$ *where* $a \cap K \neq \emptyset$ *and* $b \cap K \neq \emptyset$. *Hence* $K \oplus Z$ *is an additive subgroup of* $^*Z$.

PROOF. If $x \in a \cap K$, set $h(a) = x$; define the choice function h arbitrarily for the remaining $a \in F$. Then h is linear on the set of a where $a \cap K \neq \emptyset$.

Let $a = (a_1, a_2, \cdots)$ and $b = (b_1, b_2, \cdots)$. We set $e(a, b)$ equal to the finite cardinal of the set

$$E = (a, b) = \{i: \text{neither } a_i \text{ nor } b_i \text{ are integers or } a_i \neq -b_i\}.$$

THEOREM 8. *There exists a* $g \in F_1$ *such that* $0 \leqq |g(a, b)| \leqq e(a, b)$ *for all* $a, b \in F$.

PROOF. Let $A=\{x_1, \cdots, x_i, \cdots\}$ be a basis for $^*Z/Z$ over Q and we identify $r_ix_i+\cdots+r_jx_j$ in $^*Z/Z$ with $a\in F$ where $a_p=0$ if $p<i$ or $p>j$ and $a_p=r_p$ if $i\leqq p\leqq j$. We define $h(x_i)\in x_i$ arbitrarily and if $a=a_ix_i+\cdots+a_jx_j$ we set

$$h(a) = \sum_{p=i}^{j} [a_p h(x_p)]$$

where $[x]$ denotes the greatest integer function if $x\geqq 0$ and $[x] = -[-x]$ if $x<0$ ($[x]$ can be defined by first embedding $^*Z$ in $^*Q$). h can be shown to be a choice function on $^*Z/Z\to{}^*Z$ and hence we set

$$g(a, b) = h(a) + h(b) - h(a + b).$$

Thus

$$g(a, b) = \sum_{i\in E(a,b)} ([a_ih(x_i)] + [b_ih(x_i)] - [(a_i + b_i)h(x_i)]),$$

using familiar properties of $[x]$. The theorem now follows from the fact that for all x and y,

$$[x] + [y] - [x + y] = \begin{cases} 1, \\ 0, \\ -1. \end{cases}$$

One might ask if instead of using choice functions from $F\to{}^*Z$, could we define g with simpler functions. The next theorem gives a partial answer to this question:

THEOREM 9. *If* $h\colon F\to Z$ *and* $g(a, b)=h(a)+h(b)-h(a+b)$, *then* $g\in F_0-F_1$.

PROOF. It is not hard to show that in FgZ we have

$$(0, n)\cdot(a/n, x_n) = (a, nx_n + nh(a/n) - h(a)).$$

Hence, for each n let $x_n=-h(a/n)$. Then

$$(0, n)\cdot(a/n, x_n) = (a, -h(a))$$

for each n and each a. Thus each element $(a, -h(a))$ has infinite height in FgZ which implies $K\cap\bar{a}\neq\emptyset$ for each $a\in{}^*Z$ if the additive group of $^*Z$ were isomorphic to FgZ.

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