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TOPOLOGICALLY COMPLETE GROUPS

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ABSTRACT. Topologically complete groups are characterized by the existence of a compact subgroup such that the coset space is topologically complete and metrizable. Coset spaces of topologically complete groups and extensions of one topologically complete group by another are again topologically complete. The open mapping theorem is valid for topologically complete groups.

A space is *topologically complete* (in the sense of Čech [3]) if it is homeomorphic to a dense G_δ in a compact Hausdorff space. A metrizable space is topologically complete if and only if it has a complete metric [3]. In general, it is known that a Hausdorff space X is paracompact and topologically complete if and only if it has a proper mapping (i.e., a closed mapping with compact fibers) onto a complete metric space [5]. Another way to put this is to say there is a continuous pseudo-metric d such that every d-Cauchy net has a subnet which converges in X. We will call such a *d complete*. A topologically complete space is a G_δ in any Hausdorff space in which it is densely imbedded ([3], [4]).

For topological groups, the situation is simpler. If G is even locally topologically complete, then it is necessarily paracompact and topologically complete, and has a compact subgroup K such that G/K is metrizable and topologically complete. K cannot necessarily be taken normal, even if G is locally compact; but even so, topological completeness appears to be a useful concept in the theory of topological groups. Extensions of topologically complete groups by topologically complete groups are topologically complete, and the open mapping theorem is valid for topologically complete groups.

THEOREM 1. *If the topological group G is locally topologically complete, then G is paracompact and topologically complete. This is so if and only if G has a compact subgroup K such that $G|K$ is metrizable and topologically complete.*

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PROOF. Let U be a topologically complete open subset containing the identity e. Let $\{\mathscr{C}_n\}$ be a complete sequence of open covers of U [4]. Let U_n be an element of $\mathscr{C}_n$ containing e. Then since $\{\mathscr{C}_n\}$ is complete, any net which is eventually in each U_n has a cluster point. Now let $\{V_n\}$ be a sequence of symmetric neighborhoods of e such that $(V_{n+1})^2 \subset V_n \subset U_n$. Let $K = \bigcap_{n=1}^{\infty} V_n$. Clearly, K is a closed subgroup. Since $\{V_n\}$ has the same property as $\{U_n\}$, it is immediate that K is compact. It is also easy to see that $\{V_n\}$ is a basis of neighborhoods of K: Suppose W were an open neighborhood of K, and $V_n \not\subset W$ for any n. For each n, choose $x_n \in V_n - W$. Then $\{x_n\}$ has a cluster point $x \in K - W$, a contradiction.

Thus G/K satisfies the first axiom of countability. Since K is compact, this implies that G/K is metrizable: Let us call a neighborhood V of e *special* if $kVk^{-1} = V$ for all $k \in K$. Since K is compact, G has a special neighborhood base at e. For V special, define $\tilde{V} = \{(xK, yK) \in G/K \times G/K : y \in xVK\}$. Since $xVK = xKV$, this definition makes sense, and the $\tilde{V}$'s define a uniformity consistent with the topology of G/K. Since $\tilde{V}$ depends only on VK, this uniformity is countably generated, and G/K is metrizable.

Now since G/K is an open image of G, it follows (essentially from [6], cf. [10]) that G/K is locally topologically complete. Then (again from [6]) it follows that G/K is topologically complete. Since K is compact, the map from G to G/K is proper, and G is topologically complete. Q.E.D.

COROLLARY 1. *If G is topologically complete and E is a G_δ subset containing e, then K can be taken so that $K \subset E$.*

REMARKS. 1. K cannot necessarily be taken normal. For example, G could be a nonmetrizable, locally compact, totally disconnected, topologically simple group (such exist); or G could be the semidirect product of an uncountable torus with its full automorphism group (the latter with its discrete topology). In these two locally compact examples, of course, G has an open subgroup G_1 such that K can be taken normal in G_1. If we drop local compactness, we can obtain connected examples: Let K be a nonmetrizable, compact, connected group and G the semidirect product of K and $C(K)$, where K acts on $C(K)$ by translation.

There are, however, conditions under which K can be taken normal. We call G a *PM-group* if, for any sequence $\{V_n\}$ of neighborhoods of e, there is a sequence $\{V'_n\}$ of neighborhoods of e such that $V'_n \subset V_n$ and $\{V'_n\}$ generates a group topology. (Since it is easy to take V'_n symmetric and $(V'_{n+1})^2 \subset V'_n$, the key requirement is that, for any $x \in G$ and any n, $xV'_m x^{-1} \subset V'_n$ for m large enough.) If G is a *PM*-group, then K can be taken normal. In [2], we will show that G is a *PM*-group if G modulo its center is weakly separable. A group is *weakly separable* if it is the union of countably many left translates of each neighborhood of the identity.

2. It is easy to find a metric for G/K which corresponds to the uniformity introduced above. Let d be a continuous left-invariant pseudo-metric on G which is right invariant under K, and such that each V_n is a d-neighborhood of 1 (such exist since K is compact). Then define $\bar{d}(xK, yK)=\inf\{d(xk_1, yk_2):k_1, k_2 \in K\}$. $\bar{d}$ is the desired metric on G/K, and G is represented as a group of isometries of $(G/K, \bar{d})$. The kernel, C, is the largest compact normal subgroup contained in K, and the topology of G/C agrees with the isometry-group topology (i.e., the pointwise topology on G/K).

It is natural to ask when G/K is complete with respect to $\bar{d}$. In other words, if $\{x_\alpha\}$ is a net in G such that $x_\alpha^{-1}x_\beta$ is eventually in each V_n, when can we say that $\{x_\alpha\}$ has a cluster point? For this, we need a lemma.

LEMMA 1. *Let K be a compact subgroup of G such that G/K is metrizable, and let $\{V_n\}$ be a basis of neighborhoods of K. If $\{x_\alpha\}$ is a universal net in G (i.e., a net corresponding to a maximal filter) such that $x_\alpha^{-1}x_\beta$ is eventually in each V_n, then $\{x_\alpha\}$ is left Cauchy. A similar result holds for right Cauchyness.*

PROOF. For any special neighborhood V of e, there is $x \in G$ such that x_α is eventually in xVK. Since K is compact, we can write $K \subset \bigcup_{i=1}^{j} Vk_i$ for some k_i's in K. Since $\{x_\alpha\}$ is universal, there is an i such that x_α is eventually in $xV \cdot Vk_i = xV^2k_i = xk_iV^2$. Thus $x_\alpha^{-1}x_\beta$ is eventually in V^4. Since V^4 can be arbitrarily small, $\{x_\alpha\}$ is left Cauchy.

The answer to the above question is now clear.

COROLLARY 2. *If G has a compact subgroup K such that G/K is metrizable, then $\bar{G}$, the completion of G in the two-sided uniformity, is topologically complete.*

PROOF. $\bar{G}/K$ is also metrizable: If $\{V_n\}$ is an open neighborhood base of K in G and $V_n = G \cap V'_n$ for V'_n open in $\bar{G}$, then $V'_n \subset \bar{V}_n$. If W is a neighborhood of K in $\bar{G}$, then for some n, $W \supset V_n$. Hence $\bar{W} \supset V'_n$. Since K is compact and $\bar{G}$ regular, K has arbitrarily small closed neighborhoods. Hence $\{V'_n\}$ is a neighborhood base for K in $\bar{G}$.

Now if $\{x_\alpha\}$ is a net in $\bar{G}$ such that, for each n, $x_\alpha^{-1}x_\beta$ and $x_\alpha x_\beta^{-1}$ are eventually in V'_n, then, by Lemma 1, any universal subnet is two-sided Cauchy and hence converges. From this, it is easy to construct either a complete sequence of open covers or a complete pseudo-metric for $\bar{G}$. Q.E.D.

COROLLARY 3. *If G has a compact subgroup K such that G/K is metrizable, then G is topologically complete if and only if G is complete in its two-sided uniformity.*

PROOF. All that remains is to show that topological completeness implies two-sided completeness. The proof of this is the same as in the metrizable case: G is a dense G_δ in $\bar{G}$, which is second category. Hence $G=G\cdot G^{-1}=\bar{G}$ (see [**8**, Exercises P, Q, pp. 211–212]).

COROLLARY 4. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii), and (i)$'\Leftrightarrow$(ii)$'\Leftrightarrow$(iii)$'$.

(i) *G is topologically complete and left complete.*

(ii) *G has a left-invariant, complete pseudo-metric.*

(iii) *There is a sequence $\{V_n\}$ of neighborhoods of e such that any net $\{x_\alpha\}$ for which $x_\alpha^{-1}x_\beta$ is eventually in each V_n has a cluster point.*

(i)$'$ *G is topologically complete.*

(ii)$'$ *There is a left-invariant pseudo-metric d such that, if $d'(x,y)=d(x,y)+d(x^{-1},y^{-1})$, then d' is a complete pseudo-metric for G.*

(iii)$'$ *There is a sequence $\{V_n\}$ of neighborhoods of e such that any net $\{x_\alpha\}$ for which $x_\alpha^{-1}x_\beta$ and $x_\alpha x_\beta^{-1}$ are eventually in each V_n has a cluster point.*

Topological completeness in the sense of Čech adds nothing new to the theory of topological vector spaces. In fact:

COROLLARY 5. *If G is a topologically complete topological vector space over the reals, or over a local field k, then G is metrizable.*

PROOF. In the real case, this is clear since G has no compact subgroups. In general, since k is weakly separable, we can choose the V_n's so that they define a vector space topology (cf. [**2**]). K will then be a subspace. Since k is not compact and any one dimensional subspace of K is closed and homeomorphic to k, K must be trivial.

THEOREM 2. *If H is a closed subgroup of G, and either*: (i) *G is topologically complete, or* (ii) *G has a compact subgroup K such that G/K is metrizable and G/H is locally topologically complete, then G/H is paracompact and topologically complete.*

Note. If G is metrizable, this follows directly from [**6**]. This case was erroneously stated to be an open question in [**1**].

PROOF. In either case, K exists as in (ii). Let D be the double coset space $K\backslash G/H$, with the quotient topology. It follows as in the proof of Theorem 1 that D is metrizable: For V a special neighborhood of e, define $\tilde{V}=\{(KxH, KyH)\in D\times D : y\in KVxH\}$. The $\tilde{V}$'s define a metrizable uniformity on D. Since D is an open image of both G and G/H, it follows from [**6**] that in either case, D is topologically complete. Since the mapping from G/H to D is proper, G/H is paracompact and topologically complete.

COROLLARY 1. *If G is topologically complete and H is a closed G_δ subgroup, then G/H is metrizable.*

Proof. By Corollary 1.1, we can find $K \subset H$ such that G/K is metrizable. Then G/H is a paracompact, open image of G/K and hence is metrizable by Corollary 1.2(c) of [**9**].

Lemma 2. *Let H be a closed subgroup of G such that H is topologically complete and G/H locally topologically complete. If either*

(1) *H is normal,*

(2) *H is left-complete, or*

(3) *G is a PM-group,*

then G has a compact subgroup K such that G/K is metrizable.

Proof. It is sufficient to find a sequence $\{U_n\}$ of neighborhoods of e in G such that any net which is eventually in each U_n has a cluster point.

1. Let $\pi: G \to G/H$ be the projection. Since G/H is locally topologically complete, the proof of Theorem 1 shows that there is a sequence $\{W_n\}$ of neighborhoods of $\pi(e)$ in G/H with the desired property. Let $\{V_n\}$ be a sequence of neighborhoods of e in H, as in Lemma 1. Let $\{U_n\}$ be a sequence of open symmetric neighborhoods of e in G such that $(U_{n+1})^2 \subset U_n$, $\pi(U_n) \subset W_n$, and $U_n \cap H \subset V_n$. In case (3), we may assume also that, for any $x \in G$ and any n, $xU_m x^{-1} \subset U_n$ for m sufficiently large.

2. Let $\{x_\alpha\}$ be a net eventually in each U_n. Then, replacing $\{x_\alpha\}$ with a subnet, we may assume $\pi(x_\alpha)$ converges to $\pi(y)$ in G/H. Again replacing $\{x_\alpha\}$ with a subnet, we obtain $x_\alpha = y_\alpha h_\alpha$, where $y_\alpha \to y$ and $h_\alpha \in H$. It is sufficient to show that $\{h_\alpha\}$ has a cluster point.

3. We show that y can be taken in $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \mathrm{Cl}(U_n)$. (It is clearly permissible to replace y by any element of yH.) For each n, $y \in \mathrm{Cl}(U_{n+1}H) \subset (U_{n+1})^2 \cdot H \subset U_n \cdot H$. Therefore, we may write $yh_n = y_n$, with $y_n \in U_n$, $h_n \in H$. We must show that $\{h_n\}$ has a cluster point. For m, $n \geqq N+1$, $h_n^{-1}h_m = y_n^{-1}y_m \in (U_{N+1})^2 \cap H \subset V_N$. Therefore, in case (2), $\{h_n\}$ has a cluster point. Also $yh_n h_m^{-1} y^{-1} = y_n y_m^{-1} \in U_n \cdot U_m$. In case (1), for n, $m \geqq N+1$, $yh_n y^{-1} \cdot (yh_m y^{-1})^{-1} \in (U_{N+1})^2 \cap H \subset V_N$. Thus by Lemma 1, if $\{h_\beta\}$ is any universal subset of $\{h_n\}$, $\{h_\beta\}$ is left Cauchy and $\{yh_\beta y^{-1}\}$ is right Cauchy. It follows that $\{h_\beta\}$ is right Cauchy, hence two-sided Cauchy, hence convergent. In case (3), for any N and for n, m large enough, $h_n h_m^{-1} \in (y^{-1}U_n U_m y) \cap H \subset U_N \cap H \subset V_N$. Again, Lemma 1 implies that $\{h_n\}$ has a cluster point.

4. Now for any n, since $y \in U_{n+1}$, y_α is eventually in U_{n+1}. Since x_α is also eventually in U_{n+1}, h_α eventually is in $(U_{n+1})^2 \cap H \subset V_n$. Therefore $\{h_\alpha\}$ has a cluster point. Q.E.D.

Lemma 3. *Let H be a closed subgroup of G such that H is topologically complete and G/H locally topologically complete. If G has a compact subgroup K such that G/K is metrizable, then G is topologically complete.*

PROOF. By Corollary 1.3, it is enough to show $G=\bar{G}$. By Theorem 2, G/H is actually topologically complete. Now H is complete in its two-sided uniformity and hence closed in $\bar{G}$. Therefore, G/H has the same topology as a quotient of G and as a subspace of $\bar{G}/H$. Thus G/H is a dense G_δ in $\bar{G}/H$, and hence G is a dense G_δ in $\bar{G}$. As in the proof of Corollary 1.3, it now follows that $G=\bar{G}$.

THEOREM 3. *Let H be a closed subgroup of G such that H is topologically complete and G/H locally topologically complete. If either* (1), (2), *or* (3) *of Lemma* 2 *is true, or*

(4) *there is a continuous local section from G/H to G, then G is topologically complete.*

PROOF. All that remains is (4), the case where G is a bundle over G/H. Since the product of two topologically complete spaces is topologically complete, (4) implies that G is locally and hence globally topologically complete.

We now turn to the open mapping theorem. If $\pi: G\to H$ is a continuous homomorphism, π is called *almost open* if $\mathrm{Cl}(\pi(U))$ is a neighborhood of the identity, e, in H for each neighborhood U of the identity, e, in G. By following the general plan of [**1**], we can prove:

THEOREM 4. *If $\pi: G\to H$ is a continuous, almost open homomorphism, and G is topologically complete, then π is open.*

LEMMA 4. *Theorem* 4 *is true if G is weakly separable.*

PROOF. We may assume $H=\mathrm{Cl}(\pi(G))$. If K is a compact normal subgroup of G such that G/K is metrizable, then $\pi(K)$ is normal in H. π induces an almost open homomorphism of G/K to $H/\pi(K)$, which is open by the separable metric case of the open mapping theorem (Corollary 3.2 of [**11**], cf. the second Remark in [**1**]). It follows that π is open. (We could, for example, assume without loss of generality that π is 1-1. Then π induces a homeomorphism of K onto $\pi(K)$ and a homeomorphism of G/K onto $H/\pi(K)$. Hence π is a homeomorphism of G onto H (cf. [**2**]).)

LEMMA 5. *Let G and H be as in Theorem* 4, *$\pi: G\to H$ a continuous homomorphism, K a compact subgroup of G such that G/K is metrizable, and $\{U_n\}$ a basis of neighborhoods of K. Then π is almost open if each* $\mathrm{Cl}(\pi(U_n))$ *is a neighborhood of e.*

PROOF. It is sufficient to show that, if V is a neighborhood of e in G and $\{x_\alpha\}$ a net convergent to e in H, then some subnet of $\{x_\alpha\}$ is eventually in $\mathrm{Cl}(\pi(V))$. By hypothesis, for each neighborhood W of e in G, x_α is eventually in $\mathrm{Cl}(\pi(W\circ K))=\mathrm{Cl}(\pi(W))\circ\pi(K)$. Hence, passing to a subnet, we may write $x_\alpha = y_\alpha \pi(k_\alpha)$, where y_α is eventually in each $\mathrm{Cl}(\pi(W))$ and $k_\alpha \in K$. Again passing to a subnet, we may assume $k_\alpha \to k$. Since $x_\alpha \to e$ and $y_\alpha \to e$, clearly $\pi(k) = e$. Choose W_0 such that $W_0^2 \subset V$. Then $k_\alpha \in W_0 \circ k$, eventually, and hence $\pi(k_\alpha) \in \pi(W_0)$. Since $y_\alpha \in \mathrm{Cl}(\pi(W_0))$ eventually, it is clear that $x_\alpha \in \mathrm{Cl}(\pi(W_0)) \circ \pi(W_0) \subset \mathrm{Cl}(\pi(V))$ eventually.

Now with the same notations and assumptions, $H/\pi(K)$ is metrizable. In fact, $\mathrm{Cl}(\pi(U_n))$ is a neighborhood base for $\pi(K)$. We will say temporarily that G_0 is an *s-subgroup* of G if G_0 is the closed subgroup generated by K and countably many elements of G; similarly, H_0 is an s-subgroup of H if it is the closed subgroup generated by $\pi(K)$ and countably many elements of H. The s-subgroups actually exhaust the class of weakly separable closed subgroups containing K (respectively $\pi(K)$), but all we will need is the fact that they are weakly separable (cf. [**2**]).

LEMMA 6. *With the above notations and assumptions, if $H = \mathrm{Cl}(\pi(G))$, then for any s-subgroup H_0 of H, there is an s-subgroup G_0 of G such that π gives an open map of G_0 onto H_0.*

PROOF. We may assume $U_n = U_n \circ K$. Let V_n be the interior of $\mathrm{Cl}(\pi(U_n))$, so that $V_n = V_n \circ \pi(K)$. Let $p: H \to H/\pi(K)$ be the projection. Then since $H/\pi(K)$ is metrizable and $p(H_0)$ is separable, we may choose a large enough s-subgroup G_1 of G so that $\mathrm{Cl}(\pi(G_1)) \supset G_0$, and $\mathrm{Cl}(\pi(U_n \cap G_1)) \supset V_n \cap H_0$. Let $H_1 = \mathrm{Cl}(\pi(G_1))$. Just as in the proof of Lemma 2 of [**1**], we iterate this process and then apply the separable case (Lemma 4).

PROOF OF THEOREM 4. We may assume that $H = \mathrm{Cl}(\pi(G))$ and (for simplicity) that π is 1-1. Then since π gives a homeomorphism of K onto $\pi(K)$, it is sufficient to show that π gives a homeomorphism of G/K onto its image in $H/\pi(K)$. This will imply that π is a homeomorphism onto $\pi(G)$ and, since G is complete in its two-sided uniformity, that $\pi(G)$ is closed. Since $H/\pi(K)$ is metrizable, we need only consider sequential convergence in $H/\pi(K)$, and Lemma 6 suffices to complete the proof.

Once Theorem 4 has been proved as stated, results in Chapter V of [**7**] apply. These allow the hypothesis of both the open mapping and closed graph theorems for topologically complete groups to be relaxed a little bit.

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