A TOPOLOGICAL PROOF OF A THEOREM OF KNESER

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ABSTRACT. We give an elementary topological proof that the orthogonal groups $O(2)$ and $O(3)$ are strong deformation retracts of the space of homeomorphisms (with the compact-open topology) of R^2 and S^2, respectively. We also deform the space of bounded homeomorphisms of R^2 to S^1.

A well-known theorem of Kneser [6] says that the orthogonal groups $O(2)$ and $O(3)$ are strong deformation retracts of the space of homeomorphisms (with the compact-open topology) of R^2 and S^2, respectively. His proof, in the setting of complex analytic function theory, is unelementary and untopological and therefore not accessible to many topologists.

The main purpose of this paper is to give an elementary topological proof of his theorem. Thus we prove:

THEOREM 1. *The rotation group* $SO(2)$ *is a strong deformation retract of the space* $H^+(R^2)$ *of orientation preserving homeomorphisms of the plane.*

COROLLARY 2. $O(2)$ *is a strong deformation retract of* $H(R^2)$.

COROLLARY 3. $O(3)$ *is a strong deformation retract of* $H(S^2)$.

We also obtain the following:

COROLLARY 4. *The space* $BH(R^2)$ *of bounded homeomorphisms of the plane deforms to* S^1.

The methods used in this paper are also desirable as they function well in higher dimensions.

1. **Definitions and preliminaries.** Let R^2, S^1, and S^2 denote Euclidean 2 space, the unit spheres in Euclidean 2 and 3 space, respectively. Let B_r denote the ball of radius r in R^2. $H(X)$ will denote the space of homeomorphisms of X, with the compact-open topology. $H^+(X)$ will denote

those homeomorphisms which preserve orientation. A basis for the neighborhoods of the identity$=$id consists of sets of the form $N(C, \varepsilon)=\{h| \; |h(x)-x|<\varepsilon \text{ all } x \in C\}$, where C is compact and $|\cdot|$ denotes the usual norm. In this paper X will be a subset of R^2 or S^2, and we will usually just write H in place of $H(X)$ when it is clear what X we are regarding. By [1], [2], H will then be a topological transformation group on X and compositions, inverses, and evaluations will be continuous. If C is a compact subset of R^2, $h \in H(R^2)$, and $x \in h(C)$, then by $\text{rad}_x\{h(C)\}$ ($\text{Rad}^x\{h(C)\}$) we will mean the radius of the largest (smallest) ball centered at x which is contained in (contains) $h(C)$. If $x=$origin, it is usually omitted.

An isotopy h_t of $h \in H$ is a path in H starting at h. We say an isotopy h_t for each $h \in G \subset H$ is *canonical* if the function from $G \times I$ into H defined by $(h, t) \rightarrow h_t$ is continuous, i.e., defines a deformation of G in H.

If $0 \leq t \leq 1$ and $a<b$, $c<d$, by $T_t(a; b; c; d)$ we mean the homeomorphism of R^2 which is fixed on B_a and off B_d and takes a ray emanating from the origin onto itself as follows. T_t is fixed on $[0, a]$ and on $[d, \infty)$, and takes b to $(1-t)b+tc$ and is linear on $[a, b]$ and $[b, d]$. $T_t(a; b; c; d)$ is easily seen to be continuous viewed as a function from a subset of R^5 into $H(R^2)$.

2. Radial squeezing.

The following lemma is well known. Our version varies slightly in that the orthogonal group is to remain invariant. For ease of notation we regard R^2 as $[-\infty, \infty) \times S^1$ with $\{-\infty\} \times S^1$ squeezed to the origin. Then by the ball B_r of radius r, we mean the set $[-\infty, r] \times S^1$.

LEMMA 5.   *Let $h \in H(R^2)$ fix the origin. Then h is canonically isotopic to h_∞, with $B_{j-1} \subset h_\infty(B_j) \subset B_j$ for all $j \in J$. If $h \in O(2)$, then $h_t=h$ all $t \in [0, \infty]$.*

PROOF.   By squeezing we can assume that $B_{-1} \subset h(B_0) \subset B_0$. Inductively assume that h has been canonically isotoped to h_n, with $B_{j-1} \subset h_n(B_j) \subset B_j$ for all $|j| \leq n$.

Set

$$a_n = \text{rad } h_n^{-1}(B_{n-1}),$$

$$b_n = \min [n + 1; \text{rad } \{h_n^{-1}(B_{n+1})\}],$$

$$c_n = \min [n + 1; \text{rad } \{h_n(B_{n+1})\}],$$

$$\Phi_n = h_n \circ T_1(a_n - 1; n; a_n; b_n) \circ h_n^{-1},$$

$$h_{n+t} = \Phi_n^{-1} \circ T_{4t}(n - 1; c_n; n + 1; n + 2) \circ \Phi_n \circ h_n,$$

$$h_{n+1/4+t} = T_{4t}(n; d_n; n + 1; d_n + 1) \circ h_{n+1/4},$$

where

$$d_n = \text{Rad } h_{n+1/4}(B_{n+1}) \quad \text{for } 0 \leq t \leq \tfrac{1}{4}.$$

Thus the radial structure of h_n gives us the homeomorphism Φ_n which puts $h_n(B_n)$ inside B_{n-1}, and is fixed off B_{n+1}. We then use the radial structure given by Φ_n to slide $h_n(B_{n+1})$ outside of B_{n+1}, leaving $h_n(B_n)$ alone. A final squeeze then puts $h_n(B_{n+1})$ where it belongs. We next perform the analogous isotopy to $h_{n+1/2}$ near B_{-n} to get h_{n+1}, with

$$B_{j-1} \subset h_{n+1}(B_j) \subset B_j \quad \text{for } |j| \leqq n + 1.$$

One easily checks that $\text{limit}_{n\to\infty} h_n = h_\infty$ exists, and has the desired properties. That the isotopy is canonical follows from the continuity of the a_n, b_n, c_n, d_n.

3. **Bounded homeomorphisms.** We say that h is bounded by M if $|h(x)-x| \leqq M$ for all $x \in R^2$. We denote by $^M BH(R^2)$ the collection of all homeomorphisms bounded by M, and by $BH(R^2) = \bigcup_M {}^M BH(R^2)$, the group of all bounded homeomorphisms. If h is bounded, the Alexander isotopy gives an isotopy of h to the identity; namely

$$\begin{aligned} h_t &= \phi_t \circ h \circ \phi_t^{-1}, && 0 \leqq t < 1, \\ &= \text{id}, && t = 1, \end{aligned}$$

where $\phi_t \in H(R^2)$ is defined by $\phi_t(x) = (1-t)x$. However the Alexander isotopy is not canonical. We do have the following:

LEMMA 6. *The Alexander isotopy is canonical when restricted to* $^M BH(R^2)$.

PROOF. Given h and $t=1$ with $N(B_k, \varepsilon)$ a neighborhood of $h_1 = \text{id}$. Choose $t_0 > 1 - \varepsilon/M$. Then for $t > t_0$ and any $h \in {}^M BH(R^2)$, we have that h_t is bounded by ε, i.e., $h_t \in N(B_k, \varepsilon)$.

LEMMA 7. *Let* $h \in H^+(R^2)$ *be bounded by* M *on the first factor, i.e.,* $|\Pi_1 h(x, y) - x| \leqq M$ *for all* $(x, y) \in R^2$. *Then* h *is canonically isotopic to the identity.* (*Here* Π_i, $i=1, 2$, *means projection on the ith factor.*)

PROOF. Given h, we define a sequence of functions $f_j : R \to R$, $j \in J$, such that

(i) $f_j < f_{j+1}$,

(ii) $G_{2j-1} < h(G_{2j}) < G_{2j+1}$

where G_j is the graph of f_j, i.e., the set $\{(x, f_j(x)) | x \in R\}$ and "$<$" in (ii) means lies below in the usual sense.

Set

$f_0 = 0,$

$f_1(x_0) = \max[1; \sup\{\Pi_2 \circ h(x, 0) \mid |x - x_0| \leqq 2M\}],$

$f_{-1}(x_0) = \min[-1; \inf\{\Pi_2 \circ h(x, 0) \mid |x - x_0| \leqq 2M\}],$

$f_2(x_0) = \max[1 + f_1(x_0); \sup\{\Pi_2 \circ h^{-1}(x, f_1(x)) \mid |x - x_0| \leqq 2M\}],$

$f_{-2}(x_0) = \min[-1 + f_{-1}(x_0); \inf\{\Pi_2 \circ h^{-1}(x, f_{-1}(x)) \mid |x - x_0| \leqq 2M\}].$

For $j \geqq 1$, f_{j+2} ($f_{-(j+2)}$) is defined inductively with f_{j+1} ($f_{-(j+1)}$) in place of f_{j-1} ($f_{-(j-1)}$) in the formula for f_j (f_{-j}).

Define $\Phi_1(h) \in H^+(R^2)$ to be the vertical homeomorphism, which on each fiber $\{x\} \times R$ takes (x, j) to $(x, f_j(x))$ and is linear on the segments $[j, j+1]$, for all $j \in J$. $\Phi_1(h)$ is isotopic to the identity via $\Phi_t(h)$ in the natural way.

Set $h_t = \Phi_t(h)^{-1} \circ h \circ \Phi_t(h)$. If $R_j = \{(x, j) | x \in R\}$, then from (ii) it follows that $R_{2j-1} < h_1(R_{2j}) < R_{2j+1}$, i.e. $|\Pi_2 \circ h_1(x, y) - y| \leqq 2$ while

$$|\Pi_1 \circ h_1(x, y) - x|$$

is still bounded by M.

One easily checks that $h \rightarrow \Phi_t(h)$ is continuous and that the isotopy is canonical. The result now follows from Lemma 6.

4. **Lifting homeomorphisms.** We consider the following covering map $\lambda: R^2 \rightarrow R \times S$, defined as follows. Let $e: R \rightarrow S$ be the exponential covering map given by $e(r) = e^{\Pi i r}$. We take λ to be an approximation to id $\times e$ which equals id $\times e$ off a compact neighborhood of the point (id $\times e)(1, 0) = (1, 0)$ and which is the identity on a smaller compact neighborhood C of $(1, 0)$. Thus in "wrapping" $R^2 = R_+ \times R$ around $R_+ \times S = R \times S$, we merely place our finger over the point $(1, 0)$ which fixes C.

LEMMA 8. *Let λ be as above. Let $h \in H(R^2)$ fix the origin and $(1, 0)$. Then $h|_{R^2-0} \in H(R \times S)$ lifts (via λ) uniquely to $\hat{h} \in H(R^2)$ fixing $(1, 0)$. Furthermore h is canonically isotopic to $\hat{h}$.*

PROOF. The existence and uniqueness of $\hat{h}$ follows from Hu [4, p. 90]. Clearly $h = \hat{h}$ on $h^{-1}(C) \cap C$. We define

$$h_t = \tau \circ \phi_t^{-1} \circ \tau^{-1} \circ \hat{h}^{-1} \circ h \circ \tau \circ \phi_t \circ \tau^{-1} \quad \text{for } 0 \leq t < 1,$$
$$= \hat{h} \quad \text{for } t = 1,$$

where τ is translation by the point $(1, 0)$ and $\phi_t(x) = (1-t)x$.

Since $\tau^{-1} \circ \hat{h}^{-1} \circ h \circ \tau = $ id on a neighborhood of the origin, h_t is clearly an isotopy. That it is a canonical isotopy follows from the continuity of the functions $h \rightarrow \hat{h}$ and $h \rightarrow \text{rad}_{(1,0)}\{C \cap h^{-1}(C)\}$, which can easily be established by standard $\delta - \varepsilon$ arguments.

5. **Proof of Theorem 1.** Let $h \in H^+(R^2)$. We will give a canonical isotopy h_t, $0 \leqq t \leqq 4$, so that $h_4 \in SO(2)$ is a strict rotation. Furthermore if $h \in SO(2)$, then $h_t = h$ all t.

Step (1). By canonically translating along the line segment determined by $h(0)$ and 0, we get h_1 fixing the origin.

Step (2). We now apply Lemma 5 to h_1 to get h_2 which is small on the radial factor, as in the conclusion of Lemma 5.

Step (3). Consider $h_2(1, 0) = (r, \theta_h)$, written in polar coordinates. By canonically translating in a coordinate neighborhood of $(1, \theta_h)$ we get h_3,

with $h_3(1, 0)=(1, \theta_h)$ in polar coordinates. Let $\rho_h \in SO(2)$ be rotation by θ_h. Set $g=\rho_h^{-1} \circ h_3$. Then g fixes the origin and $(1, 0)$. We canonically isotope h_3 to ρ_h by canonically isotoping g to the identity.

Step (4). g satisfies the hypothesis of Lemma 8. Furthermore $\hat{g}$ will be bounded on the first R factor by 2 since h_3 (hence g) was so bounded radially. Thus $\hat{g}$ satisfies the hypothesis of Lemma 7. This completes the canonical isotopy of g to the identity, and of h_3 to $\rho_h=h_4$.

One easily checks that if $h \in SO(2)$, then $h_t=h$ all t. In Step (4), g will then be the identity, and so will be the isotopy from g to the identity. (*Note*. in Step (4), $\hat{g}$ may move the origin, hence the isotopy from h_3 to h_4 will also. Thus, if we wish, we could translate this part of the isotopy, which would then fix the origin.)

6. Proof of the corollaries.

PROOF OF COROLLARY 2. Just apply the analogous isotopy to the other component.

PROOF OF COROLLARY 3. Let p, q, and r correspond to the origin, ∞, and $(1, 0)$ under stereographic projection. Let $h \in H(S^2)$ be given.

Step 1. If $h(p)$ and $h(q)$ are antipodal, leave h alone. Otherwise, slide $h(p)$ and $h(q)$ along the geodesic they form until they are antipodal. More precisely, we define $\phi_t(x, y) \in H(S^1)$, $x, y \in S^1$, by $\phi_t(x, y)=$id if x and y are antipodal. If not, let a and b be the midpoints of the shorter and longer arcs from x to y. Let c and d be the midpoints of the arcs from a to b, containing x and y respectively. Then $\phi_t(x, y)$ is the linear homeomorphism of S^1 determined by $a \to a$; $b \to b$; $x \to (1-t)x+tc$; $y \to (1-t)y+td$. Consider S^2 as $S^1 \times [-1, +1]$ with each end squeezed to a point. Set $\psi_t(x, y)=\phi_t(x, y) \times$id $\in H(S^2)$. Thus we set $h_t=\psi_t(h(p), h(q)) \circ h$. h_t is clearly canonical since if $h(p)$ and $h(q)$ are almost antipodal, the isotopy is small. In much the same manner we slide $h_1(r)$ to the midpoint of the arc through $h_1(p)$, $h_1(q)$, and $h_1(r)$, getting h_2 which then uniquely determines $\theta \in O(3)$. Setting $g=\theta^{-1} \circ h_2$ fixes p, q, and r.

Step 2. Consider $g|_{S^2-\{q\}} \in H^+(R^2)$ fixing the origin and $(1, 0)$. As in the proof of Theorem 1, g is is canonically isotopic to the identity. We extend the isotopy to S^2 and thus get that h_2 (and h) is canonically isotopic to θ.

PROOF OF COROLLARY 4. Regard $S^1=[0, 1]$ with the endpoints identified. Define an embedding $i: S^1 \to BH(R^2)$ by (for $0 \leqq t < 1$)

$i(t) = $ rotation by $2\pi t$ on $B_{1/(1-t)}$,

$\qquad = $ rotation by $\{2 - r(1 - t)\}2\pi t$ on S_r, $\dfrac{1}{1-t} \leqq r \leqq \dfrac{2}{1-t}$,

$\qquad = $ id   off $B_{2/(1-t)}$

for $t=1$, $i(t)=$id.

Since $\lim_{t\to 1} i(t) = \mathrm{id}$, i is clearly an embedding. Let $h \in BH(R^2)$. We canonically isotope h into $i(S^1)$ by replacing $SO(2)$ by $i(S^1)$ in the proof of Theorem 1 and essentially performing the isotopy given by Theorem 1. However h_t might not be bounded, so we modify the isotopy slightly. In particular, applying Steps 1 and 3 we get h_2 fixing the origin with $h_2(1,0) = (1, \theta_h)$ in polar coordinates. Set $g = \rho^{-1} \circ h_2$, where $\rho \in i(S^1)$ is uniquely determined by θ_h. Then g is bounded and fixes the origin and $(1,0)$. We isotope g to the identity through bounded homeomorphisms as follows. We alter the isotopy of Lemma 5 and the first part of the isotopy of Lemma 7 to be fixed on a neighborhood of $(1,0)$. Then performing Steps 2 and 4 to g gives us $\bar{g} \in {}^M BH$, with $g = \bar{g}$ on a neighborhood N_g of $(1,0)$ with $g \to \mathrm{rad}_{(1,0)}(N_g)$ continuous. Then g is canonically isotopic to $\bar{g}$ through bounded homeomorphisms by the isotopy of Lemma 8, while $\bar{g}$ is canonically isotopic to the identity through bounded homeomorphisms (by the Alexander isotopy).

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