PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 2, December 1973

THE BAIRE ORDER OF THE FUNCTIONS CONTINUOUS ALMOST EVERYWHERE

R. D. MAULDIN

ABSTRACT. Let Φ be the family of all real-valued functions defined on the unit interval I which are continuous except for a set of Lebesgue measure zero. Let Φ_0 be Φ and for each ordinal α, let Φ_α be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma<\alpha}\Phi_\gamma$. Then Φ_{ω_1} is the Baire family generated by Φ. It is proven here that if $0<\alpha<\omega_1$, then $\Phi_\alpha\neq\Phi_{\omega_1}$. The proof is based upon the construction of a Borel measurable function h from I onto the Hilbert cube Q such that if x is in Q, then $h^{-1}(x)$ is not a subset of an F_σ set of Lebesgue measure zero.

If Φ is a family of real-valued functions defined on a set S, then the Baire family generated by Φ may be described as follows: Let $\Phi_0=\Phi$ and for each ordinal $\alpha>0$, let Φ_α be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma<\alpha}\Phi_\gamma$. Of course, $\Phi_{\omega_1}=\Phi_{\omega_1+1}$, where ω_1 denotes the first uncountable ordinal and Φ_{ω_1} is the Baire family generated by Φ; the family Φ_{ω_1} is the smallest subfamily of R^S containing Φ and which is closed under pointwise limits of sequences. The order of Φ is the first ordinal α such that $\Phi_\alpha=\Phi_{\alpha+1}$.

Let C denote the family of all real-valued continuous functions on the unit interval I. It was first proven by Lebesgue that the order of C is ω_1 [**1**]. In 1924, Kuratowski [**2**] proved that if one relaxes the continuity condition by only requiring that the original functions be continuous except for a first category set, then the Baire order of this enlarged family is 1. In 1930, Kantorovitch [**3**] showed that if one requires that the original functions be continuous except for a set of Lebesgue measure zero, then the Baire order of this family is at least 2. Recently, the author generalized this result in the following fashion [**4**].

THEOREM. *Let S be a complete separable metric space, let u be a σ-finite, complete Borel measure on S and let Φ be the family of all real-valued functions on S, whose set of points of discontinuity is of u-measure* 0. *Then* (1) *the order of Φ is* 1 *if and only if u is a purely atomic measure whose*

Received by the editors December 16, 1972.

AMS (MOS) subject classifications (1970). Primary 28A05, 26A21; Secondary 04A15, 54C50.

Key words and phrases. Lebesgue measure zero, Baire class α, universal function, Hilbert cube.

set of atoms is dispersed and (2) *if the order of* Φ *is not* 1, *the order of* Φ *is at least* 3.

In this paper Φ will denote the family of all real-valued functions defined on the unit interval I which are continuous except for a set of Lebesgue measure zero. It is shown here that the Baire order of this family is ω_1. The method of proof involves showing that there is a Borel measurable function h from I onto the Hilbert cube such that if x is a point of the Hilbert cube, then $h^{-1}(x)$ is not a subset of an F_σ set of Lebesgue measure 0. Of course, there is no such function h which is continuous or even an h such that $h^{-1}(x)$ is an F_σ set for each x. Thus, the function h is necessarily fairly complicated. We begin with a sequence of lemmas which are used to demonstrate the existence of one such function h. This function will be used to construct a transfinite sequence of "universal functions" $\{U_\alpha\}_{0<\alpha<\omega_1}$ [Theorem 2]. Finally, a diagonal type argument is applied to prove that the order of Φ is ω_1 [Theorem 4].

LEMMA 1. *Let P be a perfect subset of the interval I such that if an open set U meets P, then $\lambda(P\cap U)>0$. There is a double sequence $\{F_{np}\}_{n,p=1}^{\infty}$ of disjoint perfect subsets of P such that* (1) *each F_{np} is nowhere dense in P and if an open set U meets F_{np}, then $\lambda(U\cap F_{np})>0$, and* (2) *if n is a positive integer and U is a nonempty set open with respect to P, then there is some p such that F_{np} is a subset of U.*

PROOF. Let $\{s_n\}_{n=1}^{\infty}$ be a countable base of nonempty open sets with respect to P.

Let K_{11} be a perfect set lying in $s_1\cap s_1=s_1$ such that K_{11} is nowhere dense in P and if an open set U intersects K_{11}, then $\lambda(K_{11}\cap U)>0$. For each positive integer n and integer p, $1\leqq p\leqq n+1$, let $K_{n+1\,p}$ be $\varnothing$, if $s_{n+1}\cap s_p=\varnothing$, and, if $s_{n+1}\cap s_p\neq\varnothing$ let $K_{n+1\,p}$ be a perfect set lying in $s_{n+1}\cap s_p$ such that (1) $K_{n+1\,p}$ is nowhere dense in P, (2) $K_{n+1\,p}$ is disjoint from $(\bigcup_{r=1}^{n}\bigcup_{q=1}^{r}K_{rq})\cup(\bigcup_{i=1}^{p-1}K_{n+1\,i})$ (a union from 1 to 0 is taken to be empty) and (3) if an open set U intersects $K_{n+1\,p}$, then $\lambda(K_{n+1\,p}\cap U)>0$.

For each p, let $F_{1p}=K_{pp}$. For each positive integer pair n, p, let $F_{n+1\,p}$ be the first term of the sequence $\{K_{qp}\}_{q=p}^{\infty}$ which follows F_{np} and which is nonempty.

It follows that the double sequence $\{F_{n,p}\}_{p,n=1}^{\infty}$ has the required properties.

Now let $\{F_{(n,p)}\}_{n,p=1}^{\infty}$ be a double sequence which has the properties listed in Lemma 1, where P is the interval [0, 1].

By repeated application of Lemma 1, we have

LEMMA 2. *There is a system of sets $\{F_{(n_1,n_2,\cdots,n_{2k})}\}$, where $(n_1,\cdots,n_{2k})$ ranges over the family of all finite sequences of positive integers of even* *length such that if* $(n_1, n_2, \cdots, n_{2k-1}, n_{2k})$ *is such a sequence, then the double sequence* $\{F_{(n_1, n_2, \cdots, n_{2k-1}, n_{2k}, n, p)}\}_{n,p=1}^{\infty}$ *has the properties listed in Lemma* 1 *with respect to the set* $\{F_{(n_1, n_2, \cdots, n_{2k-1}, n_{2k})}\}$.

Let W_n be the family $\{F_{(n,p)}\}_{p=1}^{\infty}$ for each n, and for each finite sequence of positive integers $(n_1, \cdots, n_k)$, let $W_{(n_1, \cdots, n_k)}$ be the family

$$\{F_{(n_1, i_1, n_2, i_2, \cdots, n_k, i_k)}\}$$

where $(i_1, \cdots, i_k)$ ranges over all k-tuples of positive integers. Let $T_{n_1, \cdots, n_k}$ be the union of all the sets in the family $W_{(n_1, \cdots, n_k)}$.

Notice that these families have the following three properties:

(1) If $(m_1, \cdots, m_k) \neq (n_1, \cdots, n_k)$, then $T_{(n_1, \cdots, n_k)}$ and $T_{(m_1, \cdots, m_k)}$ are disjoint;

(2) Each set in $W_{(n_1, \cdots, n_k, n_{k+1})}$ is a subset of some set in $W_{(n_1, \cdots, n_k)}$; and

(3) If $F \in W_{(n_1, \cdots, n_k)}$, n is a positive integer, and U is an open set which meets F, then there is some set in the family $W_{(n_1, \cdots, n_k, n)}$ which is a subset of U.

LEMMA 3. *Let* $\{n_k\}_{k=1}^{\infty}$ *be a sequence of positive integers. The intersection of the monotonically decreasing sequence* $\{T_{(n_1, \cdots, n_k)}\}_{k=1}^{\infty}$ *is not a subset of an* F_σ *set of measure* 0.

PROOF. For each n, let A_n be a closed set of Lebesgue measure 0. Since T_{n_1} is dense in the interval I, it follows that there is some set F_{n_1, k_1} which does not intersect A_1.

Since $\lambda(F_{(n_1, k_1)}) > 0$ and $\lambda(A_2) = 0$, there is an open set which meets F_{n_1, k_1} which does not intersect A_2. It follows from property (3) that there is a set $F_{(n_1, k_1, n_2, k_2)}$ which is a subset of $F_{(n_1, k_1)}$ and does not meet A_2.

Continuing this process, we obtain a monotonically decreasing sequence $\{F_{(n_1, k_1, \cdots, n_p, k_p)}\}_{p=1}^{\infty}$ such that for each p, $F_{(n_1, k_1, \cdots, n_p, k_p)}$ does not intersect A_p. The nonempty intersection of this sequence of sets is a subset of $\bigcap_{k=j}^{\infty} T_{(n_1, \cdots, n_k)}$ which does not intersect $\bigcup_{n=1} A_n$. This completes the proof of Lemma 3.

For each k, let $H_k = \bigcup T_{n_1, \cdots, n_k}$, where the union is taken over all k-tuples of positive integers. Let $H = \bigcap_{k=1}^{\infty} H_k$. The set H is an $F_{\sigma\delta}$ set.

Let $\mathcal{N}$ denote the space of all irrational numbers between 0 and 1. Identify the space of all infinite sequences of positive integers with the space via the continued fraction expansion of the members of the space $\mathcal{N}$. If $Z \in \mathcal{N}$ let $[Z_1, Z_2, Z_3, \cdots]$ denote the sequence of integers appearing in the continued fraction expansion of Z.

LEMMA 4. *There is a Borel measurable function* f *from* H *onto* $\mathcal{N}$ *such that if* $Z \in \mathcal{N}$, *then* $f^{-1}(Z)$ *is not a subset of any* F_σ *set of Lebesgue measure* 0.

PROOF. For each $x \in H$, there is only one sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $x \in \bigcap_{k=1}^{\infty} T_{(n_1,\cdots,n_k)}$; let $f(x)$ be the irrational numbers in $\mathscr{N}$ identified with this sequence. It follows from the preceding lemma that f maps H onto $\mathscr{N}$ and if $Z \in \mathscr{N}$, then $f^{-1}(Z)$ is not a subset of an F_σ set of measure 0.

For each k-tuple $n_1, \cdots, n_k$, let $J_{(n_1,\cdots,n_k)}$ $\{Z:Z_i=n_i,\ i=1, 2, \cdots, k\}$. The sets $J_{(n_1,\cdots,n_k)}$ form an open base for the usual topology on the space $\mathscr{N}$.

We have

$$f^{-1}(J_{(n_1,\cdots,n_k)}) = \bigcup_{Z \in \mathscr{N}} \left(\bigcap_{p=1}^{\infty} T_{(n_1,\cdots,n_k,Z_1,\cdots,Z_p)} \right).$$

Thus, $f^{-1}(J_{(n_1,\cdots,n_k)})$ is an analytic set [**5**, p. 467]. It follows from Lusin's first separation theorem [**5**, p. 485] that f is Borel measurable (actually, $f^{-1}(U)$ is an $F_{\sigma\delta\sigma}$ set for each open set U).

We are now in a position to prove

THEOREM 1. *There is a Borel measurable function h from the unit interval I onto the Hilbert cube I^{ω_0} such that if $x \in I^{\omega_0}$, then $f^{-1}(x)$ is not a subset of an F_σ set of Lebesgue measure* 0.

PROOF. Let f be a function as described in Lemma 4. Let g be a con, tinuous function from $\mathscr{N}$ onto the Hilbert cube [**5**, p. 440]. The composition, $g \circ f$, maps H onto the Hilbert cube and is Borel measurable. Let $(g_1, g_2, g_3, \cdots)$ be the sequence of the natural projections of $g \circ f$. For each p, g_p is a Borel measurable function from H onto the interval I [**5**, p. 382]. For each p, let $\tilde{g}_p$ be a Borel measurable extension g_p to all of I which maps into I. Let $h = (\tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \cdots)$. The function h has the required properties.

THEOREM 2. *There exists a transfinite sequence of "universal functions" $\{U_\alpha\}_{0<\alpha<\omega_1}$ such that for each α, $0<\alpha<\omega_1$, we have*

(1) *U_α is a Baire measurable function on $I \times I$ which maps into the unit interval I*; and

(2) *if f is a function in Baire's class α which maps into I, then the set of all x such that $U_\alpha(x, y) = f(y)$, for every y in I, is not a subset of an F_σ set of Lebesgue measure zero.*

The proof essentially follows the argument in [**6**, p. 133].

PROOF. Let $\{s_n\}_{n=1}^{\infty}$ be a countable dense subset of the positive part of the unit ball of the Banach space $C(I)$.

Let

$$\begin{aligned} U_0(x, y) &= s_n(y), \quad \text{if } x = 1/n, \\ &= 0, \qquad\quad \text{otherwise.} \end{aligned}$$

It can be easily verified that U_0 is a Borel measurable function on $I \times I$ and of course it maps into the interval I. Let $h=(h_1, h_2, h_3, \cdots)$ be a function from I onto the Hilbert cube having the properties described in Theorem 1.

For each ordinal α, $0 \leqq \alpha < \omega_1$, let

$$U_{\alpha+1}(x, y) = \limsup_{p \to \infty} U_\alpha(h_n(x), y)$$

for each $(x, y) \in I \times I$; also, if α is a limit ordinal, let $\{\gamma_p\}_{p=1}^{\infty}$ be an increasing sequence of ordinals less than α which converges to α and let

$$U_\alpha(x, y) = \limsup_{p \to \infty} U_{\gamma_p}(h_p(x), y).$$

It may be proven by transfinite induction that the functions U_α, $0 < \alpha < \omega_1$, are Borel measurable and map into I.

The proof that the functions U_α are "universal" and represent each appropriate function in Baire's class α on a "large" set proceeds by transfinite induction.

First, suppose f is in Baire's class 1 and f maps I into I. Consequently, there is a sequence $(n_1, n_2, n_3, \cdots)$ of positive integers such that the sequence $\{s_{n_p}\}_{p=1}^{\infty}$ converges pointwise to f on I.

If $x \in h^{-1}(1/n_1, 1/n_2, 1/n_3, \cdots)$, then

$$U_1(x, y) = \limsup_{p \to \infty} U_0(h_p(x), y) = \limsup_{p \to \infty} s_{n_p}(y) = f(y),$$

for each y in I. Thus, the function U_1 has the second required property.

Now, suppose α is a limit ordinal, the functions U_γ, $0 < \gamma < \alpha$, have the required properties and f is a function in Baire's class α which maps I into I.

There is a sequence $\{f_p\}_{p=1}^{\infty}$ of functions, converging pointwise to f on I such that for each p, f_p is in Baire's class γ_p and f_p maps I into I.

For each p, let x_p be a number in I such that $U_{\gamma_p}(x, y) = f_p(y)$, for every y in I.

If $x \in h^{-1}(x_1, x_2, x_3, \cdots)$, then $U_\alpha(x, y) = f(y)$, for each y in I and U_α has the required properties.

A similar argument can be given for the remaining functions $U_{\alpha+1}$.

In order to prove that the Baire order of Φ is ω_1, we will employ a theorem which was published previously by the author:

THEOREM 3 [7]. *If α is an ordinal, $0 < \alpha < \omega_1$, then a function f is in Φ_α if and only if there is a function g in Baire's class α such that the set $D = \{x : f(x) \neq g(x)\}$ is a subset of an F_σ set of measure zero.*

We will now prove

THEOREM 4. *The Baire order of* Φ *is* ω_1.

PROOF. Let α be an ordinal, $0<\alpha<\omega_1$. Let U_α be a universal function having the properties stated in Theorem 2. Let $w(x)=\lim_{n\to\infty}(1-U_\alpha(x,x))^n$. The function w is a Baire function which maps I into I and there is no x such that $w(x)=U_\alpha(x, x)$. Actually, w is the characteristic function of the set of all x such that $U_\alpha(x, x)=0$.

Assume that $w\in\Phi_\alpha$. By Theorem 3, there is a function g in Baire's class α such that the set $D=\{x:w(x)\neq g(x)\}$ is a subset of an F_σ set K of Lebesgue measure 0. It is assumed here that g maps into I (this is no restriction). By Theorem 2, there is some $x\in K'$ such that $U_\alpha(x, y)=g(y)$ for all y in I. In particular, $U_\alpha(x, x)=g(x)=w(x)$, since $x\in K'$. This contradiction proves the theorem.

Question. If $0<\alpha<\omega_1$, is there a σ-ideal R_α of subsets of I of the first category which contains all the sets of Lebesgue measure 0 such that the family Φ of all functions which are continuous except for a set in this σ-ideal R_α has Baire order α? See [7], for some relationships between the classes Φ_α and the classical Baire functions of class α.

REMARK. As mentioned in the first part of this paper the Baire order of the family of all real-valued functions on I which are continuous except for a first category set is 1. This fact together with the technique employed in this paper yield the following

THEOREM. *There does not exist a Borel measurable function h from the unit interval I onto the Hilbert cube* I^{ω_0} *having the property that if* $x\in I^{\omega_0}$, *then* $f^{-1}(x)$ *is not a subset of a first category set.*

REFERENCES

1. H. Lebesgue, *Sur les fonctions représentable analytiquement*, J. Math. (6) **1** (1905).

2. C. Kuratowski, *Sur les fonctions représentables analytiquement et les ensembles de premiere categorie*, Fund. Math. **5** (1924), 75–86.

3. L. V. Kantorovitch, *Sur les suites des fonctions presque partout continues*, Fund. Math. **17** (1930), 25–28.

4. R. D. Mauldin, *On Borel measures and Baire's class* 3, Proc. Amer. Math. Soc. **39** (1973), 308–312.

5. K. Kuratowski, *Topologie*, Vol. I, PWN, Warsaw, 1958; English transl., Academic Press, New York; PWN, Warsaw, 1966. MR **19**, 873; **36** #840.

6. I. Natanson, *Theory of functions of a real variable*, GITTL, Moscow, 1950; English transl. of 1st ed., vol. 1, Ungar, New York, 1955. MR **12**, 598; **16**, 804.

7. R. D. Mauldin, *σ-ideals and related Baire systems*, Fund. Math. **71** (1971), 171–177. MR **45** #2107.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601