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ON A RESULT OF OSBORN

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ABSTRACT. The structure of certain semiprimitive rings with involution * is determined by imposing conditions on the set of *-symmetric elements and limiting the number of orthogonal *-symmetric idempotents.

An associative ring R with unit such that $1/2 \in R$ satisfies condition $C(n)$ if (i) R has an involution $*$ such that each $*$-symmetric element s of R is nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent and (ii) R has a set of n nonzero, pairwise orthogonal, $*$-symmetric idempotents whose sum is one and if $\{e_i\}_{i=1}^m$ is any set of such idempotents whose sum is one, then $m \leqslant n$. We will determine the structure of all rings with condition $C(n)$.

A ring satisfying $C(1)$ has exactly one nonzero $*$-symmetric idempotent, the one of that ring. Hence each $*$-symmetric element is either nilpotent or invertible. Osborn [4] catalogued these rings and showed that a semiprimitive ring has $C(1)$ iff R is one of

(i) a division ring,

(ii) a direct sum of two anti-isomorphic division rings with involution interchanging the summands,

(iii) the 2×2 matrices over a field with the involution fixing only the scalar matrices.

First we reduce to the case where R is semiprimitive. Then we collect some of the facts to be used for our main result (Theorem 4). Since the results of Lemma 1 and Lemma 2 are well known, their proofs are omitted.

LEMMA 1. *If each $*$-symmetric element s of R is either nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent, then the Jacobson radical of R, $\mathbf{J}(R)$, is a $*$-invariant ideal in which every $*$-symmetric element is nilpotent.*

LEMMA 2. *If each $*$-symmetric element s of R is either nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent, and $\bar{u} \in R/\mathbf{J}(R)$ is a symmetric element under the induced involution $*'$, then $\bar{u}$ is either nilpotent or some (right) multiple of $\bar{u}$ is a nonzero $*'$-symmetric idempotent.*

REMARK 1. Suppose R is a ring and for some $a \in R$, $a^2 - a$ is nilpotent. Then either a is nilpotent or for some polynomial $q(x)$ with integer

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coefficients $e = aq(a)$ is a nonzero idempotent. Furthermore, the sum of the coefficients of $q(a)$ is one. This is Lemma 1.3.2 of Herstein [1].

LEMMA 3. *Suppose each $*$-symmetric element s of R is either nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent and let $\bar{u}$ be a $*'$-symmetric idempotent of $R/\mathbf{J}(R)$. Then $\bar{u}$ can be lifted to a $*$-symmetric idempotent of R.*

PROOF. We may assume that $u^* = u$. Since $u^2 - u$ is a $*$-symmetric element in $\mathbf{J}(R)$, it is nilpotent. Remark 1 tells us that there is a $*$-symmetric idempotent $e = uq(u)$. Since the sum of the coefficients of $q(u)$ is one, $\bar{e} = \bar{u}$.

Even more can be said about lifting $*'$-symmetric idempotents.

THEOREM 1. *Suppose each $*$-symmetric element s of R is either nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent. Then if $\{\bar{u}_i\}_{i=1}^m$ is a set of $*'$-symmetric, pairwise orthogonal idempotents in $R/\mathbf{J}(R)$, then there exists a set $\{e_i\}_{i=1}^m$ of $*$-symmetric, pairwise orthogonal idempotents in R with $\bar{e}_i = \bar{u}_i$.*

PROOF. By Lemma 3, $\bar{u}_1$ can be lifted to the $*$-symmetric idempotent e_1 and $\bar{u}_2$ can be lifted to an idempotent f. Hence $e_1 f$ and fe_1 are in $\mathbf{J}(R)$. In particular, $1 - fe_1$ has an inverse in R and we may form

$$f' = (1 - fe_1)^{-1} f (1 - fe_1).$$

This is an idempotent of R and $f'e_1 = 0$. Multiplying by $1 - fe_1$ on the left, we see that $f' - f \in \mathbf{J}(R)$.

Now put $h = f' - e_1 f'$. Then $e_1 h = 0 = he_1$, $h - u_2 \in \mathbf{J}(R)$ and $h^2 = h$. Since we can assume $u_2^* = u_2$, $h^* - u_2 \in \mathbf{J}(R)$. Thus $hh^* - u_2 \in \mathbf{J}(R)$ and $e_1(hh^*) = (hh^*)e_1 = 0$. Now hh^* is not nilpotent, but $(hh^*)^2 - (hh^*) \in \mathbf{J}(R)$, so by Remark 1 some polynomial in hh^* is a $*$-symmetric idempotent e_2 and $e_2 - hh^* \in \mathbf{J}(R)$. Thus $e_2 - u_2 \in \mathbf{J}(R)$ and $e_1 e_2 = e_2 e_1 = 0$ since $e_2 = hh^* q(hh^*)$.

Suppose the first $n - 1$ elements of $\{\bar{u}_i\}_{i=1}^m$ have been lifted to a set of $\{e_i\}_{i=1}^{n-1}$ $*$-symmetric, pairwise orthogonal idempotents. By Lemma 3, we can lift $\bar{u}_n$ to an idempotent f_n. Then $(\sum_{i=1}^{n-1} e_i) f_n$ and $f_n \sum_{i=1}^{n-1} e_i$ are in $\mathbf{J}(R)$. In particular, $1 - f_n \sum_{i=1}^{n-1} e_i$ has an inverse in R. Set

$$f_n' = \left(1 - f_n \sum_{i=1}^{n-1} e_i\right)^{-1} f_n \left(1 - f_n \sum_{i=1}^{n-1} e_i\right).$$

Put $h_n = f_n' - (\sum_{i=1}^{n-1} e_i) f_n'$. As above we can construct a polynomial e_n in $h_n h_n^*$ such that e_n is a $*$-symmetric idempotent, $e_n - u_n \in \mathbf{J}(R)$ and $\{e_i\}_{i=1}^n$ is a set of pairwise orthogonal $*$-symmetric idempotents.

Hence R has at least m $*$-symmetric, pairwise orthogonal idempotents.

The results of Lemma 1 through Theorem 1 are summarized in

THEOREM 2. *If R has $C(n)$, then so does $R/\mathbf{J}(R)$.*

PROOF. $R/J(R)$ has at least n $*'$-symmetric pairwise orthogonal idempotents that it inherits from R. By Theorem 1, it can have no more.

REMARK 2. If a semiprimitive ring has $C(1)$, then there is an idempotent $f \in R$ such that fRf is a division ring (equivalently fR is a minimal right ideal R).

REMARK 3. If R is a ring with involution whose symmetric elements are all nilpotent, then R is a radical ring. This is Lemma 3 of Osborn [4].

REMARK 4. If e is a nonzero idempotent of a ring R, then in the two-sided Peirce decomposition of R relative to e,

$$R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e)$$

the ring eRe is radical free if R is radical free.

REMARK 5. Jacobson and Rickart [3] define a canonical involution $*$ in R_n (the ring of $n \times n$ matrices over R) as an involution in R_n such that $e_{ii}^* = e_{ii}$, $i = 1, 2, \ldots, n$. If $*$ is a canonical involution in R_n, then there is an involution $r \to \bar{r}$ in R and invertible elements $\delta_i \in R$ such that $\bar{\delta}_i = \delta_i$ and

$$\left(\sum \alpha_{ij}e_{ij}\right)^* = \sum \delta_j^{-1}\bar{\alpha}_{ij}\delta_i e_{ji}.$$

Let R be a simple ring with the minimum condition for right ideals and involution. Then $R = \Delta_n$, Δ a division ring. The involution is canonical except when Δ is a field Φ, $n = 2m$ and $x \to q^{-1}x'q$, where x' denotes the transpose of x, q is the diagonal $m \times m$ matrix over Φ_2 with nonzero entries $\sigma = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In this case we can regard R as S_m, where $S = \Phi_2$. If we introduce the involution $\alpha \to \bar{\alpha} = \sigma^{-1}\alpha'\sigma$ in Φ_2, then the given involution in R is canonical with all $\delta_i = 1$.

LEMMA 4. *Let R be a ring with $C(n)$ and $\{e_i\}_{i=1}^n$ a collection of pairwise orthogonal nonzero $*$-symmetric idempotents such that $\sum_{i=1}^n e_i = 1$. Then the ring $e_iR_ie_i$ has $C(1)$.*

PROOF. Look at the ring e_iRe_i in the two-sided Peirce decomposition of R relative to e_i. This is a $*$-invariant subring of R and each $*$-element of e_iRe_i is either invertible in e_iRe_i or nilpotent. That is, e_iRe_i has $C(1)$.

LEMMA 5. *Let R be a semiprimitive ring with $C(n)$. Then R has the minimum condition on the right ideals.*

PROOF. $R = \oplus\sum_{i=1}^n e_iR$, a direct sum of right R modules. But since e_iRe_i has $C(1)$ there is an $f_i \in e_iRe_i$ such that $f_i^2 = f_i$, $e_if_i = f_ie_i$ and each of f_iRf_i and $(e_i - f_i)R(e_i - f_i)$ is a division ring. Hence

$$R = \oplus \sum_{i=1}^{n} \left[f_ie_iR + (e_i - f_i)R \right]$$

a finite direct sum of minimal right ideals.

REMARK 6. It is well know that if a ring R with involution $*$ has no proper $*$-invariant ideals, then R is either a simple ring or R is a direct sum of two simple rings with the involution interchanging the summands.

REMARK 7. Let A be an associative algebra over a field Φ and suppose that a is a nonnilpotent noninvertible algebraic element of A. Then there is an idempotent $e = a^k P(a)$, where $P(a)$ lies in the subalgebra formally generated by 1 and a, such that $a^k e = a^k$ for some integer $k \geqslant 1$. (See, for example, the proof in Jacobson [**2**, p. 210, Proposition 1].)

THEOREM 3. *A simple ring R has $C(n)$ iff it is a ring of $n \times n$ matrices over* (i) *a division ring or over* (ii) *the* 2×2 *matrices over a field.*

PROOF. By Lemma 5, R has minimal right ideals. By Remark 5, R is a matrix ring with canonical involution and hence one of the two rings listed.

The main result can now be stated.

THEOREM 4. *A ring R has property $C(n)$ iff*

(i) $\mathbf{J}(R)$ *is a $*$-invariant ideal in which every $*$-symmetric element is nilpotent and*

(ii) $R/\mathbf{J}(R)$ *is the direct sum of semiprimitive rings R_{n_i} where each R_{n_i} has property $C(n_i)$, $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} n_i = n$.*

PROOF. (i) This is established in Lemma 1.

(ii) Theorem 2 shows that $R/\mathbf{J}(R)$ has $C(n)$. By Lemma 5, $R/\mathbf{J}(R)$ has the minimum condition on right ideals. Hence $R/\mathbf{J}(R)$ is the ring direct sum of matrices M_i. Since $R/\mathbf{J}(R)$ has an involution $*$, each M_i is either fixed under $*'$ and hence satisfies $C(n_i)$ for some n_i or M_i is mapped onto $M_i^{*'}$ and then $(M_i \oplus M_i^*)$ is fixed under $*'$ and satisfies $C(n_i)$ for some n_i (Remark 6).

That any collection of such semiprimitive rings put together in this fashion has the stated property is evident.

COROLLARY 1. *Let R be an associative algebra with* 1 *over the field Φ not of characteristic* 2. *If R has an involution $*$ such that for each $*$-symmetric element s of R there is a $\lambda(s) = \lambda^*(s) \in \Phi$ and $s^2 - s\lambda(s)$ is either nilpotent or invertible, then $R/\mathbf{J}(R)$ has $C(1)$ or $C(2)$.*

PROOF. Pass to $R/\mathbf{J}(R)$ as in Lemma 2 and note that each $*'$-symmetric element s of $R/\mathbf{J}(R)$ is either (i) invertible, (ii) nilpotent, or (iii) determines a nonzero $*'$-symmetric idempotent $e = s^k P_s(s)$. $R/\mathbf{J}(R)$ can have at most two nonzero pairwise orthogonal $*'$-symmetric idempotents whose sum is one. Suppose otherwise. Let $\{e_i\}_{i=1}^{3}$ be a set of $*'$-symmetric idempotents whose sum is one. Then $e_1 + 2e_2$ is a $*'$-symmetric element and $[(e_1 + 2e_2)^2 - (e_1 + 2e_2)\lambda]^k = 0$ for some positive integer k. This last expression cannot be solved for λ.

COROLLARY 2. *Let A be an associative algebraic algebra with* 1 *over the field Φ not of characteristic* 2. *Let A have an involution $*$ and a set of n nonzero, pairwise orthogonal, $*$-symmetric idempotents whose sum is one. Then A has $C(n)$ iff for each set $\{e_i\}_{i=1}^{m}$ of such idempotents whose sum is one, $m \leqslant n$.*

PROOF. We only need to show that each $*$-symmetric element s of A is nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent. But this is true by Remark 7.

COROLLARY 3. *Let A be an associative algebraic algebra with* 1 *over the field* Φ *not of characteristic* 2. *If each nonnilpotent noninvertible* $*$*-symmetric element is algebraic over* Φ, *then any finite set of pairwise orthogonal* $*'$*-symmetric idempotents in* $A/\mathrm{J}(A)$ *can be lifted to an orthogonal set of* $*$*-symmetric idempotents of* A.

PROOF. By Remark 7, each $*$-symmetric element s is either nilpotent or some (right) multiple of s is a nonzero $*$-symmetric idempotent. Then argue as in Theorem 1.

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