THE EXTENSION OF MEASURABLE FUNCTIONS

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ABSTRACT. Say a measurable space $(Y,\mathfrak{B})$ has the *extension property* (resp. the *extension property in the restricted sense*) if for every measurable space $(X,\mathfrak{S})$ and every subset A of X (resp. subset A of X with $X \setminus A$ singleton), each function f: $A \to Y$ measurable for $\mathfrak{S}(A) = \{B \cap A: B \in \mathfrak{S}\}$ may be extended to a measurable function g: $X \to Y$. A countably generated and separated $(Y,\mathfrak{B})$ has the extension property if and only if it is a standard space, i.e. it is isomorphic to a Borel subset of the real line. The discrete space $(Y,2^Y)$ has the extension property in the restricted sense if and only if the cardinality of Y is not two-valued measurable.

We shall use the following notation and terminology: if $(X,\mathfrak{S})$ is a measurable space, and $A \subset X$, let $\mathfrak{S}(A) = \{B \cap A: B \in \mathfrak{S}\}$. A measurable space $(X,\mathfrak{B})$ is *separable* if $\mathfrak{B}$ is countably generated and separated (contains singletons); $(X,\mathfrak{B})$ is *standard* if there is a complete separable (i.e. Polish) metric on X for which $\mathfrak{B}$ becomes the associated Borel σ-algebra; equivalently, $(X,\mathfrak{B})$ is isomorphic with some Borel subset of the real numbers.

The extension problem. Let there be given

(1) measurable spaces $(X,\mathfrak{S})$ and $(Y,\mathfrak{B})$,

(2) a subset A of X, and

(3) a function f: $A \to Y$, measurable from $\mathfrak{S}(A)$ to $\mathfrak{B}$.

We ask whether there is an $\mathfrak{S}$-measurable g: $X \to Y$ extending f (i.e. $g(a) = f(a)$ for $a \in A$). Say that a measurable space $(Y,\mathfrak{B})$ has the *extension property* if such an extension exists for each $(X,\mathfrak{S})$, A and f as above. If an extension exists whenever $X \setminus A$ is a singleton set, then $(Y,\mathfrak{B})$ has *the extension property in the restricted sense*.

What seem to have been the first positive results for this problem were obtained by von Alexits [1] and Sierpiński [8] in the case of real functions on subsets of Polish spaces; here, particular attention was paid to the Baire class of f and g. It is a small step from their results to the following characterization of separable spaces with the extension property:

THEOREM 1. *Among separable spaces, those $(Y,\mathfrak{B})$ having the extension property are precisely the standard spaces.*

PROOF. To prove the extension property for standard spaces, obtain an extension for functions assuming finitely many values and pass to the limit (cf. exercise 13, p. 260 of Cohn [2]); alternatively, derive the result from 2.3 Lemma 1 of Lehmann [4].

For the converse, the well-known method of Marczewski [5] and the separability of $(X, \mathfrak{B})$ allow us to consider X as a subset of the unit interval $[0, 1]$ with $\mathfrak{B} = \mathcal{C}(X)$ and $\mathcal{C}$ the Borel σ-algebra of $[0, 1]$. Let $f\colon X \to X$ be the identity function. If now f has an $\mathcal{C}$-measurable extension $g\colon I \to X$, then $X = \{x\colon g(x) = x\}$ is a Borel subset of $[0, 1]$. Thus $(X, \mathfrak{B})$ is a standard space.   Q.E.D.

Let Y be a nonempty set and let d be the metric on Y defined by the rule $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$. Then (Y, d) becomes a complete metric space under the discrete topology and has the power set 2^Y as its Borel σ-algebra. We have not solved the problem of extensions in the case where the "target" space Y is complete, but not separable. For example, we do not know if $(\mathbf{R}, 2^{\mathbf{R}})$ has the extension property, where $\mathbf{R}$ is the real line.

Recall that a cardinal number k is *measurable* if whenever a set X has cardinality k, there is a measure μ on $(X, 2^X)$ such that $\mu\{x\} = 0$ for all $x \in X$ and $\mu X = 1$; k is *two-valued measurable* if in addition μ may be chosen so as to assume only the values 0 and 1.

According to Ulam [9, p. 146], two-valued measurable cardinals must be extremely large (strongly inaccessible); still larger according to A. Tarski [3]. D. Scott [7] showed that Gödel's axiom of constructibility $V = L$ implies nonexistence of any two-valued measurable cardinals. Apparently it is as yet an open problem whether it is consistent with Zermelo-Fraenkel set theory that any two-valued measurable cardinals exist. In any case the next result shows that essentially all measurable spaces of the form $(X, 2^X)$ have the extension property in the restricted sense.

THEOREM 2. *Let X be a set with cardinality $k > 0$. Then $(X, 2^X)$ has the extension property in the restricted sense if and only if k is not two-valued measurable.*

PROOF. Suppose that k is two-valued measurable and let μ be a nontrivial 0-1 measure on $(X, 2^X)$. Define $Y = X \cup \{p\}$, where p is a new point $p \notin X$ and set $\mathfrak{S} = \{B \cup \{p\}\colon B \subset X,\ \mu B = 1\} \cup \{B\colon B \subset X,\ \mu B = 0\}$; then $\mathfrak{S}(X) = 2^X$, and the identity function $f\colon X \to X$ is $\mathfrak{S}(X)$-measurable. Suppose that $g\colon Y \to X$ is an $\mathfrak{S}$-measurable extension of f; then $g^{-1}(g(p)) = \{p, g(p)\} \notin \mathfrak{S}$, a contradiction.

Assume now that $(Y, \mathfrak{S})$ is a measurable space with $A \subset Y$ and $Y \setminus A = \{p\}$ and suppose f is a measurable function from $(A, \mathfrak{S}(A))$ to $(X, 2^X)$ with no $\mathfrak{S}$-measurable extension $g\colon Y \to X$. Then $\{p\} \notin \mathfrak{S}$ and for each $B \in \mathfrak{S}(A)$, either $B \cup \{p\} \in \mathfrak{S}$ or $B \in \mathfrak{S}$, but not both. Define a measure μ on $(A, \mathfrak{S}(A))$ by the rule $\mu B = 1$ if $B \cup \{p\} \in \mathfrak{S}$ and $\mu B = 0$ if $B \in \mathfrak{S}$. Then $f(\mu)$ is a 0-1 measure on $(X, 2^X)$ with $f(\mu)(X) = 1$; furthermore, $f(\mu)(x) = 0$: if not, then there is an $x \in X$ with $f(\mu)(x) = \mu f^{-1}(x) = 1$; then $g(p) = x$ defines an $\mathfrak{S}$-measurable extension of f: if $S \subset X$, $x \in S$, then $g^{-1}(S) = f^{-1}(S) \cup \{p\}$ and $\mu f^{-1}(S) = 1$, so that $f^{-1}(S) \cup \{p\} \in \mathfrak{S}$; if $x \notin S$, then $g^{-1}(S) = f^{-1}(S)$ and $\mu f^{-1}(S) = 0$, so that $f^{-1}(S) \in \mathfrak{S}$. Since no such extension exists, $f(\mu)$ is a nontrivial 0-1 measure on $(X, 2^X)$ and the cardinality of X is two-valued measurable.   Q.E.D.

Since measurable cardinals are weakly inaccessible [9], the continuum hypothesis implies that the cardinal of the continuum is not measurable, and the generalized continuum hypothesis implies that measurable cardinals are strongly inaccessible

and hence extraordinarily large. So one can assume that cardinalities of spaces arising in analysis are all nonmeasurable.

The *separability character* of a metric space X is the smallest cardinality of a base for the topology of X. A limited type of extension is possible for complete spaces with nonmeasurable separability character.

THEOREM 3. *Let* $(Y, \mathbb{S}, P)$ *be a finite measure space,* $A \subset Y$ *and* $f: A \to X$ *a Borel measurable function from* $(A, \mathbb{S}(A))$ *to a complete metric space* X *with nonmeasurable separability character. Then there is an* $\mathbb{S}$*-measurable* $g: Y \to X$ *such that*

$$P^*\{x \in A : f(x) \neq g(x)\} = 0.$$

PROOF. Consider P^* as a measure on $(A, \mathbb{S}(A))$ and its image $f(P^*)$ on X. By a theorem of Marczewski and Sikorski [6, Theorem III], there is a closed subset F of X which is separable and is such that $f(P^*)(F) = f(P^*)(X) = P^*(A)$. Let f_0 be f restricted to $f^{-1}(F)$ with range F and use Theorem 1 to extend f_0 to an $\mathbb{S}$-measurable $g: Y \to F$. Then $\{x \in A : f(x) \neq g(x)\} \subset A \setminus f^{-1}(F)$, which has P^*-measure $P^*(A) - P^*f^{-1}(F) = P^*(A) - f(P^*)(F) = 0$.   Q.E.D.

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