HAUSDORFF DIMENSIONS OF ESCAPING SETS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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ABSTRACT. Suppose that f and g are transcendental entire functions, each with a bounded set of singular values, and that $g \circ \varphi = \psi \circ f$, where $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ are affine. We show that the escaping sets of f and g have the same Hausdorff dimension.

Using a result of the second author, we deduce that there exists a family of transcendental entire functions for which the escaping set has Hausdorff dimension equal to one.

1. INTRODUCTION

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. The Julia set $J(f) \subset \mathbb{C}$ is the set of points where the family (f^n) is not equicontinuous (with respect to the spherical metric). For an introduction to the dynamics of transcendental entire functions, see [Be1, Be2].

A number of authors have studied the Hausdorff dimension, $\dim J(f)$, of the Julia set (see [S6] for a survey). Baker [Bak] showed that $J(f)$ contains nontrivial continua for every transcendental entire function f, so in particular $\dim J(f) \geq 1$. Although it is known [S5] that, for each $d \in (1, 2]$, there are transcendental entire functions for which the Hausdorff dimension of the Julia set is equal to d, it is a well-known open question whether the Julia set of a transcendental entire function can have Hausdorff dimension equal to 1.

More is known when we restrict the class of functions under consideration. The *Eremenko-Lyubich class* is defined by

$$\mathcal{B} = \{f : f \text{ is a transcendental entire function for which } \operatorname{sing}(f^{-1}) \text{ is bounded}\},$$

where $\operatorname{sing}(f^{-1})$ consists of the critical and asymptotic values of f. The second author proved in [S3] that

$$(1.1) \qquad \dim J(f) > 1 \quad \text{for } f \in \mathcal{B}.$$

This proof in fact constructs a subset A of the *escaping set*

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

whose closure $\overline{A}$ has Hausdorff dimension greater than one, which yields the desired conclusion due to the fact [EL] that

$$(1.2) \qquad\qquad I(f) \subset J(f) \quad \text{for } f \in \mathcal{B}.$$

It was shown more recently [RS] that the set $I(f) \cap J(f)$ contains nontrivial continua for all transcendental entire functions, and hence

$$(1.3) \qquad\qquad \dim(I(f) \cap J(f)) \geq 1.$$

In view of these results, it seems natural to ask whether the escaping set of a transcendental entire function must have Hausdorff dimension greater than one. We show that this is not the case:

1.1. Theorem (Escaping sets of dimension one). *There exists a function $f \in \mathcal{B}$ such that $\dim I(f) = 1$.*

Combined with previous results, this gives the following complete description of the possible Hausdorff dimensions of escaping sets.

1.2. Corollary (Dimensions of escaping sets). *If f is a transcendental entire function, then $\dim I(f) \in [1,2]$. Conversely, for every $d \in [1,2]$ there exists a function $f \in \mathcal{B}$ with $\dim I(f) = d$. If $d > 1$, this function can be chosen such that $\dim J(f) = d$.*

To prove Theorem 1.1 we consider a function that was studied in [S1]. Let L be the boundary of the region

$$G = \{z : \operatorname{Re}(z) > 0, -\pi < \operatorname{Im}(z) < \pi\},$$

parametrized in the clockwise direction. Then

$$(1.4) \qquad F_0 : \mathbb{C} \setminus \overline{G} \to \mathbb{C}; \quad z \mapsto \frac{1}{2\pi i} \int_L \frac{\exp(e^t)}{t - z} dt$$

can be continued analytically to a transcendental entire function $F_0 : \mathbb{C} \to \mathbb{C}$. From the properties of F_0 given in [S1], it can easily be seen that $F_0 \in \mathcal{B}$.

Consider the family

$$F_\kappa(z) := F_0(z) + \kappa, \quad \kappa \in \mathbb{C}.$$

It is shown in [S1] that

$$(1.5) \qquad\qquad \lim_{\substack{\kappa \to -\infty, \\ \kappa \in \mathbb{R}}} \dim J(F_\kappa) = 1,$$

while, by (1.1), $\dim J(F_\kappa) > 1$ for all $\kappa \in \mathbb{C}$. This implies that $\dim J(F_\kappa)$ is a nonconstant function of κ.

In contrast, we show that the Hausdorff dimension of the escaping set cannot change in a family defined in this manner. More precisely, we say that two transcendental entire functions f and g are *affinely equivalent* if there are affine functions $\varphi, \psi : \mathbb{C} \to \mathbb{C}$ such that

$$(1.6) \qquad\qquad \psi \circ f = g \circ \varphi.$$

Any two functions F_{κ_1} and F_{κ_2}, $\kappa_1, \kappa_2 \in \mathbb{C}$, are clearly affinely equivalent. (Another well-known family consisting of transcendental entire functions that are affinely equivalent to each other is the family $z \mapsto \exp(z) + \kappa$ of exponential maps. Note that, for this family, the Julia sets and escaping sets all have Hausdorff dimension two by a result of McMullen [McM].)

1.3. Theorem (Escaping dimensions and affine equivalence). *Suppose that $f, g \in \mathcal{B}$ are affinely equivalent. Then $\dim I(f) = \dim I(g)$.*

The proof uses recent results of the first author [R] on the rigidity of the dynamics near infinity for a function $f \in \mathcal{B}$. Using (1.1), (1.2) and (1.5), we obtain the following consequence of Theorem 1.3, which implies Theorem 1.1.

1.4. Corollary (Escaping dimension of F_κ). *Let F_0 be the function defined above, and let $g \in \mathcal{B}$ be affinely equivalent to F_0. Then $\dim I(g) = 1 < \dim J(g)$.*

Remarks.

(a) As far as we know, this provides the first example of an analytic family of entire functions for which the dimension of the escaping set is always strictly smaller than that of the Julia set.

(b) We note that if f has finite order, then $I(f)$ has Hausdorff dimension two [Bar, Sch], while for functions of "small" infinite order, a lower bound on the Hausdorff dimension of $I(f)$ is proved in [BKS]. It follows from these results that

$$\limsup_{r \to \infty} \frac{\log\log\log \max_{|z|=r} |f(z)|}{\log\log r} = \infty$$

for any function with $\dim I(f) = 1$. Note that for the maps F_κ we have

$$\log\log\log \max_{|z|=r} |F_\kappa(z)| \approx \log r.$$

(c) For transcendental *meromorphic* functions, the Julia set may have any Hausdorff dimension $d \in (0, 2]$ [S4]. On the other hand, (1.1) was generalized to functions having a *logarithmic singularity* over ∞ in [BRS], and extended in [BKZ] to show that in fact the *hyperbolic dimension* of such a function is strictly larger than one. As far as we know, it is an open question whether the escaping set of a transcendental meromorphic function can have Hausdorff dimension zero; this cannot occur for the Julia set [S2]. Kotus and Urbański [KU] have shown that there exist meromorphic functions with $\dim J(f) > \dim I(f)$; compare also [BK, Theorem 1.2].

(d) It is possible to prove Theorem 1.1 directly, using arguments similar to those in [S1].

To conclude our paper, we consider the notion of the *eventual dimension* of a transcendental entire function f, defined by

$$\text{edim}(f) := \inf_{R>0} \dim J_R(f),$$

where, for each $R > 0$,

(1.7) $$J_R(f) := \{z \in J(f) : |f^n(z)| \geq R \text{ for all } n \geq 1\}.$$

Remark. This eventual dimension is also implicitly used by Bergweiler and Kotus [BK] to obtain an upper bound for the Hausdorff dimension of the escaping set of certain meromorphic functions.

We show that for functions in the Eremenko-Lyubich class, the eventual dimension is an upper bound for $\dim I(f)$ and is also preserved under affine equivalence. This enables us to prove the following result.

1.5. Theorem (Eventual dimensions of entire functions)**.** *For each $d \in [1, 2]$, there exists a function $f \in \mathcal{B}$ such that $\mathrm{edim}(g) = \dim I(g) = d$ for all functions g affinely equivalent to f.*

Open questions. As far as we are aware, for all examples where $\dim I(f)$ and $\mathrm{edim}(f)$ are known, these numbers coincide. (This is also the case for the meromorphic functions considered in [BK, Theorem 1.2].)

1.6. Question. Does there exist a function $f \in \mathcal{B}$ such that $\mathrm{edim}(f) \neq \dim I(f)$?

Following [EL], two entire functions f and g are called *quasiconformally equivalent* if there are quasiconformal functions $\psi, \varphi : \mathbb{C} \to \mathbb{C}$ such that (1.6) holds. Quasiconformal equivalence classes can be considered as natural parameter spaces. It was shown in [R] that, for $f \in \mathcal{B}$, the dynamical behaviour near infinity is the same for any function quasiconformally equivalent to f.

1.7. Question. Suppose that $f, g \in \mathcal{B}$ are quasiconformally but not affinely equivalent. Do $I(f)$ and $I(g)$ have the same Hausdorff dimension?

Remark. It follows from [R] that the answer is "yes" if $\dim I(f) = 2$.

Structure of the article. In Section 2, we use the results of [R] to deduce a fact that will play an important role in our proof of Theorem 1.3, which is itself proved in Section 3. We give the proof of Corollary 1.2 in Section 4 and treat eventual dimension and the proof of Theorem 1.5 in Section 5.

2. Conjugacies on $J_R(f)$

We will require the fact (Corollary 2.2 below) that if two functions $f, g \in \mathcal{B}$ are affinely equivalent, then, for sufficiently large R, they are also quasiconformally conjugate on the set $J_R(f)$, and the dilatation of the conjugacy tends to one as $R \to \infty$. This follows from the ideas of [R] but is not explicitly stated there. We shall provide a proof for completeness, using the following result, which is a special case of [R, Proposition 3.6].

2.1. Proposition (Existence of conjugacies)**.** *Let $f \in \mathcal{B}$. Let $\varphi_\lambda : \mathbb{C} \to \mathbb{C}$, $\lambda \in \mathbb{C}$, be a family of nonconstant affine maps that depend analytically on λ. Also suppose that $\varphi_0 = \mathrm{id}$. We define $f_\lambda := f \circ \varphi_\lambda$.*

Let N be a compact subset of $\mathbb{C}$ with $0 \in N$. Then there exists a constant $R > 0$ such that, for every $\lambda \in N$, there is an injective function $\vartheta = \vartheta^\lambda : J_R(f) \to J(f_\lambda)$ with the following properties:

    (a) *$\vartheta^0 = \mathrm{id}$,*
    (b) *$\vartheta^\lambda \circ f = f_\lambda \circ \vartheta^\lambda$ and*
    (c) *for fixed $z \in J_R(f)$, the function $\lambda \mapsto \vartheta^\lambda(z)$ is analytic in λ (on the interior of N).*

Remark. The conclusion of the theorem says that the injections ϑ^λ form a *holomorphic motion* of $J_R(f)$ on the interior of N in the sense of Mañé, Sad and Sullivan. (Compare [H, Section 5.2].)

We now prove the main result of this section.

2.2. Corollary (Conjugacy near infinity). *Suppose that $f, g \in \mathcal{B}$ are affinely equivalent, and let $K > 1$. Then there exist $R > 0$ and a K-quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that*

$$\vartheta(f(z)) = g(\vartheta(z))$$

for all $z \in J_R(f)$.

Proof. By conjugating g with a Möbius transformation, we may assume for simplicity that $g = f \circ M$, where $M(z) = e^A z + B$ for suitable $A, B \in \mathbb{C}$. Define affine functions $\varphi_\lambda(z) := e^{\lambda A} z + \lambda B$ and consider the family

$$f_\lambda(z) := f(e^{\lambda A} z + \lambda B).$$

Then $f_0 = f$ and $f_1 = g$.

We can apply Proposition 2.1 to $D := (K + 1)/(K - 1)$ and $N := \{\lambda : |\lambda| \leq D\}$ and obtain a number $R > 0$ and maps $\vartheta^\lambda : J_R(f) \to J(f_\lambda)$ satisfying (a) to (c) as above.

It is well-known that a holomorphic motion such as ϑ^λ is quasiconformal as a function of z, and there is a bound on the dilatation of ϑ^λ in terms of the parameter λ. More precisely: if $\lambda \in \mathrm{int}(N)$, then ϑ^λ extends to a K_λ-quasiconformal map $\vartheta^\lambda : \mathbb{C} \to \mathbb{C}$ by [BR, Theorem 1], where

$$K_\lambda \leq \frac{D + |\lambda|}{D - |\lambda|}.$$

In particular, by the definition of D, the map ϑ_1 extends to a K-quasiconformal map $\vartheta_1 : \mathbb{C} \to \mathbb{C}$, as desired. $\square$

3. Proof of Theorem 1.3

Let f be a transcendental entire function. For $R \geq 0$, we defined the set $J_R(f)$ in the introduction; let us also set

$$I_R(f) := \{z \in I(f) : |f^n(z)| \geq R \text{ for all } n \geq 1\}.$$

3.1. Lemma (Dimension and eventual dimension of escaping sets coincide). *Let f be a transcendental entire function and $R > 0$. Then $\dim I(f) = \dim I_R(f)$ and $\dim(I(f) \cap J(f)) = \dim(I_R(f) \cap J(f))$.*

Proof. Every point in $I(f)$ eventually maps to a point in $I_R(f)$, so we have

$$I(f) = \bigcup_{j \geq 0} f^{-j}(I_R(f)),$$

and analogously for $I(f) \cap J(f)$. The conclusion follows from the preservation of Hausdorff dimension under holomorphic functions and the countable stability of Hausdorff dimension. $\square$

Proof of Theorem 1.3. Suppose that $f, g \in \mathcal{B}$ are affinely equivalent. We want to show that $\dim I(f) = \dim I(g)$. By symmetry, it is sufficient to show that $\dim I(g) \geq \dim I(f)$. We shall use the following result, due to Gehring and Väisälä [GV, Theorem 8]: if ϑ is a K-quasiconformal map and $A \subset \mathbb{C}$, then

(3.1) $\dim(\vartheta(A)) \geq \dim \vartheta(A)/K.$

(The optimal bounds on the distortion of Hausdorff dimension under quasiconformal mappings in dimension 2 are given by Astala's distortion theorem [A].)

So, let $K > 1$. It follows from Corollary 2.2 that there exist $R > 0$ and a K-quasiconformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that $\vartheta(f(z)) = g(\vartheta(z))$ for all $z \in J_R(f)$.

For $z \in I_R(f) \subset J_R(f)$, we have

$$g^n(\vartheta(z)) = \vartheta(f^n(z)) \to \infty,$$

and hence, by (3.1) and Lemma 3.1,

$$\dim I(g) \geq \dim \vartheta(I_R(f)) \geq \dim(I_R(f))/K.$$

Since K was chosen arbitrarily close to 1, this completes the proof.   □

4. Proof of Corollary 1.2

The first statement in Corollary 1.2 follows from (1.3). Recall that the second claim is that, for each $d \in [1, 2]$, there exists a function $f \in \mathcal{B}$ for which the Hausdorff dimension of the escaping set is equal to d, and that furthermore if $d > 1$, the function can be chosen such that $\dim J(f) = d$.

In the case $d = 1$, this follows from Corollary 1.4, while for $d = 2$ we can use $f(z) = \exp(z)$ and use McMullen's result [McM] that $\dim I(f) = 2$. (In fact, by [Bar, Sch], any entire function $f \in \mathcal{B}$ of finite order has the desired properties for $d = 2$.)

The remaining cases are covered by the following statement, which is a consequence of the results of [S5] and [BKS].

4.1. Proposition (Escaping sets of dimension between one and two). *For each $d \in (1, 2)$, there exists a family of functions $f_{d,K}$, $K \in \mathbb{R}$, in the class $\mathcal{B}$ such that*

$$\dim J(f_{d,K}) = \dim I(f_{d,K}) = d.$$

Proof. Set $p = (2 - d)/(d - 1)$ and define the family by

$$f_{d,K}(z) := f_p(z) - K \quad \text{for } K \in \mathbb{R},$$

where

$$f_p(z) := \frac{1}{2\pi i} \int_{L_p} \frac{\exp(e^{(\log t)^{1+p}})}{t - z} dt,$$

with L_p being the boundary of the region

$$G_p = \{z = x + iy : |y| \leq \pi x/[(1 + p)(\log(x))^p], \ x \geq 3\},$$

described in the clockwise direction, for $z \in \mathbb{C} \setminus \overline{G_p}$. As with the function F_0 defined in the introduction, f_p can be defined by analytic continuation for $z \in \overline{G_p}$.

It was shown in [S5] that $f_{d,K} \in \mathcal{B}$ and that

$$\dim J(f_{d,K}) = 1 + \frac{1}{1 + p} = d$$

for sufficiently large K. By (1.2), we have (again, for large K),

(4.1) $$\dim I(f_{d,K}) \leq \dim J(f_{d,K}) = d.$$

On the other hand, the following result was proved in [BKS]. Suppose that $f \in \mathcal{B}$, $q \geq 1$, and that, for each $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

(4.2) $$|f(z)| \leq \exp\left(\exp\left((\log|z|)^{q+\varepsilon}\right)\right) \quad \text{for } |z| \geq r_\varepsilon.$$

Then $\dim I(f) \geq 1 + \frac{1}{q}$.

The functions $f_{d,K}$ satisfy the above assumptions for $q = 1 + p = 1/(d-1)$, and hence

$$\dim I(f_{d,K}) \geq 1 + \frac{1}{q} = d$$

for all K. Together with (4.1), this implies that, for large K,

$$\dim I(f_{d,K}) = \dim J(f_{d,K}) = d,$$

as claimed. □

5. EVENTUAL DIMENSION

Recall that the *eventual dimension* of an entire function f was defined in the introduction as

$$\operatorname{edim}(f) = \inf_{R>0} \dim J_R(f).$$

We begin with the following two results.

5.1. Lemma. *Let f be a transcendental entire function. Then*

$$\dim(I(f) \cap J(f)) \leq \operatorname{edim}(f) \leq \dim J(f).$$

Proof. The second inequality holds by definition. On the other hand, by Lemma 3.1, we have

$$\dim(I(f) \cap J(f)) = \dim(I_R(f) \cap J(f)) = \dim(J_R(f) \cap I(f))$$

for all $R \geq 0$. Hence $\dim(I(f) \cap J(f)) \leq \dim J_R(f)$ for all $R \geq 0$, and thus we have $\dim(I(f) \cap J(f)) \leq \operatorname{edim}(f)$, as claimed. □

5.2. Theorem. *If $f, g \in \mathcal{B}$ are affinely equivalent, then $\operatorname{edim}(f) = \operatorname{edim}(g)$.*

Proof. The proof is very similar to the proof of Theorem 1.3. Again, we need only show that $\operatorname{edim}(g) \geq \operatorname{edim}(f)$.

Let $K > 1$. It follows from Corollary 2.2 that there exist $R > 0$ and a K-quasi-conformal map $\vartheta : \mathbb{C} \to \mathbb{C}$ such that

(5.1) $$\vartheta(f(z)) = g(\vartheta(z))$$

for all $z \in J_R(f)$.

Now let $S > 0$ and choose $R' \geq R$ sufficiently large that $|\vartheta(z)| \geq S$ whenever $|z| \geq R'$. Then it follows from (5.1) that $\vartheta(J_{R'}(f)) \subset J_S(g)$. Therefore, by (3.1) and Lemma 3.1,

$$\dim J_S(g) \geq \dim \vartheta(J_{R'}(f)) \geq \dim J_{R'}(f)/K \geq \operatorname{edim}(f)/K.$$

Since K can be chosen arbitrarily close to 1, we see that $\dim J_S(g) \geq \operatorname{edim}(f)$. Finally, since S was arbitrary, it follows that $\operatorname{edim}(g) \geq \operatorname{edim}(f)$, as required. □

We end this section by proving Theorem 1.5.

Proof of Theorem 1.5. Let $d \in [1, 2]$. By Theorems 1.3 and 5.2, we only need to show that there is a function $f \in \mathcal{B}$ with $\dim I(f) = \operatorname{edim}(f) = d$. If $d > 1$, then this follows from Corollary 1.2 together with (1.2) and Lemma 5.1.

For $d = 1$, we again consider the functions F_κ, $\kappa \in \mathbb{C}$, defined in the Introduction. It follows from (1.3), (1.5), Lemma 5.1 and Theorem 5.2 that

$$\operatorname{edim}(F_\kappa) = 1 \quad \text{for all } \kappa \in \mathbb{C}.$$

Also, by Corollary 1.4,

$$\dim I(F_\kappa) = 1 \quad \text{for all } \kappa \in \mathbb{C}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

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