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POSITIVE PERIODIC SOLUTIONS FOR A NONAUTONOMOUS NEUTRAL DELAY PREY-PREDATOR MODEL WITH IMPULSE AND HASSELL-VARLEY TYPE FUNCTIONAL RESPONSE

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Abstract. In this paper, a nonautonomous neutral delay prey-predator model with impulse and Hassell-Varley type functional response is studied. By using the continuation theorem of coincidence degree theory, easily verifiable criteria are established for the existence of positive periodic solutions to the system.

1. Introduction

It is well known that a very basic and important problem in the study of a population model with a periodic environment is the global existence and attractivity of a positive periodic solution. Much progress has been made in this direction (see for example [1]–[7] and the references cited therein). But there is little literature considering the Hassell-Varley type functional response between the predators and prey, which was introduced by Hassell and Varley in 1969 [8]. From their observations, Hassell and Varley introduced and established a Lotka-Volterra predator-prey model with Hassell-Varley type functional response as follows (see [8]):

$$\begin{aligned} x' &= rx(1-\frac{x}{k}) - \frac{cxy}{my^\gamma + x}, \\ y' &= y(-d + \frac{fx}{my^\gamma + x}), \qquad 0<\gamma<1, \end{aligned} \tag{1.1}$$

where γ is called the Hassell-Varley constant. In the typical predator-prey interaction where predators do not form groups, one can assume that $\gamma = 1$, producing the so-called ratio-dependent predator-prey system. For terrestrial predators that form a fixed number of tight groups, it is often reasonable to assume $\gamma = \frac{1}{2}$. For aquatic predators that form a fixed number of tights groups, $\gamma = \frac{1}{3}$ may be more appropriate. For more details on system (1.1), one could refer to [8]–[11] and the references cited therein.

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Recently, Wang [11] proposed the following predator-prey system with Hassell-Varley type functional response:

$$
\begin{aligned}
&N_1'(t) = N_1(t)[a(t) - b(t)N_1(t-\tau(t)) - \frac{c(t)N_2(t)}{mN_2^\gamma(t)+N_1(t)}],\\
(1.2)\qquad &N_2'(t) = N_2(t)[-d(t) + \frac{r(t)N_1(t)}{mN_2^\gamma(t)+N_1(t)}] \quad (0<\gamma<1).
\end{aligned}
$$

Under the assumption that the coefficients of system (1.2) are all continuous ω-periodic functions, by applying Mawhin's continuation theorem they obtained sufficient conditions which guarantee the existence of positive ω-periodic solutions to system (1.2).

Moreover, as Kuang [12] pointed out, it is interesting to investigate the existence of periodic solutions of neutral delay interacting population models, such as the predator-prey or competition system. In 1991, Kuang [13] studied the local stability and oscillation of the following neutral delay Gause type predator-prey system:

$$
\begin{aligned}
&x'(t) = rx(t)[1 - \frac{x(t-\tau)+\rho x'(t-\tau)}{K}] - y(t)p(x(t)),\\
(1.3)\qquad &y'(t) = y(t)[-\alpha + \beta p(x(t-\sigma))].
\end{aligned}
$$

For more details about the neutral delay system, one could refer to [12]–[17] and the references cited therein.

In addition, in population dynamics, many evolutionary processes experience short-time rapid changes after undergoing a relatively long smooth variation. For example, harvesting and stocking occur at fixed moments, and some species usually immigrate at the same time every year, etc. If we still thought of the population dynamic systems with phenomena as continuous systems, it would be unreasonable or incorrect. We should establish systems with impulsive effects. Recently, theories for impulsive differential equations have been introduced into population dynamics [4], [18], [19]. In addition, it is generally recognized that some kind of time delay is inevitable in population interactions.

Motivated by the above works, in this paper, we consider the following nonautonomous neutral delay prey-predator model with impulse and Hassell-Varley type functional response:

(1.4)

$$
\begin{aligned}
N_1'(t) &= N_1(t)[a(t) - b(t)N_1(t-\tau(t)) - g(t)N_1'(t-\tau(t))\\
&\qquad - \frac{c(t)N_2(t)}{m_2(t)N_2^\gamma(t)+m_1(t)N_1(t)}], \qquad t\neq t_k,\\
N_2'(t) &= N_2(t)[-d(t) + \frac{r(t)N_1(t)}{m_2(t)N_2^\gamma(t)+m_1(t)N_1(t)}], \quad t\neq t_k, \quad (0<\gamma<1),\\
\Delta N_1(t_k) &= N_1(t_k^+) - N_2(t_k) = \theta_{1k}N_1(t_k), \qquad k=1,2,\cdots,\\
\Delta N_2(t_k) &= N_2(t_k^+) - N_2(t_k) = \theta_{2k}N_2(t_k), \qquad k=1,2,\cdots,
\end{aligned}
$$

where $\theta_{1k}x(t_k)$ and $\theta_{2k}y(t_k)$ represent the population $x(t)$ and $y(t)$ at t_k regular harvest or stocking pulse. Throughout this paper, for system (1.4), the following conditions are assumed:

(D_1) $0 < t_1 < t_2 < \cdots$ are fixed impulsive points with $\lim_{k\to+\infty} t_k = +\infty, k \in \mathbb{N}$;

(D_2) $\{\theta_{ik}\}$ are real sequences such that $\theta_{ik} > -1$, and $\prod_{0<t_k<t}(1+\theta_{ik})$ are ω-periodic functions, $i = 1, 2, k \in \mathbb{N}$.

In recent years, the powerful and effective method of coincidence degree has been widely applied to deal with the existence of periodic solutions of differential equations and difference equations. However, to the best of our knowledge, there is no result on the existence of positive periodic solutions for the nonautonomous neutral delay prey-predator system (1.4) with impulse and Hassell-Varley type functional response in the literature. To compare with the nonneutral system, it is highly nontrivial to attack the existence of positive periodic solutions to the neutral delay system. Thus, in this paper, by utilizing the coincidence degree theorem, we present some sufficient conditions which guarantee the existence of positive periodic solutions to system (1.4).

The present paper is organized as follows: In the next section we introduce some notations and lemmas. In Section 3, we derive some sufficient conditions which ensure the existence of positive periodic solutions of system (1.4) by applying the continuation theorem of coincidence degree theory and some other techniques. Finally, as an application, we study some special cases of system (1.4).

2. Preliminaries

Consider the nonimpulsive neutral delay differential system

(2.1)
$$\begin{aligned} y_1'(t) &= y_1(t)[a(t) - B(t)y_1(t-\tau(t)) - G(t)y_1'(t-\tau(t)) - \frac{C(t)y_2(t)}{M_2(t)y_2^{\gamma}(t) + M_1(t)y_1(t)}],\\ y_2'(t) &= y_2(t)[-d(t) + \frac{R(t)y_1(t)}{M_2(t)y_2^{\gamma}(t) + M_1(t)y_1(t)}] \qquad (0<\gamma<1), \end{aligned}$$

where

$$\begin{aligned} B(t) &= b(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k}),\ G(t) = g(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k}),\\ C(t) &= c(t)\prod_{0<t_k<t}(1+\theta_{2k}),\\ M_1(t) &= m_1(t)\prod_{0<t_k<t}(1+\theta_{1k}),\ M_2(t) = m_2(t)\prod_{0<t_k<t}(1+\theta_{2k})^{\gamma},\\ R(t) &= r(t)\prod_{0<t_k<t}(1+\theta_{1k}). \end{aligned}$$

The following lemma will be used in the proof of our results.

Lemma 2.1. *Suppose that (D_1) and (D_2) hold. Then:*

(1) *If $(y_1(t), y_2(t))^T$ is a solution of (2.1), then $(N_1(t), N_2(t))^T$ is a solution of (1.4), where $N_i(t) = \prod_{0<t_k<t}(1+\theta_{ik})y_i(t), i = 1, 2$.*

(2) *If $(N_1(t), N_2(t))^T$ is a solution of (1.4), then $(y_1(t), y_2(t))^T$ is a solution of (2.1), where $y_i(t) = \prod_{0<t_k<t}(1+\theta_{ik})^{-1}N_i(t), i = 1, 2$.*

Proof. First, we prove (1). It is easy to see that $N_1(t) = \prod_{0<t_k<t}(1+\theta_{1k})y_1(t)$ and $N_2(t) = \prod_{0<t_k<t}(1+\theta_{2k})y_2(t))$ are absolutely continuous on every interval

$(t_k, t_{k+1}]$ and for any $t \neq t_k, k \in \mathbb{N}$,

$$\begin{aligned}
&N_1'(t) - N_1(t)[a(t) - b(t)N_1(t-\tau(t)) - g(t)N_1'(t-\tau(t)) \\
&\qquad\qquad - \frac{c(t)N_2(t)}{m_2(t)N_2^\gamma(t) + m_1(t)N_1(t)}] \\
&= \prod_{0<t_k<t}(1+\theta_{1k})y_1'(t) - \prod_{0<t_k<t}(1+\theta_{1k})y_1(t)[a(t) - b(t) \\
&\quad \times \prod_{0<t_k<t-\tau(t)}(1+\theta_{1k})y_1(t-\tau(t)) - g(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k})y_1'(t-\tau(t)) \\
&\quad - \frac{c(t)\prod_{0<t_k<t}(1+\theta_{2k})y_2(t)}{m_2(t)\prod_{0<t_k<t}(1+\theta_{2k})^\gamma y_2^\gamma(t) + m_1(t)\prod_{0<t_k<t}(1+\theta_{1k})y_1(t)}] \\
&= \prod_{0<t_k<t}(1+\theta_{1k})\{y_1'(t) - y_1(t)[a(t) - B(t)y_1(t-\tau(t)) - G(t)y_1'(t-\tau(t)) \\
&\quad - \frac{C(t)y_2(t)}{M_2(t)y_2^\gamma(t) + M_1(t)y_1(t)}]\} = 0
\end{aligned}$$

and

$$\begin{aligned}
&N_2'(t) - N_2(t)[-d(t) + \frac{r(t)N_1(t)}{m_2(t)N_2^\gamma(t) + m_1(t)N_1(t)}] \\
&= \prod_{0<t_k<t}(1+\theta_{2k})y_2'(t) - \prod_{0<t_k<t}(1+\theta_{2k})y_2(t)[-d(t) \\
&\quad + \frac{\prod_{0<t_k<t}(1+\theta_{1k})y_1(t)}{m_2(t)\prod_{0<t_k<t}(1+\theta_{2k})^\gamma y_2^\gamma(t) + m_1(t)\prod_{0<t_k<t}(1+\theta_{1k})y_1(t)}] \\
&= \prod_{0<t_k<t}(1+\theta_{2k})\{y_2'(t) - y_2(t)[-d(t) + \frac{R(t)y_1(t)}{M_2(t)y_2^\gamma(t) + M_1(t)y_1(t)}]\} = 0.
\end{aligned}$$

On the other hand, for any $t = t_k, k \in \mathbb{N}$ and $i = 1, 2$,

$$N_i(t_k^+) = \lim_{t \to t_k^+} \prod_{0<t_j<t}(1+\theta_{ij})y_i(t) = \prod_{0<t_j\leqslant t}(1+\theta_{ij})y_i(t)$$

and

$$N_i(t_k) = \prod_{0<t_j<t_k}(1+\theta_{ij})y_i(t_k).$$

Thus

$$N_i(t_k^+) = (1+\theta_{ik})N_i(t_k), \tag{2.2}$$

which implies that $(N_1(t), N_2(t))^T$ is a solution of (1.4).

Next, we prove (2). Since $N_i(t) = \prod_{0<t_k<t} y_i(t), i = 1, 2$, is absolutely continuous on every interval $(t_k, t_{k+1}]$ and in view of (2.2), it follows that for any $k \in \mathbb{N}$,

$$y_i(t_k^+) = \prod_{0<t_j\leqslant t_k}(1+\theta_{ik})^{-1}N_i(t_k^+) = \prod_{0<t_j\leqslant t_k}(1+\theta_{ik})^{-1}N_i(t_k) = y_i(t_k)$$

and

$$y_i(t_k^-) = \prod_{0<t_j\leqslant t_{k-1}}(1+\theta_{ij})^{-1}N_i(t_k^-) = y_i(t_k), i = 1, 2,$$

which implies that $y_i(t), i = 1, 2$, are continuous. It is easy to prove that $y_i(t), i = 1, 2$, are absolutely continuous. Similar to the proof of (1), we can check that $(y_1(t), y_2(t))^T$ is a solution of (2.1). The proof of Lemma 2.1 is completed. □

From Lemma 2.1, if we want to discuss the existence and global asymptotic stability of positive periodic solutions to the system (1.4), we only need to discuss the existence and global asymptotic of positive periodic solutions to the system (2.1).

In order to obtain the existence of positive periodic solutions to system (2.1), and for the reader's convenience, in the following we summarize a few concepts and results that will be basic for this section.

Let X, Z be real Banach spaces; let $L : \text{Dom}\, L \subset X \to Z$ be a linear mapping and $N : X \to Z$ a continuous mapping. The mapping L is called a Fredholm mapping of index zero if dim $\text{Ker}L = \text{codim Im}L < +\infty$ and $\text{Im}L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projects $P : X \to X$ and $Q : Z \to Z$ such that $\text{Im}P = \text{Ker}L, \text{Ker}Q = \text{Im}L, X = \text{Ker}L \oplus \text{Ker}P$ and $Z = \text{Im}L \oplus \text{Im}Q$, then the restriction L_P of L to $\text{Dom}L \cap \text{Ker}P$ is one-to-one and onto $\text{Im}L$, so that its (algebraic) inverse K_p: $\text{Im}L \to \text{Dom}L \cap \text{Ker}P$ is defined. Let Ω be an open bounded subset of X; the mapping N is called $L-$compact on $\overline{\Omega}$ if $QN : \overline{\Omega} \to Z$ and $K_P(I - Q)N : \overline{\Omega} \to X$ are compact. Since $\text{Im}Q$ is isomorphic to $\text{Ker}L$, there exists an isomorphism J: $\text{Im}Q \to \text{Ker}L$.

Lemma 2.2 ([20]). *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and $N : X \to Z$ be a continuous operator which is $L-$compact on $\overline{\Omega}$. Assume that*

(a) *for each $\lambda \in (0, 1), x \in \partial\Omega \cap \, DomL, Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap KerL, QNx \neq 0$;*
(c) $\deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap DomL$.

Lemma 2.3 ([18]). *Suppose that $g \in PC^1_\omega = \{x : x \in C^1(\mathbb{R}, \mathbb{R}), x(t + \omega) \equiv x(t)\}$; then*

$$0 \leqslant \max_{s\in[0,\omega]} g(s) - \min_{s\in[0,\omega]} g(s) \leqslant \frac{1}{2}\int_0^\omega |g'(s)|ds.$$

For convenience, we shall introduce the notations

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(s)ds, \quad f^l = \min_{s\in[0,\omega]} f(s), \quad f^u = \max_{s\in[0,\omega]} f(s),$$

where $f(t)$ is a ω-periodic function. The following two numbers are also needed:

$$H_1 = \ln[2\overline{a}(\frac{1-\tau'}{B-\Psi'})^u] + \frac{2\overline{a}\Psi^u}{(B-\Psi')^l} + \frac{1}{2}\omega(\overline{|a|} + \overline{a}),$$
$$H_2 = \frac{1}{\gamma}\ln\frac{(\overline{R} - \overline{d}M_1^l)e^{H_1}}{M_2^l} + \frac{1}{2}\omega(\overline{|d|} + \overline{d}).$$

3. Existence of positive periodic solutions

In this section, we investigate the existence conditions of periodic solutions for system (2.1). First, we always make the following assumptions for the system (2.1):

(D_3) $a, d \in C(\mathbb{R}, \mathbb{R})$, $B, C, R, M_1, M_2 \in C(\mathbb{R}, [0, +\infty))$ are all ω-periodic functions. In addition $\overline{a} > 0$, $\overline{d} > 0$.

(D_4) $B(t) > \Psi'(t)$, where $\Psi(t) = \frac{G(t)}{1-\tau'(t)}$ and $\tau'(t) < 1$, $G \in C^1(\mathbb{R}, [0, +\infty))$ and $\tau \in C^2(\mathbb{R}, \mathbb{R})$ are ω-periodic functions.

(D_5) $\overline{R} > \overline{d}M_1^u$.

(D_6) $1 - G^u e^{H_1} > 0$.

(D_7) $\overline{a} > \overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2}$.

Our main results are stated in the following theorems.

Theorem 3.1. *Assume that the conditions $(D_3) - (D_7)$ hold. Then (2.1) has at least one ω-periodic solution.*

Proof. Let $y_1(t) = e^{u_1(t)}, y_2(t) = e^{u_2(t)}$. Then system (2.1) becomes

$$
\begin{aligned}
u_1'(t) &= a(t) - B(t)e^{u_1(t-\tau(t))} - G(t)e^{u_1(t-\tau(t))}u_1'(t-\tau(t)) \\
&\quad - \frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}, \\
u_2'(t) &= -d(t) + \frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}} \qquad (0 < \gamma < 1).
\end{aligned} \tag{3.1}
$$

It is easy to see that if system (3.1) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^T$, then $(y_1^*(t), y_2^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)})^T$ is a positive ω-periodic solution of (2.1). Therefore, to complete the proof, we only need to prove that (3.1) has at least one ω-periodic solution.

Take

$$X = \{u = (u_1(t), u_2(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2) : u_i(t+\omega) = u_i(t), t \in \mathbb{R}, i = 1, 2\}$$

and

$$Z = \{u = (u_1(t), u_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : u_i(t+\omega) = u_i(t), t \in \mathbb{R}, i = 1, 2\}$$

and define

$$|u|_\infty = \max_{t\in[0,\omega]}\{|u_1(t)| + |u_2(t)|\}, \quad ||u|| = |u|_\infty + |u'|_\infty.$$

Then X and Z are Banach spaces when they are endowed with the norms $||\cdot||$ and $|\cdot|_\infty$, respectively. Let $L : X \to Z$ and $N : X \to Z$ be

$$Lu = L(u_1(t), u_2(t))^T = (u_1'(t), u_2'(t))^T$$

and

$$Nu = N(u_1(t), u_2(t))^T = (\Gamma_{1,u}(t), \Gamma_{2,u}(t))^T,$$

where

$$
\begin{aligned}
\Gamma_{1,u}(t) &= a(t) - B(t)e^{u_1(t-\tau(t))} - G(t)e^{u_1(t-\tau(t))}u_1'(t-\tau(t)) \\
&\quad - \frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}, \\
\Gamma_{2,u}(t) &= -d(t) + \frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}.
\end{aligned}
$$

With these notations, system (3.1) can be written in the form

$$Lu = Nu, \quad u \in X.$$

Evidently

$$\begin{aligned}\mathrm{Ker}L &= \{u : u = (u_1(t), u_2(t))^T \equiv h \in \mathbb{R}^2, t \in [0, \omega]\},\\ \mathrm{Im}L &= \{(u_1(t), u_2(t))^T \in Z : \int_0^\omega u_i(s)ds = 0, i = 1, 2\},\end{aligned}$$

and $\dim\mathrm{Ker}L = 2 = \mathrm{codim}\mathrm{Im}L$. Hence, $\mathrm{Im}L$ is closed in Z, and L is a Fredholm mapping of index zero. Define

$$Pu = P(u_1(t), u_2(t))^T = \frac{1}{\omega}\int_0^\omega u(t)dt = (\frac{1}{\omega}\int_0^\omega u_1(t)dt, \frac{1}{\omega}\int_0^\omega u_2(t)dt)^T \quad (\forall u = (u_1(t), u_2(t))^T \in X),$$

$$Qu = Q(u_1(t), u_2(t))^T = \frac{1}{\omega}\int_0^\omega u(t)dt = (\frac{1}{\omega}\int_0^\omega u_1(t)dt, \frac{1}{\omega}\int_0^\omega u_2(t)dt)^T \quad (\forall u = (u_1(t), u_2(t))^T \in Z).$$

It is easy to verify that P and Q are two continuous projections such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : \mathrm{Dom}L \cap \mathrm{Ker}P \to \mathrm{Im}\, L$ is invertible, and the generalized inverse $K_P :\ \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ can be written as

$$K_P(u) = \int_0^t u(s)ds - \frac{1}{\omega}\int_0^\omega\int_0^t u(s)dsdt.$$

Then $QN : X \to Z$ and $K_p(I - Q)N : X \to X$ read

$$QNu = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega [a(t) - B(t)e^{u_1(t-\tau(t))} - \Psi'(t)e^{u_1(t-\tau(t))} \\ \qquad -\frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}]dt \\ \frac{1}{\omega}\int_0^\omega [-d(t) + \frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}]dt \end{bmatrix},$$

$$K_p(I-Q)Nu = \begin{bmatrix} \int_0^t [a(s) - B(s)e^{u_1(s-\tau(s))} - \Psi'(s)e^{u_1(s-\tau(s))} \\ \qquad -\frac{C(s)e^{u_2(s)}}{M_2(s)e^{\gamma u_2(s)} + M_1(s)e^{u_1(s)}}]ds \\ \qquad -\Psi(t)e^{u_1(t-\tau(t))} + \Psi(0)e^{u_1(-\tau(0))} \\ \int_0^t [-d(s) + \frac{R(s)e^{u_1(s)}}{M_2(s)e^{\gamma u_2(s)} + M_1(s)e^{u_1(s)}}]ds \end{bmatrix}$$

$$- \begin{bmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t [a(s) - B(s)e^{u_1(s-\tau(s))} - \Psi'(s)e^{u_1(s-\tau(s))} \\ \qquad -\frac{C(s)e^{u_2(s)}}{M_2(s)e^{\gamma u_2(s)} + M_1(s)e^{u_1(s)}}]dsdt \\ \qquad -\frac{1}{\omega}\int_0^\omega [\Psi(t)e^{u_1(t-\tau(t))} - \Psi(0)e^{u_1(-\tau(0))}]dt \\ \frac{1}{\omega}\int_0^\omega\int_0^t [-d(s) + \frac{R(s)e^{u_1(s)}}{M_2(s)e^{\gamma u_2(s)} + M_1(s)e^{u_1(s)}}]dsdt \end{bmatrix}$$

$$-\begin{bmatrix} (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega [a(s)-B(s)e^{u_1(s-\tau(s))}-\Psi'(s)e^{u_1(s-\tau(s))} \\ \qquad -\frac{C(s)e^{u_2(s)}}{M_2(s)e^{\gamma u_2(s)}+M_1(s)e^{u_1(s)}}]ds \\ (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega [-d(s)+\frac{R(s)e^{u_1(s)}}{M_2(s)e^{\gamma u_2(s)}+M_1(s)e^{u_1(s)}}]ds \end{bmatrix}.$$

Obviously, QN and $K_P(I-Q)N$ are continuous by the Lebesgue theorem. By using the Arzela-Ascoli theorem, it is not difficult to prove that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are compact for any open bounded set $\Omega\subset X$. Therefore, N is $L-$compact on $\overline{\Omega}$ for any open bounded set $\Omega\subset X$.

Now we reach the position where we search for an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.2). Corresponding to the operator equation $Lu=\lambda Nu,\ \lambda\in(0,1)$, we have

$$\begin{aligned} u_1'(t) &= \lambda[a(t)-B(t)e^{u_1(t-\tau(t))}-G(t)e^{u_1(t-\tau(t))}u_1'(t-\tau(t)) \\ &\qquad -\frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}], \\ u_2'(t) &= \lambda[-d(t)+\frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}] \qquad (0<\gamma<1). \end{aligned} \tag{3.2}$$

Assume that $x=x(t)\in X$ is a solution of system (3.2) for a certain $\lambda\in(0,1)$. By integrating system (3.2) over the interval $[0,\omega]$, we can derive

$$\int_0^\omega [a(t)-B(t)e^{u_1(t-\tau(t))}-G(t)e^{u_1(t-\tau(t))}u_1'(t-\tau(t)) -\frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}]dt=0, \tag{3.3}$$

$$\int_0^\omega [-d(t)+\frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}]dt=0. \tag{3.4}$$

Note that

$$\begin{aligned} \int_0^\omega G(t)e^{u_1(t-\tau(t))}u_1'(t-\tau(t))dt &= \int_0^\omega \frac{G(t)}{1-\tau'(t)}(e^{u_1(t-\tau(t))})'dt \\ &= [\Psi(t)e^{u_1(t-\tau(t))}]_0^\omega - \int_0^\omega \Psi'(t)e^{u_1(t-\tau(t))}dt \\ &= -\int_0^\omega \Psi'(t)e^{u_1(t-\tau(t))}dt, \end{aligned}$$

which, together with (3.3), yields

$$\int_0^\omega [(B(t)-\Psi'(t))e^{u_1(t-\tau(t))}+\frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}]dt=\omega\overline{a}. \tag{3.5}$$

From (3.2)–(3.4), it follows that

(3.6)
$$\begin{aligned}\int_0^\omega |\frac{d}{dt}[u_1(t)+\lambda\Psi(t)e^{u_1(t-\tau(t))}]|dt \\ &= \lambda\int_0^\omega |a(t)-(B(t)-\Psi'(t))e^{u_1(t-\tau(t))}-\frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}|dt \\ &\leqslant \omega(\overline{|a|}+\overline{a}),\end{aligned}$$

(3.7)
$$\begin{aligned}\int_0^\omega |u_2'(t)|dt &= \lambda\int_0^\omega |-d(t)+\frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}|dt \\ &\leqslant \omega(\overline{|d|}+\overline{d}).\end{aligned}$$

Let $t=v(s)$ be the inverse function of $s=t-\tau(t)$. It is easy to see that $B(v(s)), \Psi'(v(s))$ and $\tau'(v(s))$ are all ω-periodic functions. Further, it follows from (3.5) that

$$\begin{aligned}\omega\overline{a} &\geqslant \int_0^\omega (B(t)-\Psi'(t))e^{u_1(t-\tau(t))}dt = \int_{-\tau(0)}^{\omega-\tau(\omega)} \frac{(B(v(s))-\Psi'(v(s)))e^{u_1(s)}}{1-\tau'(v(s))}ds \\ &= \int_0^\omega \frac{(B(v(s))-\Psi'(v(s)))e^{u_1(s)}}{1-\tau'(v(s))}ds = \int_0^\omega \frac{(B(v(t))-\Psi'(v(t)))e^{u_1(t)}}{1-\tau'(v(t))}dt,\end{aligned}$$

which, together with (3.5), gives

$$2\omega\overline{a} \geqslant \int_0^\omega [\frac{(B(v(t))-\Psi'(v(t)))e^{u_1(t)}}{1-\tau'(v(t))}+(B(t)-\Psi'(t))e^{u_1(t-\tau(t))}]dt.$$

According to the mean value theorem of differential calculus, there exists $\zeta\in[0,\omega]$ such that

$$\frac{(B(v(\zeta))-\Psi'(v(\zeta)))e^{u_1(\zeta)}}{1-\tau'(v(\zeta))}+(B(\zeta)-\Psi'(\zeta))e^{u_1(\zeta-\tau(\zeta))} \leqslant 2\overline{a},$$

which implies

$$u_1(\zeta) \leqslant \ln[2\overline{a}(\frac{1-\tau'}{B-\Psi'})^u], \tag{3.8}$$

$$e^{u_1(\zeta-\tau(\zeta))} \leqslant \frac{2\overline{a}}{(B-\Psi')^l}. \tag{3.9}$$

It follows from (3.6), (3.8), (3.9) and Lemma 2.3 that, for any $t\in[0,\omega]$,

$$\begin{aligned}u_1(t)+\lambda\Psi(t)e^{u_1(t-\tau(t))} &\leqslant u_1(\zeta)+\lambda\Psi(\zeta)e^{u_1(\zeta-\tau(\zeta))} \\ &\quad +\frac{1}{2}\int_0^\omega |\frac{d}{dt}[u_1(t)+\lambda\Psi(t)e^{u_1(t-\tau(t))}]|dt \\ &\leqslant \ln[2\overline{a}(\frac{1-\tau'}{B-\Psi'})^u]+\frac{2\overline{a}\Psi^u}{(B-\Psi')^l}+\frac{1}{2}\omega(\overline{|a|}+\overline{a}) = H_1.\end{aligned}$$

Since $\lambda\Psi(t)e^{u_1(t-\tau(t))}\geqslant 0$, one can get

$$u_1(t) \leqslant H_1, \quad t\in[0,\omega]. \tag{3.10}$$

Noting that $u = (u_1(t), u_2(t))^T \in X$, then there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t) \quad \text{and} \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \qquad i = 1, 2. \tag{3.11}$$

Since the function $\dfrac{e^{u_1}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}$ is increasing on u_1, by (3.10), (3.11) and (3.4), we obtain

$$\frac{\omega \overline{R} e^{H_1}}{M_2^l e^{\gamma u_2(\eta_2)} + M_1^l e^{H_1}} \geqslant \int_0^\omega \frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}} dt = \omega \overline{d},$$

i.e.

$$u_2(\eta_2) \leqslant \frac{1}{\gamma} \ln \frac{(\overline{R} - \overline{d} M_1^l)e^{H_1}}{M_2^l},$$

which, together with (3.7) and Lemma 2.3, implies

$$\begin{aligned} u_2(t) &\leqslant u_2(\eta_2) + \frac{1}{2}\int_0^\omega |u_2'(t)|dt \\ &\leqslant \frac{1}{\gamma} \ln \frac{(\overline{R} - \overline{d} M_1^l)e^{H_1}}{M_2^l} + \frac{1}{2}\omega(\overline{|d|} + \overline{d}) = H_2. \end{aligned} \tag{3.12}$$

In addition, in view of (3.2), (3.10) and (3.12), for any $t \in [0, \omega]$, we have

$$\begin{aligned} |u_1'(t)| &= \lambda |a(t) - B(t)e^{u_1(t-\tau(t))} - G(t)e^{u_1(t-\tau(t))}u_1'(t - \tau(t)) \\ &\qquad - \frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}}| \\ &\leqslant a^u + B^u e^{H_1} + G^u e^{H_1}|u_1'|^u + (\frac{C}{M_2})^u e^{(1-\gamma)H_2}, \end{aligned}$$

which implies

$$|u_1'|^u \leqslant \frac{1}{1 - G^u e^{H_1}}[a^u + B^u e^{H_1} + (\frac{C}{M_2})^u e^{(1-\gamma)H_2}] \triangleq H_3. \tag{3.13}$$

In addition, from (3.5), (3.11) and (3.12), we get

$$\begin{aligned} \omega(\overline{B} - \overline{\Psi'})e^{u_1(\xi_1)} &\geqslant \omega\overline{a} - \int_0^\omega \frac{C(t)e^{u_2(t)}}{M_2(t)e^{\gamma u_2(t)} + M_1(t)e^{u_1(t)}} dt \\ &\geqslant \omega\overline{a} - \omega\overline{(\frac{C}{M_2})} e^{(1-\gamma)H_2}, \end{aligned}$$

that is

$$u_1(\xi_1) \geqslant \ln[\overline{a} - \overline{(\frac{C}{M_2})} e^{(1-\gamma)H_2}] - \ln(\overline{B} - \overline{\Psi'}),$$

which, together with (3.13) and Lemma 2.3, implies

$$\begin{aligned} u_1(t) &\geqslant u_1(\xi_1) - \frac{1}{2}\int_0^\omega |u_1'(t)|dt \\ &\geqslant \ln[\overline{a} - \overline{(\frac{C}{M_2})} e^{(1-\gamma)H_2}] - \ln(\overline{B} - \overline{\Psi'}) - \frac{1}{2}\omega H_3 \triangleq H_4. \end{aligned} \tag{3.14}$$

Since the function $\dfrac{e^{u_1}}{me^{\gamma u_2}+e^{u_1}}$ is increasing on u_1, by (3.14), (3.7) and (3.4), we obtain

$$\frac{\omega\overline{R}e^{H_4}}{M_2^u e^{\gamma u_2(\xi_2)}+M_1^u e^{H_4}} \leqslant \int_0^\omega \frac{R(t)e^{u_1(t)}}{M_2(t)e^{\gamma u_2(t)}+M_1(t)e^{u_1(t)}}dt=\omega\overline{d},$$

i.e.

$$u_2(\xi_2) \geqslant \frac{1}{\gamma}\ln\frac{(\overline{R}-\overline{d}M_1^u)e^{H_4}}{M_2^u}, \tag{3.15}$$

which, together with (3.6) and Lemma 2.3, implies

$$\begin{aligned} u_2(t) &\geqslant u_2(\xi_2)-\frac{1}{2}\int_0^\omega |u_2'(t)|dt \\ &\geqslant \frac{1}{\gamma}\ln\frac{(\overline{R}-\overline{d}M_1^u)e^{H_4}}{M_2^u}-\frac{1}{2}\omega(\overline{|d|}+\overline{d}) \triangleq H_5. \end{aligned} \tag{3.16}$$

In addition, in view of (3.2), for any $t\in[0,\omega]$, we have

$$|u_2'|^u \leqslant d^u+(\frac{R}{M_1})^u \triangleq H_6. \tag{3.17}$$

It follows from (3.10), (3.12)–(3.14), (3.16)–(3.17) that we have

$$||u|| \leqslant |H_1|+|H_2|+|H_3|+|H_4|+|H_5|+|H_6| \triangleq H_0. \tag{3.18}$$

Obviously, the H_0's are independent of λ.

Consider the following algebraic equations:

$$\begin{aligned} &\overline{a}-(\overline{B}-\overline{\Psi'})e^{u_1}-\frac{1}{\omega}\int_0^\omega \frac{C(t)e^{u_2}}{M_2(t)e^{\gamma u_2}+M_1(t)e^{u_1}}dt=0, \\ &\frac{1}{\omega}\int_0^\omega \frac{R(t)e^{u_1}}{M_2(t)e^{\gamma u_2}+M_1(t)e^{u_1}}dt=\overline{d}. \end{aligned} \tag{3.19}$$

If system (3.19) has a solution or a number of solutions $u^*=(u_1^*,u_2^*)^T$, then a similar argument to (3.10), (3.12) and (3.14), (3.16) shows that

$$\begin{aligned} u_1^* &\leqslant \ln\frac{a}{\overline{B}-\overline{\Psi'}} \leqslant H_1, \\ u_2^* &\leqslant \frac{1}{\gamma}\ln\frac{(\overline{R}-\overline{d}M_1^l)e^{H_1}}{M_2^l} \leqslant H_2, \\ u_1^* &\geqslant \ln[\overline{a}-\overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2}]-\ln(\overline{B}-\overline{\Psi'}) \geqslant H_4, \\ u_2^* &\geqslant \frac{1}{\gamma}\ln\frac{(\overline{R}-\overline{d}M_1^u)e^{H_4}}{M_2^u} \geqslant H_5. \end{aligned}$$

Hence

$$||u^*||=||(u_1^*,u_2^*)^T||=\max\{|u_1^*|,|u_2^*|\}<H_0. \tag{3.20}$$

Set $\Omega=\{u=(u_1,u_2)^T\in X:||u||<H_0\}$. Then, $Lu\neq\lambda Nu$ for $u\in\partial\Omega$ and $\lambda\in(0,1)$; that is, Ω verifies requirement (a) of Lemma 2.2.

For $u \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}^2$ with $||u|| = H_0$, if system (3.19) has a solution or a number of solutions, from (3.20) we obtain

$$QNu = \left[\begin{array}{l} \overline{a} - (\overline{B} - \overline{\Psi'})e^{u_1} - \frac{1}{\omega}\int_0^{\omega} \frac{C(t)e^{u_2}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt \\ \frac{1}{\omega}\int_0^{\omega} \frac{R(t)e^{u_1}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt - \overline{d} \end{array} \right] \neq 0.$$

For $u \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}^2$ with $||u|| = H_0$, if system (3.19) does not contain a solution, then naturally

$$QNu = \left[\begin{array}{l} \overline{a} - (\overline{B} - \overline{\Psi'})e^{u_1} - \frac{1}{\omega}\int_0^{\omega} \frac{C(t)e^{u_2}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt \\ \frac{1}{\omega}\int_0^{\omega} \frac{R(t)e^{u_1}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt - \overline{d} \end{array} \right] \neq 0.$$

Thus, condition (b) in Lemma 2.2 is satisfied.

To complete the proof, we will prove that condition (c) of Lemma 2.2 is satisfied. We define the mapping $\phi : \mathrm{Dom}L \times [0, 1] \to X$ by

$$\phi(u_1, u_2, \mu) = \left[\begin{array}{l} \overline{a} - (\overline{B} - \overline{\Psi'})e^{u_1} - \overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2} \\ \frac{\overline{R}e^{H_4}}{M_2^u e^{\gamma u_2} + M_1^u e^{H_4}} - \overline{d} \end{array} \right]$$

$$+ \mu \left[\begin{array}{l} \overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2} - \frac{1}{\omega}\int_0^{\omega} \frac{C(t)e^{u_2}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt \\ \frac{1}{\omega}\int_0^{\omega} \frac{R(t)e^{u_1}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt - \frac{\overline{R}e^{H_4}}{M_2^u e^{\gamma u_2} + M_1^u e^{H_4}} \end{array} \right],$$

where $\mu \in [0, 1]$ is a parameter. We will show that if $u = (u_1, u_2)^T \in \partial\Omega \cap \mathrm{Ker}L$, $u = (u_1, u_2)^T$ is a constant vector in $\mathbb{R}^2$ with $\max\{|u_1|, |u_2|\} = H_0$, then $\phi(u_1, u_2, \mu) \neq 0$. Otherwise, suppose that $u = (u_1, u_2)^T \in \mathbb{R}^2$ with $\max\{|u_1|, |u_2|\} = H_0$ satisfying $\phi(u_1, u_2, \mu) = 0$; that is,

$$\overline{a} - (\overline{B} - \overline{\Psi'})e^{u_1} - \overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2} + \mu[\overline{(\frac{C}{M_2})}e^{(1-\gamma)H_2}$$
$$- \frac{1}{\omega}\int_0^{\omega} \frac{C(t)e^{u_2}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt] = 0,$$
$$\frac{\overline{R}e^{H_4}}{M_2^u e^{\gamma u_2} + M_1^u e^{H_4}} - \overline{d} + \mu[\frac{1}{\omega}\int_0^{\omega} \frac{R(t)e^{u_1}}{M_2(t)e^{\gamma u_2} + M_1(t)e^{u_1}}dt - \frac{\overline{R}e^{H_4}}{M_2^u e^{\gamma u_2} + M_1^u e^{H_4}}] = 0.$$

Similarly to the arguments of (3.19), (3.20) shows that

$$||u|| = \max\{|u_1|, |u_2|\} \leqslant H_0,$$

which is a contradiction.

Thus, by the property of topological degree and taking J, we obtain the identity mapping

$$\begin{aligned} \deg\{JQN, \Omega \cap \mathrm{Ker}L, (0, 0)^T\} &= \deg\{\phi(u_1, u_2, 1), \Omega \cap \mathrm{Ker}L, (0, 0)^T\} \\ &= \deg\{\phi(u_1, u_2, 0), \Omega \cap \mathrm{Ker}L, (0, 0)^T\} \neq 0. \end{aligned}$$

Now we have proved that Ω verifies all the requirements in Lemma 2.2. Hence, $Lu = Nu$ has at least one solution, $u^*(t) = (u_1^*(t), u_2^*(t))^T$, in $\mathrm{Dom}L \cap \overline{\Omega}$. That is to say, system (2.1) has at least one positive ω-periodic solution. Accordingly, system (1.4) has at least one ω-periodic solution with strictly positive components. The proof of Theorem 3.1 is finished. □

Remark 1. After the above discussion, one can easily find that Theorem 3.1 is true for the following general delayed system:

(3.21)

$$\begin{aligned} y_1'(t) &= y_1(t)[a(t) - B(t)y_1(t-\tau(t)) - G(t)y_1'(t-\tau(t)) \\ &\qquad - \frac{C(t)y_2(t-\sigma(t))}{M_2(t)y_2^\gamma(t-\sigma(t)) + M_1(t)y_1(t-\sigma(t))}], \\ y_2'(t) &= y_2(t)[-d(t) + \frac{R(t)y_1(t-\sigma(t))}{M_2(t)y_2^\gamma(t-\sigma(t)) + M_1(t)y_1(t-\sigma(t))}] \qquad (0<\gamma<1), \end{aligned}$$

where

$$\begin{aligned} &B(t) = b(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k}),\ G(t) = g(t)\prod_{0<t_k<t-\tau(t)}(1+\theta_{1k}), \\ &C(t) = c(t)\prod_{0<t_k<t}(1+\theta_{2k}), \\ &M_1(t) = m_1(t)\prod_{0<t_k<t-\sigma(t)}(1+\theta_{1k}),\ M_2(t) = m_2(t)\prod_{0<t_k<t-\sigma(t)}(1+\theta_{2k})^\gamma, \\ &R(t) = r(t)\prod_{0<t_k<t}(1+\theta_{1k}). \end{aligned}$$

In addition, one can easily find that time delays $\tau(t)$ and $\sigma(t)$ do not necessarily remain nonnegative. Moreover, Theorem 3.1 will remain valid for systems (2.1) and (3.21) if the delayed terms are replaced by a term with discrete time delays, state-dependent delays, or deviating argument. Time delays of any type or deviating arguments have no effect on the existence of positive periodic solutions.

Remark 2. From the proof of Theorem 3.1, we see that Theorem 3.1 is also valid if $g(t) \equiv 0$. Consequently, we can obtain the following corollary.

Corollary 3.1. *Assume that conditions* (D_3), $(D_5), (D_7)$ *hold. Then the following delayed prey-predator model with impulse and Hassell-Varley type functional response*

(3.22)

$$\begin{aligned} N_1'(t) &= N_1(t)[a(t) - b(t)N_1(t-\tau(t)) - \frac{c(t)N_2(t)}{m_2(t)N_2^\gamma(t) + m_1(t)N_1(t)}], \qquad t \neq t_k, \\ N_2'(t) &= N_2(t)[-d(t) + \frac{r(t)N_1(t)}{m_2(t)N_2^\gamma(t) + m_1(t)N_1(t)}], \quad t \neq t_k \quad (0<\gamma<1), \\ \Delta N_1(t_k) &= N_1(t_k^+) - N_2(t_k) = \theta_{1k}N_1(t_k), \qquad k = 1, 2, \cdots, \\ \Delta N_2(t_k) &= N_2(t_k^+) - N_2(t_k) = \theta_{2k}N_2(t_k), \qquad k = 1, 2, \cdots, \end{aligned}$$

has at least one ω*-periodic solution with strictly positive components.*

If the impulse terms θ_{ik} $(k = 1, 2, \cdots)$ vanish and $m_2(t) \equiv m, m_1(t) \equiv 1$, then system (3.22) reduces to system (1.2), which was studied by Wang in [11]. Thus, from Corollary 3.1, for system (1.2), we have the following result.

Corollary 3.2. *Assume that the following conditions hold:*

(D_3^*) $a, d \in C(\mathbb{R}, \mathbb{R})$, $b, c, r \in C(\mathbb{R}, [0, +\infty))$ *are all* ω*-periodic functions. In addition* $\overline{a} > 0$, $\overline{d} > 0$ *and* m *is a positive constant;*

(D_5^*) $\overline{r} > \overline{d}$;

(D_7^*) $\overline{a} > \frac{\overline{c}}{m} e^{(1-\gamma)H_2^*}$, *where* $H_2^* = \frac{1}{\gamma} \ln \frac{(\overline{r} - \overline{d})e^{H_1^*}}{m} + \frac{1}{2}\omega(\overline{|d|} + \overline{d})$, $H_1^* = \ln[2\overline{a}(\frac{1-\tau'}{B})^u] + \frac{1}{2}\omega(\overline{|a|} + \overline{a})$.

Then system (1.2) *has at least one* ω*-periodic solution with strictly positive components.*

In [11], Wang obtained the following result.

Theorem A (Theorem 3.1 of [11])**.** *Assume the following three assumptions hold:*

(C1) $\tau'(t) < 1$ *for* $t \in \mathbb{R}$.

(C2) $\overline{r} > \overline{d}$, $m\overline{a} > \overline{c}$.

(C3) *The algebraic equation set* $(*) \triangleq \{\overline{a} - \overline{b}u - \frac{\overline{c}v}{mv^\gamma + u} = 0, -\overline{d} + \frac{\overline{r}u}{mv^\gamma + u} = 0\}$ *has a finite number of real-valued positive solutions.*

Then system (1.2) *has at least one positive periodic solution.*

Remark 3. Compared to the corresponding result (Theorem 3.1 of [11]), it is easy to see that (D_5^*) is just the first inequality of (C2) and that (D_7^*) is different from the second inequality of (C2). It is worthwhile to point out that from Corollary 3.2 of this paper, we can easily see that condition (C3) in paper [11] is superfluous and could be removed.

Next consider the following neutral state-dependent delayed prey-predator model with impulse and Hassell-Varley type functional response:

$$
\begin{aligned}
N_1'(t) &= N_1(t)[a(t) - b(t)N_1(t - \tau(t, N_1(t), N_2(t))) - g(t)N_1'(t - \tau(t, N_1(t), N_2(t))) \\
&\quad - \frac{c(t)N_2(t - \sigma(t, N_1(t), N_2(t)))}{m_2(t)N_2^\gamma(t - \sigma(t, N_1(t), N_2(t))) + m_1(t)N_1(t - \sigma(t, N_1(t), N_2(t)))}], \quad t \neq t_k, \\
N_2'(t) &= N_2(t)[-d(t) \\
&\quad + \frac{r(t)N_1(t - \sigma(t, N_1(t), N_2(t)))}{m_2(t)N_2^\gamma(t - \sigma(t, N_1(t), N_2(t))) + m_1(t)N_1(t - \sigma(t, N_1(t), N_2(t)))}], \\
&\qquad t \neq t_k, \\
\Delta N_1(t_k) &= N_1(t_k^+) - N_2(t_k) = \theta_{1k}N_1(t_k), \qquad k = 1, 2, \cdots, \\
\Delta N_2(t_k) &= N_2(t_k^+) - N_2(t_k) = \theta_{2k}N_2(t_k), \qquad k = 1, 2, \cdots \quad (0 < \gamma < 1),
\end{aligned} \tag{3.23}
$$

where $\tau(t, N_1, N_2)$ and $\sigma(t, N_1, N_2)$ are continuous functions and ω-periodic functions with respect to t.

Theorem 3.2. *Assume that conditions* $(D_1) - (D_7)$ *hold. Then system* (3.23) *has at least one* ω*-periodic solution with strictly positive components.*

4. Discussion

In this paper, we studied the combined effects of periodicity of ecological and environmental parameters of a neutral delay prey-predator model with impulse and Hassell-Varley type functional response. By using the continuation theorem of coincidence degree theory, easily verifiable criteria are established for the existence of positive periodic solutions to the system. By Theorem 3.1 and Lemma 2.1, we can see that system (1.4) (or system (2.1)) will have at least one ω-periodic solution with strictly positive components if b (the density-dependent coefficient of the prey species) is sufficiently large and the neutral coefficient g is sufficiently small; r (the convert rate of the predator species) multiplies $\prod_{0<t_k<t}(1+\theta_{1k})$ (the impulse of the prey species) is sufficiently large; the predator natural mortality rate d is sufficiently small; a (the intrinsic growth rate of prey species) is sufficiently large; and c (the capturing rate of the predator species) multiplies $\prod_{0<t_k<t}(1+\theta_{2k})$ (the impulse of the predator species) is sufficiently small.

It is worthwhile to point out that τ (the time delay due to gestation) plays an important role in determining the existence of positive periodic solutions of system (1.4) (or system (2.1)).

From the results of this paper, one can find that the neutral term effects are quite significant.

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