PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 5, May 2017, Pages 2059–2073
http://dx.doi.org/10.1090/proc/13338
Article electronically published on January 11, 2017

NONOSCILLATION THEOREMS FOR SECOND-ORDER LINEAR DIFFERENCE EQUATIONS VIA THE RICCATI-TYPE TRANSFORMATION

JITSURO SUGIE AND MASAHIKO TANAKA

(Communicated by Mourad Ismail)

ABSTRACT. A nonoscillation problem is dealt with the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n,$$

where $\{b_n\}$ and $\{c_n\}$ are positive sequences. For all sufficiently large $n \in \mathbb{N}$, the ratios c_n/c_{n-1} and c_{n-1}/b_n play an important role in the results obtained. To be precise, our nonoscillation criteria are described in terms of the sequence

$$q_n = \frac{c_{n-1}}{b_n}\frac{c_n}{b_{n+1}}\frac{c_n}{c_{n-1}} = \frac{c_n^2}{b_n b_{n+1}}.$$

These criteria are compared with those that have been reported in previous researches by using some specific examples. Figures are attached to facilitate understanding of the concrete examples.

1. Introduction

We consider the second-order linear difference equation

$$c_n x_{n+1} + c_{n-1} x_{n-1} = b_n x_n, \qquad n = 1, 2, \dots, \tag{1}$$

where $\{b_n\}$ and $\{c_n\}$ are sequences satisfying $b_n > 0$ for $n \in \mathbb{N}$ and $c_n > 0$ for $n \in \mathbb{N} \cup \{0\}$, respectively (as can be seen from the proof of our theorems below, we have only to assume that the sequences $\{b_n\}$ and $\{c_n\}$ are positive for n sufficiently large). Needless to say, equation (1) has the trivial solution $\{x_n\}$; that is, $x_n = 0$ for $n \geq 0$. Nontrivial solutions of (1) are divided into two groups. A nontrivial solution of (1) is said to be *oscillatory* if, for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n x_{n+1} \leq 0$. Otherwise, it is said to be *nonoscillatory*. Hence, if $\{x_n\}$ is a nonoscillatory solution of (1), then there exists an $N \in \mathbb{N}$ such that $x_n > 0$ for $n \geq N$ or $x_n < 0$ for $n \geq N$. It is clear that if $\{x_n\}$ is a solution of (1), then $\{-x_n\}$ is also a solution of (1). Hence, it is sufficient to consider that a nonoscillatory solution $\{x_n\}$ of (1) continues being positive for all large n.

The purpose of this paper is to give sufficient conditions for all nontrivial solutions of (1) to be nonoscillatory. Our conditions are expressed with the relation between the sequences $\{b_n\}$ and $\{c_n\}$. If there is a nonpositive subsequence

Received by the editors February 20, 2016 and, in revised form, June 19, 2016.
2010 *Mathematics Subject Classification.* Primary 39A06, 39A21; Secondary 39A10.
Key words and phrases. Linear difference equations, nonoscillation, Riccati transformation, Sturm's separation theorem.
The first author's work was supported in part by Grant-in-Aid for Scientific Research No. 25400165 from the Japan Society for the Promotion of Science.

$\{b_{n_k}\} \subset \{b_n\}$, then all nontrivial solutions of (1) are oscillatory. Hence, it is natural to assume that the sequence $\{b_n\}$ is positive.

For $n \in \mathbb{N}$, let

$$p_n = c_n + c_{n-1} - b_n \quad \text{and} \quad r_n = c_n.$$

Then, equation (1) becomes the self-adjoint difference equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n = 0, \tag{2}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$. The continuous counterpart of (2) is a second-order differential equation of the form

$$\big(r(t)x'\big)' + p(t)x = 0, \tag{3}$$

where r, $p\,[a,\infty) \to \mathbb{R}$ are continuous functions, $r(t) > 0$ for $t \geq a$.

Oscillation theory for equation (3) has been studied by a number of researchers from a long time ago (for example, see the books [2,5,22,23]). Since the 1980s, oscillation and nonoscillation criteria came to be reported flourishingly for a generalized equation

$$\big(r(t)\phi(x')\big)' + p(t)\phi(x) = 0 \tag{4}$$

of (3). Here, $\phi(z)$ is a real-valued nonlinear function defined by

$$\phi(z) = \begin{cases} |z|^{p-2}z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0 \end{cases}$$

for $z \in \mathbb{R}$ with $p > 1$ a fixed real number. Equation (4) is often called a *half-linear* differential equation of *self-adjoint* type. For example, we can refer the reader to the books [2,6,8] and the references cited therein. From the end of the last century, many authors were motivated by the results about equations (3) and (4), and they developed oscillation theory for equation (2) and the discrete counterpart of (4). For example, we can refer to [7,10,11,19,21,24].

On the other hand, we can find some conditions which guarantee that all nontrivial solutions of (1) are oscillatory (or nonoscillatory) in a series of papers of Hooker et al. [14,15,18] (see also the books [1, Chap. 6], [9, Chap. 7], [16, Chap. 6]). For the sake of simplicity, they denoted $c_n^2/(b_n b_{n+1})$ by q_n for $n \in \mathbb{N}$. Their typical and fundamental results are as follows.

Theorem A. *If* $q_n \geq 1/(4-\varepsilon)$ *for some* $\varepsilon > 0$ *and for all sufficiently large* n, *then all nontrivial solutions of* (1) *are oscillatory.*

Theorem B. *If* $q_n \leq 1/4$ *for all sufficiently large* n, *then all nontrivial solutions of* (1) *are nonoscillatory.*

Theorem C. *If* $q_{n_k} \geq 1$ *for a sequence* $\{n_k\}$ *tending to* ∞, *then all nontrivial solutions of* (1) *are oscillatory.*

Theorems A and B have a good balance. These results are called "oscillation theorem" and "nonoscillation theorem", respectively. The constant $1/4$ often appears as a critical value that divides oscillation and nonoscillation of solutions of second-order linear differential equations (for example, see [12, 17, 20, 23]). Also, several generalizations of Theorem A have been given by Hooker et al. [14,15,18]. Results which generalized Theorem B seem to be fewer compared with those of

Theorem A though there are a lot of nonoscillation comparison theorems for difference equations including equation (1). We can find only a few nonoscillation theorems for equation (1) (or (2)) in [3, 4, 13].

In this paper, we will derive the following nonoscillation theorem about equation (1) by considering the behavior of (q_{2k-1}, q_{2k}) or (q_{2k}, q_{2k+1}) with $k \in \mathbb{N}$.

Theorem 1. *Suppose that there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ there is a sequence $\{\alpha_k\}$ with $\alpha_k > 1$ and either*

$$\frac{\alpha_k}{\alpha_k - 1} q_{2k-1} + \alpha_{k+1} q_{2k} \leq 1 \tag{5}$$

or

$$\frac{\alpha_k}{\alpha_k - 1} q_{2k} + \alpha_{k+1} q_{2k+1} \leq 1. \tag{6}$$

Then all nontrivial solutions of (1) *are nonoscillatory.*

2. Basic knowledge for proving Theorem 1

To prove Theorem 1, we need only to use two well-known fundamental facts; that is, Sturm's separation theorem and the Riccati transformation method. For Sturm's separation theorem, see [9, pp. 321–322] for example. From Sturm's separation theorem it follows that if one nontrivial solution of (1) (or (2)) is nonoscillatory, then all nontrivial solutions are nonoscillatory. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1). Then we can define

$$z_n = \frac{b_{n+1} x_{n+1}}{c_n x_n}$$

with $n \geq M$ for some $M \in \mathbb{N} \cup \{0\}$. The sequence $\{z_n\}$ satisfies the first-order nonlinear difference equation

$$q_n z_n + \frac{1}{z_{n-1}} = 1, \qquad n = M+1, M+2, \dots, \tag{7}$$

where $\{q_n\}$ is the sequence given in Section 1. Equation (7) is called a difference equation of *Riccati*-type. From this transformation it turns out that a nonoscillatory solution $\{x_n\}$ of (1) corresponds to a positive solution $\{z_n\}$ of (7) and the converse is true. Hence, by virtue of Sturm's separation theorem, we see that all nontrivial solutions of (1) are nonoscillatory if and only if there exists an integer $N \geq M$ such that equation (7) has a solution $\{z_n\}$ satisfying $z_n > 0$ for all $n \geq N$. We therefore have only to find a positive solution of (7) in order to prove Theorem 1.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Suppose that the inequality (5) holds. Then

$$\alpha_{k+1} \leq \frac{1}{q_{2k}} \left(1 - \frac{\alpha_k}{\alpha_k - 1} q_{2k-1} \right)$$

for all $k \geq N$. Consider a solution $\{z_n\}$ of (7) satisfying $z_{2N-2} \geq \alpha_N$. Since $\alpha_N > 1$, we see that

$$z_{2N-1} = \frac{1}{q_{2N-1}} \left(1 - \frac{1}{z_{2N-2}} \right) \geq \frac{1}{q_{2N-1}} \left(1 - \frac{1}{\alpha_N} \right) = \frac{\alpha_N - 1}{\alpha_N q_{2N-1}} > 0.$$

Hence, we have

$$z_{2N} = \frac{1}{q_{2N}}\left(1 - \frac{1}{z_{2N-1}}\right) \geq \frac{1}{q_{2N}}\left(1 - \frac{\alpha_N}{\alpha_N - 1}\, q_{2N-1}\right) \geq \alpha_{N+1} > 1.$$

Similarly, we can check that

$$z_n \geq \begin{cases} \dfrac{\alpha_k - 1}{\alpha_k q_{2k-1}} & \text{if } n = 2k-1, \\ \alpha_{k+1} & \text{if } n = 2k \end{cases}$$

with $k \geq N$. Hence, the sequence $\{z_n\}$ is a positive solution of (7). We therefore conclude that all nontrivial solutions of (1) are nonoscillatory.

Suppose that the inequality (6) holds. Consider a solution $\{z_n\}$ of (7) satisfying $z_{2N-1} \geq \alpha_N$. Then, as in the proof of the case that (5) holds, we see that z_n is positive for $n \geq 2N-1$. Hence, all nontrivial solutions of (1) are nonoscillatory. $\square$

Let $\alpha_k = 2$ with $k \geq N$ for some $N \in \mathbb{N}$. Then we have the following corollary of Theorem 1.

Corollary 2. *Suppose that there exists an $N \in \mathbb{N}$ such that either*

(8) $$q_{2k-1} + q_{2k} \leq \frac{1}{2}$$

or

(9) $$q_{2k} + q_{2k+1} \leq \frac{1}{2}$$

with $k \geq N$. Then all nontrivial solutions of (1) are nonoscillatory.

Remark 1. If $q_n \leq 1/4$ for all sufficiently large n, then it is clear that the inequalities (8) and (9) are satisfied. Hence, Corollary 2 contains Theorem B completely.

Remark 2. From the fact that the arithmetic mean of two positive numbers is not less than their geometric mean, if the inequality (8) holds, then

(10) $$\sqrt{q_{2k-1}q_{2k}} \leq \frac{1}{4}$$

is satisfied for $k \geq N$. However, we cannot change (8) to (10) in Corollary 2. In fact, let $b_n = 1$ and

$$c_n = \begin{cases} 1/4 & \text{if } n = 2k-1, \\ 1 & \text{if } n = 2k \end{cases}$$

with $k \in \mathbb{N}$. Then we have

$$q_n = \begin{cases} 1/16 & \text{if } n = 2k-1, \\ 1 & \text{if } n = 2k. \end{cases}$$

Hence, by Theorem C, all nontrivial solutions are oscillatory.

Corollary 3. *Suppose that there exists an $N \in \mathbb{N}$ such that either*

(11) $$q_{2k-1} < 1 \quad \textit{and} \quad q_{2k} \leq (1 - \sqrt{q_{2k-1}})(1 - \sqrt{q_{2k+1}})$$

or

(12) $$q_{2k} < 1 \quad \textit{and} \quad q_{2k+1} \leq (1 - \sqrt{q_{2k}})(1 - \sqrt{q_{2k+2}})$$

with $k \geq N$. Then all nontrivial solutions of (1) are nonoscillatory.

Proof. Suppose that the inequality (11) holds. Let

$$\alpha_k = \frac{1}{1-\sqrt{q_{2k-1}}}$$

for any $k \geq N$. Then it is clear that $\alpha_k > 1$. Since

$$\frac{\alpha_k}{\alpha_k - 1} = \frac{1}{\sqrt{q_{2k-1}}} \quad \text{and} \quad \alpha_{k+1} = \frac{1}{1-\sqrt{q_{2k+1}}},$$

the inequality (5) coincides with

$$\sqrt{q_{2k-1}} + \frac{q_{2k}}{1-\sqrt{q_{2k+1}}} \leq 1;$$

namely, the inequality (11). Hence, all nontrivial solutions of (1) are nonoscillatory by Theorem 1.

Suppose that the inequality (12) holds. Let

$$\alpha_k = \frac{1}{1-\sqrt{q_{2k}}} > 1$$

for any $k \geq N$. Then the inequality (6) coincides with the inequality (12). Hence, all nontrivial solutions of (1) are nonoscillatory by Theorem 1. □

3. Comparison with previous studies

To illustrate our results, we give some examples in this section. But, before that, we introduce an interesting related research which was proved by Abu-Risha [3].

Theorem D. *All nontrivial solutions of* (1) *are nonoscillatory if and only if there is an eventually positive sequence* $\{\xi_n\}$ *such that*

$$\left(q_{n+1}\xi_{n+1} + \frac{1}{\xi_n}\right)\left(q_n\xi_n + \frac{1}{\xi_{n-1}}\right) \leq 1. \tag{13}$$

Although the inequality (13) is a necessary and sufficient condition for nonoscillation of (1), it is expressed implicitly. For this reason, Abu-Risha also presented an explicit condition concerning (q_n, q_{n+1}) as follows.

Corollary E. *All nontrivial solutions of* (1) *are nonoscillatory if there is an* $N \in \mathbb{N}$ *such that*

$$\left(\sqrt{q_{n+1}} + \sqrt{q_n}\right)\left(\sqrt{q_n} + \sqrt{q_{n-1}}\right) \leq 1 \tag{14}$$

holds for $n \geq N$.

Remark 3. Let $\xi_n = 1/\sqrt{q_n}$. Then Corollary E follows from Theorem D.

We first give an example of Corollary 2.

Example 1. Let $c_0 = 1$ and let

$$c_n = \begin{cases} 2\sqrt{6} & \text{if } n = 4k-3, \\ 2\sqrt{2} & \text{if } n = 4k-2, \\ 7 & \text{if } n = 4k-1, \\ 1 & \text{if } n = 4k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 4 & \text{if } n = 4k-3, \\ 16 & \text{if } n = 4k-2, \\ 4 & \text{if } n = 4k-1, \\ 25 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Then all nontrivial solutions of (1) are nonoscillatory.

Since

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 0.375 & \text{if } n = 4k-3, \\ 0.125 & \text{if } n = 4k-2, \\ 0.49 & \text{if } n = 4k-1, \\ 0.01 & \text{if } n = 4k, \end{cases} \tag{15}$$

we obtain

$$q_{2k-1} + q_{2k} = 0.5$$

with $k \in \mathbb{N}$. Hence, the inequality (8) holds. Thus, by Corollary 2, all nontrivial solutions of (1) are nonoscillatory.

Let us denote by $\{x_n\}$ a solution of (1) with the sequences $\{b_n\}$ and $\{c_n\}$ that were given in Example 1 (see Figure 1). To make the motion of a solution of (1) more visible, we connect the dots x_{n-1} and x_n with a line segment and draw a line graph.

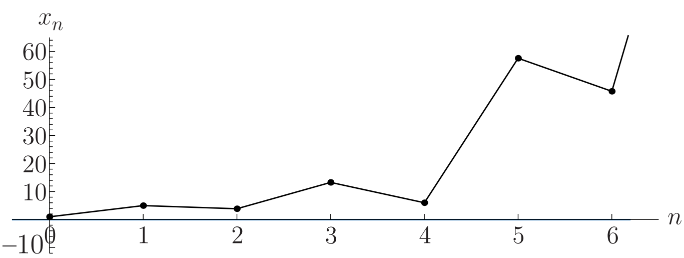

FIGURE 1. This line graph displays the motion of a solution $\{x_n\}$ of (1) given in Example 1. The initial condition of the solution is $(x_0, x_1) = (1, 5)$.

Figure 1 shows that $x_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, this solution $\{x_n\}$ is nonoscillatory. Recall that if equation (1) has a nontrivial solution which is nonoscillatory, then all nontrivial solutions are nonoscillatory. To be specific, we also simulate a solution $\{z_n\}$ of (7) (see Figure 2). This solution corresponds to the solution of (1) described in Figure 1.

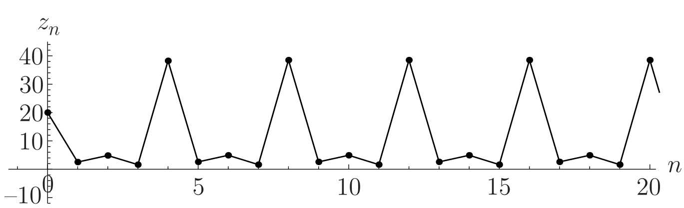

FIGURE 2. Riccati's equation (7) has a positive solution $\{z_n\}$ when the sequence $\{q_n\}$ satisfies (15). The initial condition of the solution is $z_0 = 20$.

Remark 4. The inequality (9) does not hold in Example 1. In fact,

$$q_{4k-2} + q_{4k-1} = 0.615 > 0.5$$

for any $k \in \mathbb{N}$. We can apply Corollary 2 to equation (1) when the pair (q_{2k-1}, q_{2k}) (or (q_{2k}, q_{2k+1})) is in the triangular region

$$R \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0 \text{ and } x + y \leq 1/2\}$$

even if the pair (q_{2k}, q_{2k+1}) (or (q_{2k-1}, q_{2k})) is outside the region R for $k \in \mathbb{N}$ sufficiently large.

As can be seen from (15), the sequence $\{q_n\}$ is periodic with period 4. Let

$$P_1 = (q_{4k-3}, q_{4k-2}) = (0.375, 0.125), \quad P_2 = (q_{4k-2}, q_{4k-1}) = (0.125, 0.49),$$
$$P_3 = (q_{4k-1}, q_{4k}) = (0.49, 0.01), \quad P_4 = (q_{4k}, q_{4k+1}) = (0.01, 0.375).$$

By plotting these points in the first quadrant of the plane $\mathbb{R}^2$, the following figure is obtained:

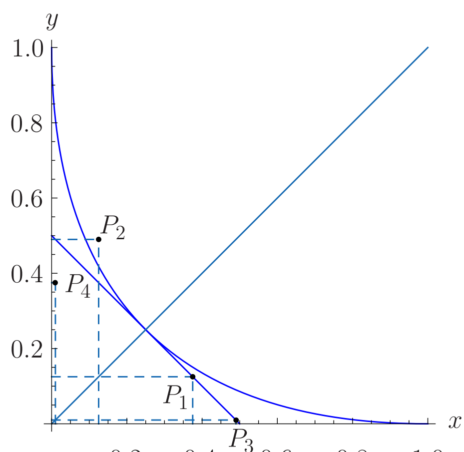

FIGURE 3. The points P_1 and P_3 are on the straight line $x + y = 1/2$. The point P_2 is outside the region R. The point P_4 is within the region R.

From (15) it follows that

$$\sqrt{q_{n+1}} + \sqrt{q_n} = \begin{cases} \sqrt{2}/4 + \sqrt{6}/4 & \text{if } n = 4k-3, \\ 0.7 + \sqrt{2}/4 & \text{if } n = 4k-2, \\ 0.1 + 0.7 & \text{if } n = 4k-1, \\ \sqrt{6}/4 + 0.1 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence, we have

$$\left(\sqrt{q_{n+1}} + \sqrt{q_n}\right)\left(\sqrt{q_n} + \sqrt{q_{n-1}}\right) = \begin{cases} 1.017654429348457\cdots & \text{if } n = 4k-2, \\ 0.8428427124746188\cdots & \text{if } n = 4k-1, \\ 0.5698979485566356\cdots & \text{if } n = 4k, \\ 0.6880989335750164\cdots & \text{if } n = 4k+1 \end{cases}$$

with $k \in \mathbb{N}$. There is no $N \in \mathbb{N}$ where the inequality (14) is satisfied for all $n \geq N$. Hence, Corollary E is not applicable to Example 1.

Remark 5. To apply Corollary E, both pairs (q_{2k-1}, q_{2k}) and (q_{2k}, q_{2k+1}) have to be in the region

$$S \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{R}^2 \colon x > 0,\ y > 0 \ \text{ and } \ \sqrt{x} + \sqrt{y} \leq 1\right\} \supset R$$

(see Figure 3 again).

Next, we give an example of Corollary 3.

Example 2. Let $c_0 = 4$ and let

$$c_n = \begin{cases} 5\sqrt{5} & \text{if } \ n = 4k-3, \\ 2 & \text{if } \ n = 4k-2, \\ \sqrt{2} & \text{if } \ n = 4k-1, \\ 4 & \text{if } \ n = 4k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 20 & \text{if } \ n = 4k-3, \\ 25 & \text{if } \ n = 4k-2, \\ 1 & \text{if } \ n = 4k-1, \\ 5 & \text{if } \ n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Then all nontrivial solutions of (1) are nonoscillatory.

It is easy to check that

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 0.25 & \text{if } \ n = 4k-3, \\ 0.16 & \text{if } \ n = 4k-2, \\ 0.4 & \text{if } \ n = 4k-1, \\ 0.16 & \text{if } \ n = 4k. \end{cases} \tag{16}$$

Hence, we obtain $q_{2k-1} < 1$ and

$$q_{2k} = 0.16 < (1 - \sqrt{0.25})(1 - \sqrt{0.4}) = (1 - \sqrt{q_{2k-1}})(1 - \sqrt{q_{2k+1}})$$

for all $k \in \mathbb{N}$, and therefore, the inequality (11) holds. Thus, by Corollary 3, all nontrivial solutions of (1) are nonoscillatory.

We give two simulations to illustrate Example 2. One is the line graph of a solution $\{x_n\}$ of (1) with the sequences $\{b_n\}$ and $\{c_n\}$ that were given in Example 2 (see Figure 4). The other is the line graph of a solution $\{z_n\}$ of (7) (see Figure 5). This solution corresponds to the solution of (1) described in Figure 4.

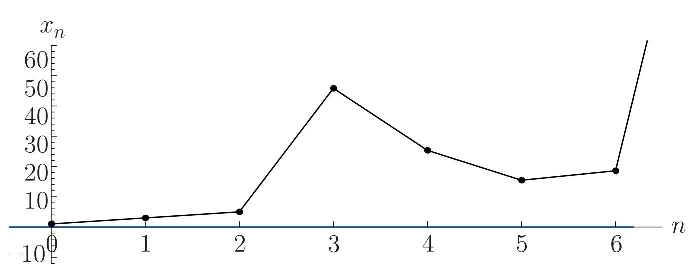

FIGURE 4. This line graph displays the motion of a solution x_n of (1) given in Example 2. The initial condition of the solution is $(x_0, x_1) = (1, 3)$.

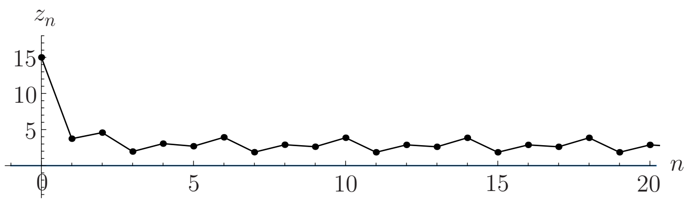

FIGURE 5. Riccati's equation (7) has a positive solution $\{z_n\}$ when the sequence $\{q_n\}$ satisfies (16). The initial condition of the solution is $z_0 = 15$.

Remark 6. The inequality (12) does not hold in Example 2, because

$$q_{4k-1} = 0.4 > 0.36 = (1 - \sqrt{0.16})^2 = (1 - \sqrt{q_{4k-2}})(1 - \sqrt{q_{4k}})$$

with $k \in \mathbb{N}$. We can apply Corollary 3 to equation (1) when the triple $(q_{2k-1}, q_{2k}, q_{2k+1})$ (or $(q_{2k}, q_{2k+1}, q_{2k+2})$) is in the domain

$$V \overset{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 : 0 < x < 1,\ y > 0,\ z > 0 \ \text{and} \ y \leq (1 - \sqrt{x})(1 - \sqrt{z})\}$$

even if the triple $(q_{2k}, q_{2k+1}, q_{2k+2})$ (or $(q_{2k-1}, q_{2k}, q_{2k+1})$) is outside the region V for $k \in \mathbb{N}$ sufficiently large (see Figure 6).

As can be seen from (16), the sequence $\{q_n\}$ is periodic with period 4. Let

$$\begin{aligned} P_1 &= (q_{4k-3}, q_{4k-2}, q_{4k-1}) = (0.25, 0.16, 0.4), \\ P_2 &= (q_{4k-2}, q_{4k-1}, q_{4k}) = (0.16, 0.4, 0.16), \\ P_3 &= (q_{4k-1}, q_{4k}, q_{4k+1}) = (0.4, 0.16, 0.25), \\ P_4 &= (q_{4k}, q_{4k+1}, q_{4k+2}) = (0.16, 0.25, 0.16). \end{aligned} \tag{17}$$

By plotting these points in the first octant of three-dimensional space $\mathbb{R}^3$, the following figure is obtained:

Remark 7. We cannot apply Corollary 2 to Example 2, because both inequalities (8) and (9) are not satisfied. In fact, from (16) it follows that

$$q_{4k-1} + q_{4k} = 0.4 + 0.16 > 0.5$$

and

$$q_{4k-2} + q_{4k-1} = 0.16 + 0.4 > 0.5$$

for all $k \in \mathbb{N}$.

From (16) it follows that

$$\sqrt{q_{n+1}} + \sqrt{q_n} = \begin{cases} 0.4 + 0.5 & \text{if } n = 4k - 3, \\ \sqrt{10}/5 + 0.4 & \text{if } n = 4k - 2, \\ 0.4 + \sqrt{10}/5 & \text{if } n = 4k - 1, \\ 0.5 + 0.4 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence, we have

$$\left(\sqrt{q_{n+1}} + \sqrt{q_n}\right)\left(\sqrt{q_n} + \sqrt{q_{n-1}}\right) = \begin{cases} 0.9292099788303083\cdots & \text{if } n = 4k - 2, \\ 1.065964425626941\cdots & \text{if } n = 4k - 1, \\ 0.9292099788303083\cdots & \text{if } n = 4k, \\ 0.81 & \text{if } n = 4k + 1 \end{cases}$$

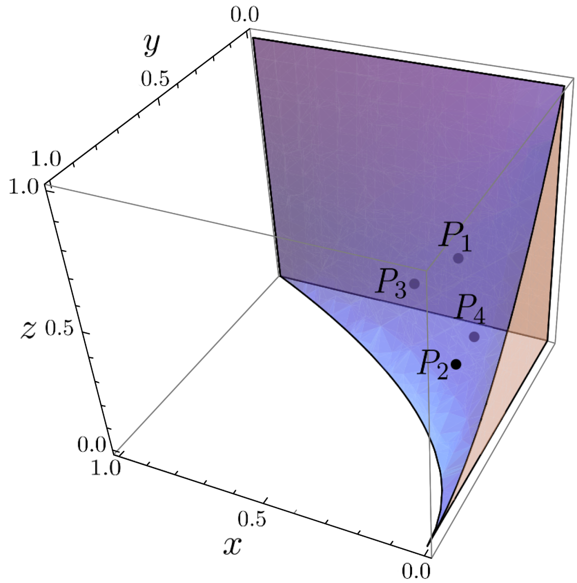

FIGURE 6. The points P_1, P_3 and P_4 are in the domain V. However, the point P_2 is outside the domain V.

with $k \in \mathbb{N}$. There is no $N \in \mathbb{N}$ where the inequality (14) is satisfied for all $n \geq N$. Hence, Corollary E is not available for Example 2.

Remark 8. To apply Corollary E, both triples $(q_{2k-1}, q_{2k}, q_{2k+1})$ and $(q_{2k}, q_{2k+1}, q_{2k+2})$ have to be in the domain

$$W \stackrel{\text{def}}{=} \left\{(x, y, z) \in \mathbb{R}^3 \colon x > 0,\ y > 0,\ z > 0 \ \text{ and } \ (\sqrt{x} + \sqrt{y})(\sqrt{y} + \sqrt{z}) \leq 1\right\} \supset V$$

(see Figure 7).

4. FURTHER NONOSCILLATION CRITERIA

In Section 3, we have focused on the behavior of the pair (q_{2k-1}, q_{2k}) (or (q_{2k}, q_{2k+1})) and the triple $(q_{2k}, q_{2k+1}, q_{2k+2})$ (or $(q_{2k-1}, q_{2k}, q_{2k+1})$) with $k \in \mathbb{N}$. In this section let us examine the influence which a set of many more elements gives to nonoscillation of (1). The following result is a generalization of Theorem 1.

Theorem 4. *Suppose that there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ there are two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ with $\alpha_k > 1$ and $\beta_k > 1$. If*

$$\frac{\alpha_k}{\alpha_k - 1} q_{4k-3} + \beta_k q_{4k-2} \leq 1 \tag{18}$$

and

$$\frac{\beta_k}{\beta_k - 1} q_{4k-1} + \alpha_{k+1} q_{4k} \leq 1, \tag{19}$$

then all nontrivial solutions of (1) *are nonoscillatory.*

Proof. From (18) and (19) it follows that

$$\beta_k \leq \frac{1}{q_{4k-2}}\left(1 - \frac{\alpha_k}{\alpha_k - 1}\, q_{4k-3}\right)$$

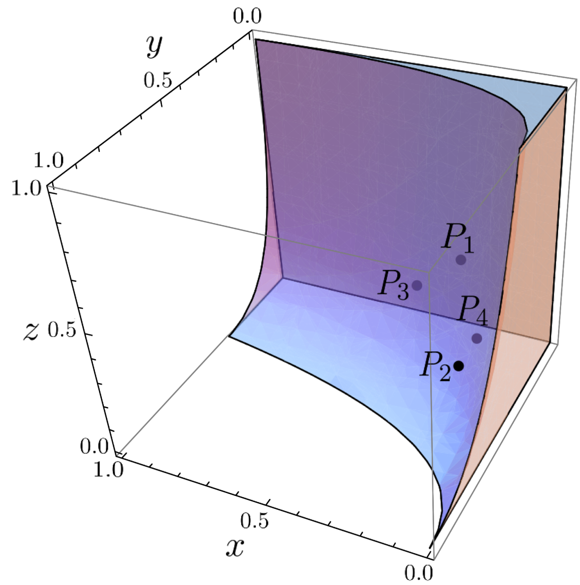

FIGURE 7. Let P_1, P_2, P_3 and P_4 be the points given in (17). The points P_1, P_3 and P_4 are in the domain W. However, the point P_2 is outside the domain W.

and

$$\alpha_{k+1} \leq \frac{1}{q_{4k}}\left(1 - \frac{\beta_k}{\beta_k - 1}\, q_{4k-1}\right)$$

for all $k \geq N$. Consider a solution $\{z_n\}$ of (7) satisfying $z_{4N-4} \geq \alpha_N > 1$. Then we can check that

$$\begin{aligned}
z_{4N-3} &= \frac{1}{q_{4N-3}}\left(1 - \frac{1}{z_{4N-4}}\right) \geq \frac{1}{q_{4N-3}}\left(1 - \frac{1}{\alpha_N}\right) = \frac{\alpha_N - 1}{\alpha_N q_{4N-3}} > 0,\\
z_{4N-2} &= \frac{1}{q_{4N-2}}\left(1 - \frac{1}{z_{4N-3}}\right) \geq \frac{1}{q_{4N-2}}\left(1 - \frac{\alpha_N}{\alpha_N - 1}\, q_{4N-3}\right) \geq \beta_N > 1,\\
z_{4N-1} &= \frac{1}{q_{4N-1}}\left(1 - \frac{1}{z_{4N-2}}\right) \geq \frac{1}{q_{4N-1}}\left(1 - \frac{1}{\beta_N}\right) = \frac{\beta_N - 1}{\beta_N q_{4N-1}} > 0,\\
z_{4N} &= \frac{1}{q_{4N}}\left(1 - \frac{1}{z_{4N-1}}\right) \geq \frac{1}{q_{4N}}\left(1 - \frac{\beta_N}{\beta_N - 1}\, q_{4N-1}\right) \geq \alpha_{N+1} > 1.
\end{aligned}$$

We inductively obtain

$$z_n \geq \begin{cases} \dfrac{\alpha_k - 1}{\alpha_k q_{4k-3}} & \text{if } n = 4k-3,\\ \beta_k & \text{if } n = 4k-2,\\ \dfrac{\beta_k - 1}{\beta_k q_{4k-1}} & \text{if } n = 4k-1,\\ \alpha_{k+1} & \text{if } n = 4k \end{cases}$$

with $k \geq N$. Hence, the sequence $\{z_n\}$ is a positive solution of (7). We therefore conclude that all nontrivial solutions of (1) are nonoscillatory. □

In the same way, we have the following result (we omit the proof).

Theorem 5. *Suppose that there exists an $N \in \mathbb{N}$ such that for any $k \geq N$ there are two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ with $\alpha_k > 1$ and $\beta_k > 1$. If*

$$\frac{\alpha_k}{\alpha_k - 1} q_{4k-2} + \beta_k q_{4k-1} \leq 1 \tag{20}$$

and

$$\frac{\beta_k}{\beta_k - 1} q_{4k} + \alpha_{k+1} q_{4k+1} \leq 1, \tag{21}$$

then all nontrivial solutions of (1) *are nonoscillatory.*

Remark 9. If the inequalities (18) and (19) are satisfied for $k \in \mathbb{N}$ sufficiently large, then the inequality (5) also holds. In fact, let

$$\gamma_k = \begin{cases} \alpha_\ell & \text{if } k = 2\ell - 1, \\ \beta_\ell & \text{if } k = 2\ell \end{cases}$$

with $k \in \mathbb{N}$. Then, by (18) and (19), we obtain

$$\frac{\gamma_k}{\gamma_k - 1} q_{2k-1} + \gamma_{k+1} q_{2k} \leq 1;$$

namely, the inequality (5). Similarly, if the inequalities (20) and (21) are satisfied for $k \in \mathbb{N}$ sufficiently large, then the inequality (6) also holds. Hence, Theorems 4 and 5 also extend Theorem 1.

Let p be a real number that is larger than 1 and let p^* be the conjugate number of p; namely,

$$\frac{1}{p} + \frac{1}{p^*} = 1.$$

Then p^* is also greater than 1. We choose constants $\alpha > 1$ and $\beta > 1$ as the two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ in Theorems 4 and 5, respectively. Then the inequalities (18)–(21) become

$$\alpha^* q_{4k-3} + \beta q_{4k-2} \leq 1, \tag{22}$$

$$\beta^* q_{4k-1} + \alpha q_{4k} \leq 1, \tag{23}$$

$$\alpha^* q_{4k-2} + \beta q_{4k-1} \leq 1, \tag{24}$$

$$\beta^* q_{4k} + \alpha q_{4k+1} \leq 1, \tag{25}$$

respectively. Hence, we have the following corollaries of Theorems 4 and 5.

Corollary 6. *Suppose that there exists an $N \in \mathbb{N}$ such that both* (22) *and* (23) *hold for $k \geq N$. Then all nontrivial solutions of* (1) *are nonoscillatory.*

Corollary 7. *Suppose that there exists an $N \in \mathbb{N}$ such that both* (24) *and* (25) *hold for $k \geq N$. Then all nontrivial solutions of* (1) *are nonoscillatory.*

Here we give an example of Corollary 6.

Example 3. Let $c_0 = \sqrt{6}$ and let

$$c_n = \begin{cases} 3 & \text{if } n = 4k-3, \\ \sqrt{3} & \text{if } n = 4k-2, \\ 2 & \text{if } n = 4k-1, \\ \sqrt{6} & \text{if } n = 4k \end{cases} \quad \text{and} \quad b_n = \begin{cases} 9 & \text{if } n = 4k-3, \\ 3 & \text{if } n = 4k-2, \\ 9 & \text{if } n = 4k-1, \\ 2 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Then all nontrivial solutions of (1) are nonoscillatory.

In Example 3, the sequence $\{q_n\}$ satisfies

$$q_n = \frac{c_n^2}{b_n b_{n+1}} = \begin{cases} 1/3 & \text{if } n = 4k-3, \\ 1/9 & \text{if } n = 4k-2, \\ 2/9 & \text{if } n = 4k-1, \\ 1/3 & \text{if } n = 4k. \end{cases} \tag{26}$$

Let $\alpha = 2$ and $\beta = 3$. Then we obtain

$$\alpha^* q_{4k-3} + \beta q_{4k-2} = 2 \times \frac{1}{3} + 3 \times \frac{1}{9} = 1$$

and

$$\beta^* q_{4k-1} + \alpha q_{4k} = \frac{3}{2} \times \frac{2}{9} + 2 \times \frac{1}{3} = 1;$$

namely, the inequalities (22) and (23) are satisfied for all $k \in \mathbb{N}$. Hence, by Corollary 6, all nontrivial solutions of (1) are nonoscillatory.

From (26) it follows that

$$\sqrt{q_{n+1}} + \sqrt{q_n} = \begin{cases} (1+\sqrt{3})/3 & \text{if } n = 4k-3, \\ (\sqrt{2}+1)/3 & \text{if } n = 4k-2, \\ (\sqrt{3}+\sqrt{2})/3 & \text{if } n = 4k-1, \\ 2\sqrt{3}/3 & \text{if } n = 4k \end{cases}$$

with $k \in \mathbb{N}$. Hence, we have

$$\left(\sqrt{q_{n+1}} + \sqrt{q_n}\right)\left(\sqrt{q_n} + \sqrt{q_{n-1}}\right) = \begin{cases} 0.7328615680805721\cdots & \text{if } n = 4k-2, \\ 0.8439726791916834\cdots & \text{if } n = 4k-1, \\ 1.210997720618484\cdots & \text{if } n = 4k, \\ 1.051566846126417\cdots & \text{if } n = 4k+1 \end{cases}$$

with $k \in \mathbb{N}$. There is no $N \in \mathbb{N}$ where the inequality (14) is satisfied for all $n \geq N$. Corollary E is inapplicable to Example 3.

Remark 10. We cannot apply Corollary 2 to Example 3, because both inequalities (8) and (9) are not satisfied. In fact, from (26) we see that

$$q_{4k-1} + q_{4k} = 2/9 + 1/3 > 1/2$$

and

$$q_{4k} + q_{4k+1} = 1/3 + 1/3 > 1/2$$

for all $k \in \mathbb{N}$.

Remark 11. For any $k \in \mathbb{N}$,

$$\begin{aligned} q_{4k} &= 1/3 > 0.2234107369952512\cdots \\ &= (3-\sqrt{2})(3-\sqrt{3})/9 = (1-\sqrt{q_{4k-1}})(1-\sqrt{q_{4k+1}}) \end{aligned}$$

and

$$\begin{aligned} q_{4k+1} &= 1/3 > 0.2817664872069162\cdots \\ &= 2(3-\sqrt{3})/9 = (1-\sqrt{q_{4k}})(1-\sqrt{q_{4k+2}}). \end{aligned}$$

Hence, both inequalities (11) and (12) are not satisfied, and therefore, Corollary 3 cannot be applied to Example 3.

References

[1] Ravi P. Agarwal, *Difference equations and inequalities*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, Inc., New York, 2000. Theory, methods, and applications. MR1740241

[2] Ravi P. Agarwal, Said R. Grace, and Donal O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht, 2002. MR2091751

[3] M. H. Abu-Risha, *Oscillation of second-order linear difference equations*, Appl. Math. Lett. **13** (2000), no. 1, 129–135, DOI 10.1016/S0893-9659(99)00156-1. MR1750979

[4] Shao Zhu Chen and Lynn H. Erbe, *Riccati techniques and discrete oscillations*, J. Math. Anal. Appl. **142** (1989), no. 2, 468–487, DOI 10.1016/0022-247X(89)90015-2. MR1014591

[5] Earl A. Coddington and Norman Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. MR0069338

[6] Ondřej Došlý, *Half-linear differential equations*, Handbook of differential equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 161–357. MR2166491

[7] Ondřej Došlý and Pavel Řehák, *Nonoscillation criteria for half-linear second-order difference equations*, Comput. Math. Appl. **42** (2001), no. 3-5, 453–464, DOI 10.1016/S0898-1221(01)00169-9. Advances in difference equations, III. MR1838006

[8] Ondřej Došlý, *Half-linear differential equations*, Handbook of differential equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 161–357. MR2166491

[9] Saber Elaydi, *An introduction to difference equations*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2005. MR2128146

[10] Hassan A. El-Morshedy, *Oscillation and nonoscillation criteria for half-linear second order difference equations*, Dynam. Systems Appl. **15** (2006), no. 3-4, 429–450. MR2367656

[11] Simona Fišnarová, *Oscillatory properties of half-linear difference equations: two-term perturbations*, Adv. Difference Equ., posted on 2012, 2012:101, 16, DOI 10.1186/1687-1847-2012-101. MR3078366

[12] Einar Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234–252. MR0027925

[13] Don B. Hinton and Roger T. Lewis, *Spectral analysis of second order difference equations*, J. Math. Anal. Appl. **63** (1978), no. 2, 421–438. MR0611455

[14] John W. Hooker, Man Kam Kwong, and William T. Patula, *Oscillatory second order linear difference equations and Riccati equations*, SIAM J. Math. Anal. **18** (1987), no. 1, 54–63, DOI 10.1137/0518004. MR871820

[15] John W. Hooker and William T. Patula, *Riccati type transformations for second-order linear difference equations*, J. Math. Anal. Appl. **82** (1981), no. 2, 451–462, DOI 10.1016/0022-247X(81)90208-0. MR629769

[16] Walter G. Kelley and Allan C. Peterson, *Difference equations*, Academic Press, Inc., Boston, MA, 1991. An introduction with applications. MR1142573

[17] Adolf Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen* (German), Math. Ann. **42** (1893), no. 3, 409–435, DOI 10.1007/BF01444165. MR1510784

[18] Man Kam Kwong, John W. Hooker, and William T. Patula, *Riccati type transformations for second-order linear difference equations. II*, J. Math. Anal. Appl. **107** (1985), no. 1, 182–196, DOI 10.1016/0022-247X(85)90363-4. MR786022

[19] Manjun Ma, *Dominant and recessive solutions for second order self-adjoint linear difference equations*, Appl. Math. Lett. **18** (2005), no. 2, 179–185, DOI 10.1016/j.aml.2004.03.005. MR2121277

[20] Zeev Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85** (1957), 428–445. MR0087816

[21] Pavel Řehák, *Hartman-Wintner type lemma, oscillation, and conjugacy criteria for half-linear difference equations*, J. Math. Anal. Appl. **252** (2000), no. 2, 813–827, DOI 10.1006/jmaa.2000.7124. MR1800179

[22] William T. Reid, *Ordinary differential equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1971. MR0273082

[23] Charles A. Swanson, *Comparison and oscillation theory of linear differential equations*, Academic Press, New York-London, 1968. Mathematics in Science and Engineering, Vol. 48. MR0463570

[24] Michal Veselý and Petr Hasil, *Oscillation and nonoscillation of asymptotically almost periodic half-linear difference equations*, Abstr. Appl. Anal. (2013), Art. ID 432936, 12. MR3066297

Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: jsugie@riko.shimane-u.ac.jp

Department of Mathematics, Shimane University, Matsue 690-8504, Japan
E-mail address: qut4527@yahoo.co.jp