CONCERNING CONTINUA IN THE PLANE*

BY

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In this paper a study will be made of plane continua. Part I deals with continua which constitute the boundary of a connected domain and is concerned in particular with (1) properties of domains which are consequences of certain conditions imposed upon their boundaries, (2) properties of the boundaries of domains which are consequences of conditions imposed upon the domains, and (3) conditions under which the boundary of a domain is accessible from that domain. Part II is concerned with the cut points and end points of continua.

I wish to acknowledge my indebtedness to Professor R. L. Moore, to whom the success of this investigation should be largely attributed. Credit is due him for the suggestion of most of the problems treated in Part I; and it is his stimulating personality, constant encouragement, and many helpful suggestions and criticisms which have attracted my interest to this field of mathematics and have made possible the solution of the problems treated in this paper.

I. Domains and their boundaries

Definitions. A domain D is said to have property S† provided it is true that for any positive number ϵ, D may be expressed as the sum of a finite number of connected point sets each of diameter less than ϵ. A point set K will be said to be uniformly connected im kleinen with reference to every one of its bounded subsets provided it is true that if M is any bounded point set whatever and ϵ is any positive number, then there exists a positive number δ_ϵ such that every two points which are common to M and K and whose distance apart is less than δ_ϵ lie together in a connected subset of K of diameter less than ϵ. A boundary point P of a domain D is accessible from all sides from D‡ provided it is true that if A and B are any two points

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† Cf. R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237.

‡ Cf. A. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, zweiter Teil, Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsbände, vol. 2 (1908), p. 215

of the boundary of D and AXB is an arc such that $AXB-(A+B)$ is a subset of D and such that AXB separates D into two domains D_1 and D_2, then P is accessible from every one of the domains D_1, D_2 to whose boundary it belongs. Two point sets will be said to be mutually separated if they are mutually exclusive and neither contains a limit point of the other. The point P of a continuum M will be called a cut point of M provided the set $M-P$ is not connected, i.e., is the sum of two mutually separated sets.

Notation. In this paper wherever a symbol X is used to denote a point set, the symbol $\overline{X}$ will be used to denote the set X plus all those points which are limit points of X. And wherever a symbol AB is used to designate a simple continuous arc, the symbol (AB) will be used to denote the point set $AB-(A+B)$.

THEOREM 1. *In order that a bounded domain D should have property S it is necessary and sufficient that every point of the boundary of D should be accessible from all sides from D.*

Proof. I shall first show that the condition is necessary. Suppose D is a domain having property S and P is a point of its boundary. Let A and B be two points of the boundary of D, and let AXB be an arc from A to B such that (AXB) is a subset of D, and such that AXB separates D into two domains D_1 and D_2. In an unpublished paper,* C. M. Cleveland has proved the following theorem. In order that a bounded domain D should have property S it is necessary and sufficient that (1) every maximal connected subset of the boundary of D should be a continuous curve, and (2) for any given positive number ϵ, there should be not more than a finite number of these continuous curves of diameter greater than ϵ. Now since D has property S, it follows that the boundary of D satisfies conditions (1) and (2) of Cleveland's theorem. And since the boundary of D satisfies these conditions, it can easily be shown by methods almost identical with those used by R. L. Moore to prove Theorem 4 of his paper *Concerning connectedness im kleinen and a related property*† that the boundary of D_1, and also that of D_2, must satisfy these conditions. Hence, it follows by Cleveland's theorem that each of the domains D_1 and D_2 must have property S. Now let R denote either one of the domains D_1, D_2 which has the point P in its boundary. It is sufficient, then, to show that P is accessible from R.

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* Cf. an abstract of a paper by the author and C. M. Cleveland, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 109.

† Loc. cit.

Let R be expressed as the sum of n_1 connected point sets K_{11}, K_{12}, $\cdots$, K_{1n_1}, all of diameter less than 1/3. Let G_1 denote this collection of point sets. Let S_1 denote the collection of all those elements of G_1 which have P for a limit point, and let T_1 denote the sum of all the point sets of the collection S_1. There exists a circle C_1 having P as center and neither containing nor enclosing any point of $R-T_1$. Let X_1 denote a point common to T_1 and to the interior of C_1. Let I_1 denote the sum of all those elements of S_1 which can be joined to that element of S_1 which contains X_1 by a connected subset of R lying wholly within C_1. Every point of I_1 which is a limit point of $R-I_1$ lies within a circle c such that c plus its interior belongs to R and is of diameter less than 1/9. Add to I_1 the interiors of all such circles (c), and let R_1 denote the domain thus obtained. Clearly R_1 is of diameter less than 1, and P is a boundary point of R_1. Now let R be expressed as the sum of n_2 connected point sets K_{21}, K_{22}, K_{23}, $\cdots$, K_{2n_2}, all of diameter less than 1/6 and also less than the radius of C_1. Let G_2 denote this collection of point sets. And let T_2 and C_2 be point sets which, with respect to G_2, correspond to T_1 and C_1 selected previously with respect to G_1. Let X_2 be a point common to T_2, to the interior of C_2, and to I_1. Let I_2 denote the sum of all those elements of S_2 which can be joined to the element of S_2 which contains X_2 by a connected subset of R lying wholly within C_2. Clearly I_2 is a subset of I_1, and hence also of R_1. Every point of I_2 which is a limit point of $R-I_2$ lies within some circle c such that c plus its interior belongs to R and to R_1 and is of diameter less than 1/18. Add to I_2 the interiors of all such circles (c), and let R_2 denote the domain thus obtained. Clearly R_2 is a subset of R_1, is of diameter less than $\frac{1}{2}$, and has the point P in its boundary. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of R: R_1, R_2, R_3, $\cdots$, such that for every positive integer n, R_{n+1} has P in its boundary and is a subset of R_n, and such that the diameter of R_n approaches zero as a limit as n increases indefinitely.

Now let Q denote any point of R. For each positive integer n, let P_n denote a point of R_n. There exists an arc QP_1 lying in R, and for each n, there exists an arc P_nP_{n+1} lying in R_n. It is easy to see* that the point set $P+QP_1+P_1P_2+P_2P_3+\cdots$ is closed and that it contains as a subset an arc QP such that $QP-P$ is a subset of R. Hence, P is accessible from R, and since R is either one of the domains D_1, D_2 which has P in its boundary, it follows that P is accessible from all sides from D.

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* Cf., for example, R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 155, paragraph beginning with line five.

The condition is also sufficient. Suppose D is a bounded domain such that its boundary, M, is accessible from all sides from D. Condition (I) will be said to exist if some maximal connected subset of the boundary of D fails to be a continuous curve; Condition (II) will be said to exist if it is true that for some positive number ϵ, there exist infinitely many maximal connected subsets of M of diameter greater than ϵ. Suppose Condition (II) exists, and let G denote the set of all those maximal connected subsets of M which are of diameter greater than ϵ. Since the sum of all the continua of G is bounded and G contains infinitely many elements, it follows* that there exists a continuum T of diameter at east as great as ϵ which is the sequential limiting set of some sequence $T_1, T_2, T_3, \cdots$ of elements of G. There exist two points E and F of T whose distance apart is $\geqq \epsilon$. Let C_1 be a circle with E as center and of radius $\frac{3}{4}\epsilon$. Let C_2 be a circle with E as center and of radius $\frac{1}{4}\epsilon$. Then since E is within C_2 and F is without C_1, there exists a positive integer d such that for every $n>d$, T_n contains a point x_n within C_2 and a point y_n without C_1. It follows from a theorem due to Janiszewski† that for every $n>d$, T_n contains a subcontinuum t_n which contains at least one point on each of the circles C_1 and C_2 and is a subset of the point set H consisting of the circles C_1 and C_2 together with all those points of the plane which lie between C_1 and C_2. For every positive integer i, let M_i denote t_{d+i}. The continuum T contains a subcontinuum M_∞ which has at least one point on each of the circles C_1 and C_2, is a subset of H, and is the sequential limiting set of the sequence $M_1, M_2, M_3, \cdots$. Now suppose Condition (I) exists. It follows directly from a theorem of R. L. Moore's‡ that there exist circles C_1 and C_2, and that M contains a countable infinity of continua M_∞, $M_1, M_2, M_3, \cdots$, having exactly the same properties as the point sets of the same notation whose existence was shown as a consequence of Condition (II). Hence, we see that the existence of either Condition (I) or Condition (II) leads to exactly the situation described above.

Let A (Fig. 1) denote a point common to M_∞ and C_2, and B a point common to M_∞ and C_1. Since M is accessible from D, it follows that there exists an arc AB such that (AB) is a subset of D. It can be shown that there exists a bounded complementary domain R of the point set $AB+M_\infty$ such

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* See an abstract of a paper by R. G. Lubben, *Concerning limiting sets*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 14.

† *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique, (2), vol. 16 (1912).

‡ *A report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 296–297.

that every point of the arc AB belongs to the boundary of R. It can be shown that the arc AB separates D into two domains D_1 and D_2 such that D_1 lies wholly within R, and D_2 lies wholly in K, the unbounded complementary domain of the boundary of R. Since no member of the sequence of continua $M_1, M_2, M_3, \cdots$ has a point in common with $AB+M_\infty$, it follows that for every positive integer i, M_i lies either wholly in R or wholly in K. Hence, either R or K must contain infinitely many members of the sequence $M_1, M_2, M_3, \cdots$.

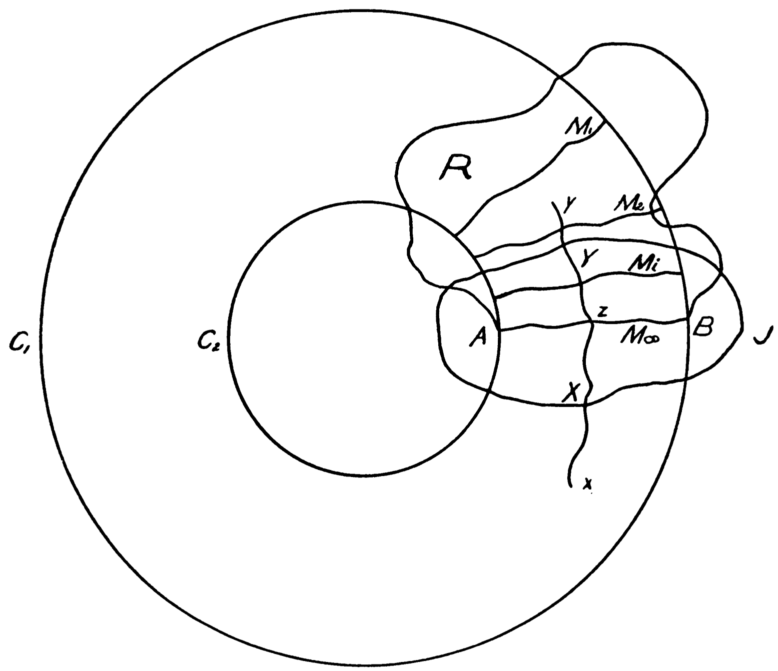

Fig. 1

Suppose R contains infinitely many members of this sequence. Then every point of M_∞ is a limit point of a set of points common to D and R. And since all such points belong to D_1, it follows that every point of M_∞ is a boundary point of D_1 and, by hypothesis, is therefore accessible from D_1. Let O be a point of K and let P be a point of M_∞ distinct from A and from B. Then since the arc AB does not of itself separate the plane, there exists an arc OP which contains no point of the arc AB. On OP, in the order from O to P, let z denote the first point belonging to M_∞. Then the point set $Oz-z$ is a subset of K. Now either (a) there exists a point x on the arc Oz such that the arc zx of Oz contains no point of M, or (b) z is a limit point of a set of points common to D and K. In case (b), since all points common to K and D

belong to D_2, then z is a boundary point of D_2 and is, therefore, accessible from D_2. Hence, if x is a point of D_2, there exists an arc xz such that $xz-z$ is a subset of D_2. Hence, in either case, (a) or (b), there exists an arc xz such that $xz-z$ is a subset of K and contains no point whatever of M. It was shown above that z is accessible from D_1. Hence, if y is a point of D_1, there exists an arc yz such that $yz-z$ is a subset of D_1. The two arcs xz and yz can have in common only the point z.

Let I denote the point set consisting of M_∞ plus all of its bounded complementary domains. Let L denote the closed point set $I+M_1+M_2+M_3+\cdots$. It can easily be shown that neither of the points x and y belongs to I. Now I does not separate the plane, and it is a maximal connected subset of the closed set L. By a theorem of R. L. Moore's* it follows that there exists a simple closed curve J such that J encloses I and contains no point of L, and every point on or within J is at a distance from some point of I less than the minimum distance from x to I and also less than the minimum distance from y to I. Hence both x and y are without J. On the arc zx, in the order from z to x, and on zy, in the order from z to y, let X and Y respectively denote the first points belonging to J. Denote the two arcs of J from X to Y by XTY and XSY respectively, and let R_1 and R_2 denote the interiors of the closed curves $XzYTX$ and $XzYSX$ respectively. On the arc XzY there exist points E, U, H, and G in the order X, E, U, z, H, G, Y such that within some circle which has z as center and which neither contains nor encloses any point of the arc AB there exist arcs EFG and UCH such that (EFG) and (UCH) are subsets of R_1 and R_2 respectively. Since E and U lie in K, and H and G lie in R, it follows that both (EFG) and (UCH) must contain a point in common with M_∞. But (EFG) belongs to R_1, and (UCH) belongs to R_2. Hence R_1 contains a point u of M_∞, and R_2 contains a point v of M_∞. Let C_u and C_v be circles having u and v respectively as centers and such that C_u belongs to R_1 and C_v belongs to R_2. Now since J encloses M_∞ and contains no point of L, there exists a positive number d_1 such that for every integer $n>d_1$, M_n lies wholly within J. There exists a positive number d_2 such that for every integer $n>d_2$, M_n has a point within C_u and also a point within C_v. Let i be an integer which is greater than each of these two numbers d_1 and d_2. Then M_i lies wholly within J and contains at least one point in each of the domains R_1 and R_2. Hence, since it is connected, M_i must contain a point p of the arc XzY. Since M_i has no point in common with M_∞, therefore $p\neq z$. Hence p must belong either to (zX) or to (zY). But p

* *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), p. 469.

belongs to M, and neither (zX) nor (zY) contains any point whatever of M. Thus in case R contains infinitely many members of the sequence M_1, M_2, $M_3, \cdots$, the supposition that either Condition (I) or Condition (II) exists leads to a contradiction. In an entirely similar manner the same supposition leads to a contradiction in case K contains infinitely many members of this sequence.

Since neither Condition (I) nor Condition (II) can exist, then (1) every maximal connected subset of the boundary of D is a continuous curve, and (2) for every positive number ϵ, there are not more than a finite number of these maximal connected subsets of diameter greater than ϵ. Since D is bounded, it follows from Cleveland's theorem quoted above that D has property S.

THEOREM 2. *If the domain D is uniformly connected im kleinen with reference to every one of its bounded subsets, then every point of the boundary of D is accessible from D.*

Proof. Let P denote a point of the boundary of D. Let C_1 be a circle having P as center and of diameter less than 1. For every point x common to D and the interior of C_1, let K_x denote the greatest connected point set which contains x and is common to D and the interior of C_1. Let G_1 denote the collection of all such sets (K_x). Since D is uniformly connected im kleinen with respect to every one of its bounded subsets, it follows that if C is any circle concentric with and within C_1, then there are not more than a finite number of elements of G_1 which have points on or within C and have a limit point on C_1. Hence there exists a circle k_1, concentric with and within C_1, such that k_1 neither contains nor encloses any point of any element of G_1 which does not have P for a limit point. Let S_1 denote the finite collection of all those elements of G_1 which have points on or within k_1, and let T_1 denote the sum of all the point sets of this collection. Let X_1 be a point common to T_1 and the interior of k_1. Let D_1 denote the element of S_1 which contains X_1. Clearly D_1 is a domain which (1) is a subset of D and of the interior of C_1, (2) has P in its boundary, and (3) contains every point common to D and the interior of C_1 which can be joined to X_1 by an arc which is also a subset of D and of the interior of C_1. Now let C_2 denote a circle which is concentric with C_1 and of diameter less than $\frac{1}{2}$ and also less than the radius of k_1. Let G_2, k_2, S_2, T_2, X_2, and D_2 be collections and sets which with respect to C_2 correspond to G_1, k_1, S_1, T_1, X_1, and D_1 selected above with respect to C_1, with the additional condition that X_2 shall belong to D_1. Then D_2 is a domain which (1) is a subset of D, of D_1, and of the interior of C_2, (2) has P in its boundary, and (3) contains every point common to D and the interior of C_2 which can be joined to X_2 by an arc which is also a subset of D and of the interior of C_2. This process may be continued indefinitely, and thus we obtain a sequence of subdomains of D: $D_1, D_2, D_3, \cdots$, such that for every positive integer n, D_{n+1} has P in its boundary and is a subset of D_n, and such that the diameter of D_n approaches zero as a limit as n increases indefinitely. By an argument which is identical with the third paragraph of the proof of Theorem 1, it follows that if A is any point of D, then there exists an arc AP such that $AP-P$ is a subset of D. Hence, every point of the boundary of D is accessible from D.

THEOREM 3. *In order that a domain D should be uniformly connected im kleinen with reference to every one of its bounded subsets it is necessary and sufficient that* (1) *every maximal connected subset of the boundary of D should be either a point, a simple closed curve, or an open curve, and* (2) *if ϵ is any positive number and J is any simple closed curve, there should be not more than a finite number of maximal connected subsets of the boundary of D which have points within J and are of diameter greater than ϵ.*

Proof. I shall show that the conditions are necessary. This may be done with the use of methods only slightly different from those used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries*† to prove the proposition that every bounded, simply connected, and uniformly connected im kleinen domain is bounded by a simple closed curve. I shall simply indicate the modifications necessary in his argument to establish Theorem 3.

Suppose the domain D is uniformly connected im kleinen with reference to every one of its bounded subsets. Then by an argument almost identical with that used by Moore to show that the boundary of his domain in the above mentioned proposition is a continuous curve, it follows that every maximal connected subset of the boundary of D is a continuous curve, and that if J is any simple closed curve and ϵ is any positive number, then there are not more than a finite number of these maximal connected subsets of the boundary of D which have points within J and are of diameter greater than ϵ. Now let M denote any definite maximal connected subset of the boundary of D which consists of more than one point. I shall show that M must be either a simple closed curve or an open curve. Let the points $P_1, P_2, P_3, \cdots$, the arcs $A_1B_1, A_2B_2, A_3B_3, \cdots$, and the point set N^* be selected and defined with respect to M exactly as was done by Moore in the paragraph beginning

† Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–70.

near the bottom of page 366 of his paper. I shall now show that M is neither a simple continuous arc nor a ray of an open curve. Suppose the contrary is true. Then if M is an arc, let A and B denote its end points, and if M is a ray, let A denote its end point. Let X be a point of M which is distinct from A and from B, and let C be a circle with X as center and not enclosing or containing either A or B. Within C and on M there exist points E, U, W, and G in the order A, E, U, X, W, G, and within C there exist arcs EFG and UVW having only their end points in common with M and such that if R_1 and R_2 denote the interiors of the closed curves $EFGWXUE$ and $UVWXU$ respectively, then R_1 and R_2 are mutually exclusive domains each of which lies wholly within C. Since under this supposition, M can contain no simple closed curve, it readily follows that X must be a limit point of a set of points K_1 common to D and R_1 and also of a set K_2 common to D and R_2. But clearly this is impossible, since D is uniformly connected im kleinen with reference to every one of its bounded subsets. It follows, then, that M is neither an arc nor a ray of an open curve.

Now suppose M is bounded. In this case, since M cannot be an arc, it follows by exactly the same argument given by Moore in the first paragraph of page 369 of his paper that M is a simple closed curve. Suppose M is unbounded. Since M cannot be a ray of an open curve, it readily follows that both of the sequences of points $A_1, A_2, A_3, \cdots$ and $B_1, B_2, B_3, \cdots$ must be infinite and that neither of these sequences can have a limit point. It follows that N^* is a closed point set which is identical with M and which, evidently, must be an open curve. Hence the conditions are necessary.

C. M. Cleveland† has proved that the conditions are sufficient.

THEOREM 4. *If K denotes the set of all the cut points of the boundary M of a complementary domain D of a continuous curve, then $D+K$ is uniformly connected im kleinen.*

Proof. By a theorem due to Miss Torhorst,‡ M is a continuous curve. Suppose $D+K$ is not uniformly connected im kleinen. Then, for some positive number ϵ, D contains two infinite sequences of points, X_1, X_2, $X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$, such that (1) for every positive integer n, the distance from X_n to Y_n is less than $1/n$, (2) for no integer n is it true that X_n and Y_n lie together in some connected subset of $D+K$ of diameter less than ϵ, and (3) there exists in M a point P which is the sequential limiting

† In unpublished work.

‡ *Ueber den Rand der einfach zusammenhängenden ebenen Gebiete*, Mathematische Zeitschrift, vol. 9 (1921), p. 64 (73).

set of each of these two sequences of points. Let C be a circle having P as center and of diameter $\epsilon/2$. Let us first suppose that D is bounded. It follows by a theorem of R. L. Moore's* that D has property S. Hence, D is expressible as the sum of a finite number of connected point sets K_1, $K_2, K_3, \cdots, K_n$, all of diameter less than $\epsilon/5$. Let $K_{m_1}, K_{m_2}, K_{m_3}, \cdots, K_{m_m}$ denote those sets of this sequence which have P as a limit point. Clearly $K_{m_1}+K_{m_2}+\cdots+K_{m_m}$ is a subset of the interior of C. Since P is not a limit point of $D-(K_{m_1}+K_{m_2}+\cdots+K_{m_m})$, there exists a positive integer i such that both X_i and Y_i belong to $K_{m_1}+K_{m_2}+\cdots+K_{m_m}$. Let N_x and N_y denote sets of this sequence which contain X_i and Y_i respectively. Let R_x and R_y denote the maximal connected subsets of D which contain N_x and N_y respectively and lie within C. Clearly, the domains R_x and R_y can have no points in common. The point P belongs to the boundary of each of these domains, and by the method used in the proof of Theorem 1, it can be shown that P is accessible from each of them. Hence, there exist arcs X_iP and Y_iP such that X_iP-P and Y_iP-P are subsets of R_x and R_y respectively. There exists an arc t from X_i to Y_i which is a subset of D. The point set $t+X_iP+Y_iP$ contains a simple closed curve J which contains P and lies, except for the point P, wholly in D. Let I and E denote the interior and exterior respectively of J. If either I or E, say I, contains no point of M, then since D contains points of I, it follows that I is a subset of D, and clearly in this case X_i and Y_i can be joined by a connected subset of D of diameter less than ϵ, contrary to supposition. And if both I and E contain points of M, then clearly P is a cut point of M and belongs to K. And in this case R_x+R_y+P is a connected subset of $D+K$ which contains both X_i and Y_i and is of diameter less than ϵ, contrary to supposition. Thus, in any case, the supposition that $D+K$ is not uniformly connected im kleinen leads to a contradiction. The case where D is unbounded may be treated in a slightly modified manner.

THEOREM 5. *In order that the simply connected bounded domain D should become uniformly connected im kleinen upon the addition of a single point O of its boundary B it is necessary and sufficient that* (1) *if K be any maximal connected subset of $B-O$, then $K+O$ is a simple closed curve, and* (2) *there be not more than a finite number of these curves of B of diameter greater than any preassigned positive number.*

Proof. The conditions are necessary. Suppose D is a bounded domain with connected boundary B, and O is a point of B such that $D+O$ is uniformly

* *Concerning connectedness im kleinen and a related property*, loc. cit., Theorem 4.

connected im kleinen. Then B is a continuous curve. For suppose it is not. Then B contains a point P which is distinct from O and at which B is not connected im kleinen. Then by an argument identical with that used by R. L. Moore in his paper *A characterization of Jordan regions by properties having no reference to their boundaries*,† in the paragraph beginning at the bottom of page 365, with the additional condition that the circle K used in his argument be taken of radius less than $\frac{1}{2}$ the distance between O and P, it can be shown that this supposition leads to a contradiction. Hence, B is a continuous curve.

Let K denote a maximal connected subset of $B-O$. Then since $B-K$ is closed, it follows that K is connected im kleinen. Now let an inversion of the plane be performed about some circle which has O as center. Since $K+O$ is closed and connected, it follows that K^*, the image of K, is unbounded, closed, connected, and connected im kleinen. Since the inversion does not act upon the point O, and since $D+O$ is uniformly connected im kleinen, it can readily be shown that D^*, the image of D, is uniformly connected im kleinen with reference to every one of its bounded subsets. Therefore, by Theorem 3, it follows that K^* is an open curve, and hence, that $K+O$ is a simple closed curve. Therefore, condition (1) is necessary. Now since every maximal connected subset K of $B-O$ is a simple closed curve minus one point, every such set K contains an arc of diameter greater than $\frac{1}{2}$ the diameter of K. By a theorem of R. L. Wilder's,‡ B cannot contain, for any given positive number ϵ, more than a finite number of mutually exclusive arcs all of diameter greater than ϵ. In view of this result, it follows that for any positive number ϵ, $B-O$ cannot contain an infinite number of maximal connected subsets each of diameter greater than ϵ. Hence, condition (2) is necessary.

The conditions are also sufficient. Suppose D is a bounded domain with connected boundary B which satisfies conditions (1) and (2) in the statement of this theorem. Clearly, B must be a continuous curve. Unless the point O is a cut point of B, then B is a simple closed curve and D is its interior. In this case D itself is uniformly connected im kleinen. Hence, unless this theorem is true, O must be a cut point of B. No other point is a cut point of B. For let P denote any other point of B. Let K denote the maximal connected subset of $B-O$ which contains P, and let J denote the point set $K+O$. By hypothesis, J is a simple closed curve. Hence, $J-P$ is connected. But $B-K$ is connected, and since the connected sets $J-P$ and

† Loc. cit.

‡ *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358, Theorem 6.

$B-K$ have the point O in common, their sum S is connected. But $S=B-P$. Hence, P is not a cut point of B. It follows that O is the only cut point of B, and therefore, by Theorem 4, $D+O$ is uniformly connected im kleinen.

THEOREM 6. *In order that a continuous curve M should be the boundary of a connected domain it is necessary and sufficient that if J denotes any simple closed curve of M, then* (1) *M is a subset either of $J+I$ or of $J+E$, where I and E denote the interior and exterior respectively of J, and* (2) *if A and B are any two points of J, then $M-(A+B)$ is not connected.*

Proof. The conditions are necessary. That condition (1) is necessary is evident. Now let A and B denote any two points of J, where J is any simple closed curve contained in the boundary M of a complementary domain D of a continuous curve. Since A and B are accessible from D, it readily follows that there exists an arc AXB such that (AXB) is a subset of D. Now $M+D$ lies wholly either in J plus its interior I, or in J plus its exterior E, suppose in $J+I$. Then there exists an arc AYB such that (AYB) is a subset of E. Let t and t' denote the two arcs of J from A to B. Then the simple closed curve $AXBYA$ encloses one of these arcs minus $A+B$, say $t-(A+B)$, and neither contains nor encloses any point of $t'-(A+B)$. Since M has in common with the curve $AXBYA$ only the points A and B, it follows that $M-(A+B)$ is not connected. Hence the conditions are necessary.

The conditions are also sufficient. Let M denote a continuous curve which satisfies conditions (1) and (2) of this theorem. Let K denote the unbounded complementary domain of M, and let N denote its boundary. Now N contains a simple closed curve J, or otherwise M† is the boundary of K and the theorem is true. By hypothesis, M is a subset either of $J+I$ or of $J+E$, where I and E denote the interior and exterior respectively of J.

Case I. Suppose M is a subset of $J+E$. I shall show that in this case $N\equiv M$, i.e., that M is the boundary of K. Suppose M contains a point P which does not belong to N. Then let R denote the complementary domain of N which contains P and let C denote its boundary. By a theorem of R. L. Moore's‡ it follows that C is a simple closed curve. Since R is bounded, C encloses P; and P belongs to E, the exterior of J. Hence, J contains a point Q which does not belong to C. The curve C does not enclose Q, for Q is a boundary point of K, the unbounded complementary domain of M.

† R. L. Moore, *Concerning continuous curves in the plane,* Mathematische Zeitschrift, vol. 15 (1922), pp.254–260, Theorem 5.

‡ Loc. cit., Theorem 4.

Hence Q lies in the exterior of C. But C encloses P and, by hypothesis, M is a subset either of C plus its interior or of C plus its exterior. Thus the supposition that $N \neq M$ leads to a contradiction. Hence M is the boundary of the connected domain K.

Case II. Suppose M is a subset of $J+1$. With the aid of hypothesis (2) it is shown that there exists a point O which does not belong to M and which is within J but is not within any other simple closed curve belonging to M. Let C be a circle having O as center and not enclosing or containing any point of M. Let an inversion of the plane be performed about C. If X is a point set, let X' denote the image of X under this inversion. Now M' is a subset of $J'+I'$, and I' is the exterior of J'. Let K' denote the unbounded complementary domain of M', and let N' denote its boundary. Then N' contains J', and by an argument identical with that used in Case I it can be shown that M' is the boundary of the connected domain K'. Hence, it follows that M is the boundary of the connected domain K, where K is the point set of which K' is the image under this inversion of the plane.

THEOREM 7. *If the point P of a continuous curve M belongs to the boundary of no complementary domain of M, then for every positive number ϵ, M contains a simple closed curve which encloses P and is of diameter less than ϵ.*

Proof. Let P denote a point of a continuous curve M which belongs to the boundary of no complementary domain of M, and let ϵ denote any positive number. Let C be a circle having P as center and of diameter less than $\epsilon/2$, and such that the exterior of C contains at least one point of M. Let N denote the maximal connected subset of M which contains P and is contained in C plus its interior. By a theorem of H. M. Gehman's,* N is a continuous curve. The curve N contains a point A which belongs to C. Any arc whatever from A to P must contain at least one point of N which is distinct from A and from P. For suppose there exists an arc t from A to P which has only the points A and P in common with N. Since M is connected im kleinen, it readily follows that P is not a limit point of $M-N$. Hence, there exists a point X on t such that the arc PX of t has only the point P in common with M. Hence, the connected set $PX-P$ belongs to some complementary domain of M, and P must be a boundary point of that domain. But P is not a boundary point of any complementary domain of M. It follows, then, that every arc from A to P contains a point of N which is

* *Concerning the subsets of a plane continuous curve,* Annals of Mathematics, (2), vol. 27 (1925), pp. 29–46, Theorem 4, Lemma B.

distinct from A and from P. By a theorem proved by C. M. Cleveland,* it follows that N contains a simple closed curve J which encloses either A or P. The curve J cannot enclose A, because A belongs to C, and J is a subset of C plus its interior. Hence J must enclose P. Since it is contained in C plus its interior, J is of diameter less than ϵ.

II. Cut points and end points

In this section, I shall make a study of the properties of the cut points and end points of a given plane continuum. More particularly, I shall study the connected subsets of the set of all the cut points and end points of a continuum, and I shall establish some very fundamental properties of such sets, both internal properties and properties relative to the remainder of the continuum.

Definitions. The term *cut point* will be used as defined in Part I. The term *end point*, as applied to a continuous curve, will be used in the sense as defined by R. L. Wilder,† i.e., a point P of a continuous curve M will be called an end point of M provided it is true that if t is any arc of M having P as one of its extremities, then $M-(t-P)$ contains no connected subset which contains P. As applied to continua in general, I shall define the term *end point* as follows. The point P of a continuum M will be called an end point of M provided it is true that if N is any subcontinuum of M which contains P, then P is not a limit point of any connected subset of $M-N$. It is obvious that this definition will allow as many, if not more, points of a continuum to be end points as would the following extension of Wilder's definition: the point P of a continuum M is said to be an end point of M provided it is true that if H is any subcontinuum of M which contains P, then P belongs to no connected subset of $M-(H-P)$. The term *acyclic continuous curve* will be used, after Gehman, to designate a continuous curve which contains no simple closed curve.

R. L. Moore has shown‡ that no subcontinuum K of a given continuum M can contain an uncountable set of points each of which is a cut point of M but not of K. It follows from this theorem that no simple closed curve K can contain more than a countable number of cut points of any continuum

* This theorem is to the effect that if A and P are distinct points of a continuous curve N and every arc from A to P contains at least one point of N distinct from A and from P, then N contains a simple closed curve which separates A from P. Cf. an abstract of a paper by C. M. Cleveland, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 311.

† *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358.

‡ *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

which contains K. Extensive use will be made of these results in the proofs given in this section.

THEOREM 8. *If H is any connected subset of a continuum M, then not more than a countable number of points of $\bar{H}-H$ are cut points of M.*

Proof. Let T denote the set of all those points of $\bar{H}-H$ which are cut points of M. Clearly no point of T is a cut point of $\bar{H}$. Hence, by R. L. Moore's theorem quoted above, it follows that T must be countable.

THEOREM 9. *If K denotes the set of all the cut points and H the set of all the end points of a continuum M, and if T is any countable subset of M, then every bounded, closed, and connected subset of $K+H+T$ is an acyclic continuous curve.*

Proof. Let N denote any bounded continuum which is a subset of $K+H+T$. I shall first show that N is a continuous curve. Suppose N is not a continuous curve. Then by R. L. Moore and R. L. Wilder's* characterization of continua which are not continuous curves it follows that there exist two concentric circles k_1 and k_2 and that N contains a countable infinity of mutually exclusive continua N_∞, N_1, N_2, N_3, $\cdots$, such that (1) each of these continua contains at least one point on each of the circles k_1 and k_2, (2) the set N_∞ is the sequential limiting set of the sequence of sets N_1, N_2, N_3, $\cdots$, and (3) there exists a connected subset L of N which contains all of the sets of the sequence N_1, N_2, N_3, $\cdots$, but which contains no point whatever of the set N_∞. Now clearly $\bar{L}-L$ contains the continuum N_∞. Hence, by Theorem 8, N_∞ can contain not more than a countable number of points of K. And since every point of N_∞ is a limit point of L, a connected subset of $M-N_\infty$, it follows that no point whatever of N_∞ can belong to H. Hence, since T is countable and N_∞ is a subset of $K+H+T$, it follows that N_∞ is countable. But this is absurd. Thus the supposition that N is not a continuous curve leads to a contradiction.

Now suppose N contains a simple closed curve J. Then clearly no point of J can belong to H. And by R. L. Moore's theorem, only a countable number of points of J can belong to K. Hence, since T is countable, J must be countable. But this is impossible. It follows, then, that N is an acyclic continuous curve.

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* See R. L. Moore, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 103, and R. L. Wilder, Fundamenta Mathematicae, vol. 7 (1925), p. 371.

THEOREM 10. *If K is any closed and connected subset of the set of all the cut points of a bounded continuum M, and H is any connected subset of $M-K$, then $\overline{H}$ and K have at most one point in common. And if H is a maximal connected subset of $M-K$, then $\overline{H}$ and K have exactly one point in common.*

Proof. Suppose, on the contrary, that for some closed and connected subset K of the set of all the cut points of a bounded continuum M, $M-K$ contains a connected set H such that $\overline{H}$ and K have two points A and B in common. Now since, by Theorem 9, K is a continuous curve, it follows that K contains an arc t from A to B. By Theorem 8, t contains only a countable number of points of $\overline{H}$. Hence, t contains an interior point O which does not belong to $\overline{H}$. Let C denote a circle enclosing O and not enclosing or containing any point of $\overline{H}$. Within C there exist points E, G, U, and W on t in the order A, E, U, O, W, G, B, and arcs EFG and UVW having only their end points in common with t and such that if D_1 and D_2 denote the interiors of the closed curves $EFGWOUE$ and $UVWOU$ respectively, then D_1 and D_2 are mutually exclusive domains each of which lies within C. Let N denote the continuum $\overline{H}+t$. Let X and Y denote points of D_1 and D_2 respectively, and let Z denote a point belonging to the unbounded complementary domain of M. It is readily seen that every arc from X to Y contains at least one point of N, and that not both X and Y can be joined to Z by an arc which contains no point of N. Let v denote one of the points X, Y which cannot be so joined to Z, and let u denote the other one of the points X, Y. Let R_v denote that complementary domain of N which contains v. Let β denote the boundary of R_v. Then let R_u denote that complementary domain of β which contains u, and let α denote its boundary. R. L. Moore has shown* that under these conditions α contains no cut point of itself. But since R_v contains that one of the domains D_1 and D_2 which contains v and R_u contains the one which contains u, it readily follows that α contains the arc WOU of t. But WOU belongs to K, and every point of K is a cut point of M. Hence α contains an uncountable set of points each of which is a cut point of M but not of α, and since α is a continuum, this conclusion is contradictory to R. L. Moore's theorem quoted above. Thus the supposition that $\overline{H}$ and K have more than one point in common leads to a contradiction.

Now if H is any *maximal* connected subset of $M-K$, it is clear that K must contain at least one limit point of H. And in view of the above argument it follows that $\overline{H}$ and K must have exactly one point in common.

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* *Concerning the sum of a countable number of mutually exclusive continua in the plane*, Fundamenta Mathematicae, vol. 6 (1924), p. 190.

THEOREM 11. *If L denotes the set of all the cut points of a bounded continuum M, T is any countable subset of M, K is any closed and connected subset of $L+T$, and H is any connected subset of $M-K$, then K contains at most one limit point of H.*

Theorem 11 may be proved by an argument only slightly different from that given in the proof of Theorem 10.

THEOREM 12. *In order that the point P of a continuous curve M should be an end point of M it is necessary and sufficient that no arc of M should have P as one of its interior points.*

Proof. The condition is sufficient. Let P denote any point of M which is not an end point of M. I will show that every such point is an interior point of some arc of M. From the definition of an end point it follows that M contains some arc t having extremities at P and some other point A of M and such that $M-(t-P)$ contains a connected set N which contains P. Let X denote a point of N which is distinct from P. Let K denote the maximal connected subset of $M-t$ which contains X. I will show that P is a limit point of K. Suppose, on the contrary, that P is not a limit point of K. Let T denote the set of points common to N and K. Since M is connected im kleinen at every one of its points and t is closed, it readily follows (1) that $N-T$ contains no limit point of T, and (2) that T contains no limit point of $N-T$. Hence, N is expressible as the sum of two mutually separated point sets T and $N-T$. But this is impossible, because N is connected. It follows, then, that P is a limit point of K. Now K is a domain with respect to M,† for t is a closed set of points. And the boundary U of K with respect to M is a subset of t. Hence U contains no continuum of condensation. By a theorem of R. L. Wilder's‡ it follows that every point of U is accessible in M from K. I have just shown that P belongs to U. Hence, if B denotes a point of K, there exists an arc BP such that $BP-P$ is a subset of K. The arcs t and BP have in common only the point P. Hence, their sum, $t+BP$, is an arc APB from A to B which contains P as an interior point. I have shown, then, that every point of M which is not an end point of M is an interior point of some arc of M. It follows that every point of M which is not an interior point of any arc of M is an end point of M.

The condition is also necessary.§ For suppose some arc APB of M contains as an interior point the point P which is an end point of M. Clearly

† Cf. R. L. Wilder, *Concerning continuous curves*, loc. cit., Section I.

‡ Loc. cit., Theorem 1.

§ R. L. Wilder gives a proof of this part of the theorem for the special case of an acyclic continuous curve. His method of argument, however, is not applicable to the general case here treated.

this is impossible, because the arc PB of APB is a connected subset of $M-(AP-P)$ which contains P.

I will remark that Theorem 12 shows the equivalence of Wilder's definition of an end point of a continuous curve and the following one: the point P of a continuous curve M is said to be an end point of M provided it is true that if t is any arc of M having P as one of its extremities, then P is not a limit point of any connected subset of $M-t$. This latter definition for the case of a continuous curve is analogous to the one I have given above for continua in general.

THEOREM 13. *If K is a connected subset of the set of all the cut points of a continuous curve M, then in order that $\overline{K}$ should be an acyclic continuous curve it is necessary and sufficient that every point of $\overline{K}$ should be either a cut point or an end point of M.*

Proof. That the condition is sufficient is a corollary to Theorem 9. I will show that it is necessary. Suppose K is a connected set of cut points of a continuous curve M such that $\overline{K}$ is an acyclic continuous curve. Let P denote a point of $\overline{K}$ which is not an end point of M. I will show that P is a cut point of M. Let U denote a point of K which is distinct from P. Then $\overline{K}$ contains an arc t from U to P. Every point of t, except possibly the point P, is a cut point of M. For suppose t contains an interior point O which is not a cut point of M. Then O does not belong to K. Since, by a theorem of R. L. Wilder's,* every connected subset of an acyclic continuous curve is arcwise connected, it follows that $K+P$ contains an arc t_0 from U to P which does not contain O. Then the sum of the arcs t_0+t contains a simple closed curve, contrary to the hypothesis that $\overline{K}$ is acyclic. Hence, every point of t, except possibly the point P, is a cut point of M. Now since P is not an end point of M, it follows by Theorem 12 that M contains an arc APB having P as one of its interior points. Not both of the arcs AP and PB of APB can contain an interval in common with t which contains P, because P is an end point of t. Suppose AP has no interval in common with t which contains P. Then AP and t have in common only the point P. For suppose they have in common a point V which is distinct from P. The interval VP of AP contains a point Q which does not belong to t. In the order from Q to P and from Q to A respectively on AP, let X and Y denote the first points belonging to t. The simple closed curve formed by the arc XY of t plus the arc XQY of AP contains a segment XY every point of which is a cut point

* *Concerning continuous curves*, loc. cit., Theorem 20.

of M. Clearly this is impossible. Hence, it follows that AP and t have in common only the point P.

Now suppose, contrary to this theorem, that P is not a cut point of M. Then by a theorem of R. L. Moore's,* $M-P$ contains an arc b from U to A. The sum of the arcs $AP+t+b$ contains a simple closed curve J which contains a segment of t, every point of which is a cut point of M. This is absurd, and thus the supposition that P is not a cut point of M leads to a contradiction. It follows, then, that every point of $\overline{K}$ is either a cut point or an end point of M.

THEOREM 14. *If K denotes the set of all the cut points of a continuous curve M, then for every positive number ϵ, K contains not more than a finite number of mutually exclusive continua each of diameter greater than ϵ.*

Proof. Suppose Theorem 14 is not true. Then there exists a positive number ϵ such that K contains infinitely many mutually exclusive continua each of diameter greater than ϵ. Since by Theorem 9, every closed and connected subset of K is a continuous curve, it follows that K contains infinitely many mutually exclusive arcs each of diameter greater than ϵ. Let $t_1, t_2, t_3, \cdots$ denote some sequence of these arcs which has a sequential limiting set t. It is evident that t contains two points A and B whose distance apart is $\geqq \epsilon$. Now since M is uniformly connected im kleinen, there exists a positive number δ_ϵ such that every two points of M whose distance apart is less than δ_ϵ are end points of an arc of M of diameter less than $\frac{1}{4}\epsilon$. There exists a positive number d such that for every integer $n>d$, t_n contains a point X_n and a point Y_n whose distances from A and B respectively are less than $\frac{1}{2}\delta_\epsilon$. Let i and j denote two integers greater than d. Then X_i and X_j and also Y_i and Y_j can be joined by an arc of M of diameter less than $\frac{1}{4}\epsilon$. Let X_iX_j and Y_iY_j denote these two arcs. It is readily seen that the sum of the arcs $t_i+t_j+X_iX_j+Y_iY_j$ contains a simple closed curve J which contains an interval of the arc t_i. But every point of t_i is a cut point of M. Thus the supposition that Theorem 14 is false leads to a contradiction.

THEOREM 15. *If K is any closed and connected subset of the set of all the cut points of a continuous curve M, then for every positive number ϵ, $M-K$ contains not more than a finite number of maximal connected subsets of diameter greater than ϵ.*

Proof. Suppose Theorem 15 is not true. Then there exists a positive number ϵ such that $M-K$ contains an infinite collection G of maximal

* *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

connected subsets each of diameter greater than ϵ. By Theorem 10, K contains exactly one limit point of each set of the collection G. For each set g of G let X denote the limit point of g which belongs to K, and let H denote the set of all such points (X) thus defined. Now if H contains infinitely many distinct points, then K contains a point A which is a limit point of H. And if H contains only a finite number of points, then H contains a point A which is a limit point of each of an infinite number of distinct sets of the collection G. Let us first suppose that A is a limit point of H. Then H contains an infinite sequence of points $X_1, X_2, X_3, \cdots$ which has A as its sequential limit point. For every positive integer n, let G_n denote an element of G which has X_n as a limit point. The sequence of sets $G_1, G_2, G_3, \cdots$ have a sequential limiting set L which contains A. And since every element of G is of diameter greater than ϵ, it follows that L contains a point B whose distance from A is $\geqq \epsilon/3$. Now since M is connected im kleinen, it can easily be shown that B must belong to K. Let C_1 and C_2 be circles having A and B respectively as centers and each of diameter less than $\epsilon/10$. The sequence of points $X_1, X_2, X_3, \cdots$ contains an infinite subsequence $X_{n_1}, X_{n_2}, X_{n_3}, \cdots$ every point of which is within C_1. There exists a circle C_b having B as center and such that every point of M which is enclosed by C_b can be joined to B by an arc common to M and to the interior of C_2. There exists an integer i such that G_{n_i} contains a point V within C_b. Hence, M contains an arc t from V to B which lies within C_2. On t, in the order from V to B, let E denote the first point belonging to K. Then E is a limit point of G_{n_i}. But X_{n_i} is also a limit point of G_{n_i}, and X_{n_i} lies within C_1. Hence, K contains two distinct limit points of G_{n_i}. But this is contrary to Theorem 10. A similar conclusion is reached when it is assumed that A is a limit point of each of an infinite number of distinct elements of G. Thus the supposition that Theorem 15 is false leads to a contradiction.

THEOREM 16. *If the bounded continuum M has the property that every connected subset of M is arcwise connected, and K is any maximal connected subset of the set of all the cut points of M, and H denotes the set of all those limit points of K which K does not contain, then every point of H is an end point of M.*

Proof. Suppose, on the contrary, that H contains a point P which is not an end point of M. Now by a theorem of R. L. Wilder's,* M is a continuous curve. Hence by Theorem 12 M contains an arc APB having P as one of its interior points. Let U denote a point of K. By hypothesis $K+P$ contains an arc t from U to P. Now P is not a cut point of M, for otherwise it would

* *Concerning continuous curves*, loc. cit., Theorem 18.

belong to K. In view of this fact, it follows by an argument almost identical with the latter part of the proof of Theorem 13, beginning with the fifteenth sentence, that this situation leads to an absurdity. Hence, every point of H is an end point of M.

THEOREM 17. *Under the same hypothesis as in Theorem* 16, $K+H$ *is an acyclic continuous curve, and every point of* H *is an end point both of* M *and of the curve* $(K+H)$.

THEOREM 18. *If* K *is any closed and connected subset of the set of all the cut points of a continuum* M, *then* K *contains at least one subcontinuum which belongs to the boundary of some single complementary domain of* M.

Proof. The complementary domains of M are countable. Let them be ordered $D_1, D_2, D_3, \cdots$, and let their respective boundaries be ordered $B_1, B_2, B_3, \cdots$. It is a consequence of a theorem of R. L. Moore's* that K is a subset of the point set $B_1+B_2+B_3+\cdots$. Let $A_1, A_2, A_3, \cdots$ denote the point sets common to $B_1, B_2, B_3, \cdots$ respectively, and to K. Then for every positive integer n, A_n is a closed point set. Now $K=A_1+A_2+A_3+\cdots$. It is well known that no continuum is expressible as the sum of a countable number of closed point sets each of which is totally disconnected. Hence for some positive integer i, A_i is not totally disconnected and therefore contains a continuum H. The continuum H belongs to B_i, the boundary of D_i.

THEOREM 19. *In order that the point* P *of a bounded continuum* M *should be a cut point of* M *it is necessary and sufficient that* P *should be a cut point of the boundary of some complementary domain of* M.

Proof. R. L. Moore has shown† that this condition is necessary. I shall show that it is sufficient. Suppose P is a cut point of the boundary N of a complementary domain D of a bounded continuum M.

Case I. Suppose D is bounded. Then let B denote the outer boundary‡ of D. R. L. Moore has shown§ that B has no cut point. Hence, $B-P$, in case P belongs to B, or B, in case P does not belong to B, must be a subset

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* *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924). pp. 203–213.

† Loc. cit.

‡ If D is a bounded domain, the *outer boundary* of D is the boundary of the unbounded complementary domain of $\overline{D}$. If D_1 and D_2 are mutually exclusive domains, the outer boundary of D_1 *with respect to* D_2 is the boundary of that complementary domain of D_1 which contains D_2. Cf. R. L. Moore, *Concerning the separation of point sets by curves*, loc. cit., footnote to p. 475.

§ *Concerning the sum of a countable number of mutually exclusive continua in the plane*, loc. cit.

either of S_1 or of S_2, where S_1 and S_2 denote two mutually separated point sets into which, by hypothesis, N is divided by the omission of the point P. Suppose it belongs to S_1. Then let R denote the complementary domain of the continuum S_1+P which contains D. Since no point of S_2 belongs to S_1+P, and since every point of S_2 is a limit point of D, it follows that R contains S_2. Then S_2+P is a continuum which lies, except for the point P, wholly in R. By a theorem of R. L. Moore's,* there exists a simple closed curve J which contains P, encloses S_2, and lies, except for the point P, wholly in R. The curve J does not enclose or contain any point of $B-P$. Since J encloses S_2, it follows that $J-P$ contains a point of D. And since $J-P$ is connected and contains no point of N, then $J-P$ must be a subset of D. Hence, $J-P$ contains no point whatever of M. But S_2 belongs to the interior of J, and $B-P$ to the exterior of J, and J contains in common with M only the point P. It readily follows that P is a cut point of M.

Case II. Suppose D is unbounded. It is easily shown that there exists a ray r of an open curve which has exactly one point A, distinct from P, in common with N and lies except for the point A wholly in D. Now by hypothesis, $N-P$ is expressible as the sum of two mutually separated point sets S_1 and S_2, one of which, say S_1, contains the point A. The set $D-(r-A)$ is connected. Let R denote that complementary domain of the continuum S_1+r+P which contains $D-(r-A)$. The domain R is simply connected and contains S_2. Then by R. L. Moore's theorem quoted above there exists a simple closed curve J which encloses S_2, contains P, and lies except for the point P wholly in R. Just as in Case I it follows that $J-P$ is a subset of D and therefore contains no point of M. But J encloses S_2 and does not enclose or contain the point A. It follows that P is a cut point of M, and the theorem is proved.

THEOREM 20. *In order that the point P of a continuous curve M should be an end point of M it is sufficient (but not necessary) that P should be an end point of the boundary of some complementary domain of M.*

Proof. Let P denote a point of M which is an end point of N, the boundary of some complementary domain D of M. Suppose, contrary to this theorem, that P is not an end point of M. Then, by Theorem 12, M contains an arc APB having P as one of its interior points. Now either (1) each of the segments (AP) and (PB) of APB contains a point of N, or (2) one of these segments contains no point whatever of N. I will show that in either case P must belong to some simple closed curve of M. Suppose case (1) is true.

* *Concerning the separation of point sets by curves*, loc. cit., Theorem 3.

Then let X and Y denote points of N which belong to the segments AP and PB respectively of APB. Since P is not a cut point of N, it follows that $N-P$ contains an arc t from X to Y. The sum of the arcs t and APB contains a simple closed curve which contains P. Now suppose case (2) is true. Let S denote one of the segments (AP), (PB) of the arc APB which contains no point of N. Then S belongs to some complementary domain R of N. It follows from a theorem of R. L. Moore's* that the boundary of R is a simple closed curve which belongs to M. Clearly this curve must contain P. Hence, in any case, M contains a simple closed curve J which contains P. Let I and E denote the interior and exterior respectively of J. Then D is a subset either of I or of E, say of I. Let K denote the complementary domain of N which contains E. By R. L. Moore's theorem just cited, the boundary C of K is a simple closed curve which belongs to N. Clearly C must contain P. But by hypothesis P is an end point of N, and therefore, by Theorem 12, can belong to no simple closed curve of N. Thus the supposition that P is not an end point of M leads to a contradiction, and the theorem is proved.

THEOREM 21. *The set of all the end points of a continuous curve is totally disconnected.*†

Proof. Let K denote the set of all the end points of a continuous curve M. Suppose K contains a connected set H which consists of more than one point. Then from Theorem 12 and Theorem 7 it follows that every point of H must belong to the boundary of some complementary domain of M. Let D_1 denote a complementary domain of M which has the point A of H on its boundary. Now if H is a subset of the boundary of D_1, then by a theorem of R. L. Wilder's,‡ H is arcwise connected, and it easily follows that some point of H must be an interior point of some arc of M, contrary to Theorem 12. Hence, there exists a complementary domain D_2 of M which has on its boundary a point B of H which does not belong to the boundary of D_1. Let N denote the boundary of D_1. Let R denote the complementary domain of N which contains D_2. By R. L. Moore's theorem mentioned above, the boundary of R is a simple closed curve J. It is easily seen that J separates A from B. Hence, since H is connected, J must contain a point

* *Concerning continuous curves in the plane*, loc. cit., Theorem 4.

† My attention has been called to the fact that K. Menger has recently proved a proposition similar to Theorem 21. However, he uses the term *end point* in a different sense. Cf. K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272–306.

‡ Loc. cit., Theorem 20.

of H. But this is contrary to Theorem 12. It follows that K is totally disconnected.

THEOREM 22. *If K, H, and N respectively denote the set of all the cut points, end points, and simple closed curves of a continuous curve M, then $K+H+N=M$.*

Proof. Let P denote a point of M, if there be any, which is neither a cut point nor an end point of M. I shall show that P belongs to a simple closed curve of M and therefore belongs to N. Since P is not an end point of M, it follows by Theorem 12 that P is an interior point of some arc APB of M. And since P is not a cut point of M, it follows by R. L. Moore's theorem mentioned above that $M-P$ contains an arc t from A to B. On the arcs PA and PB of APB, in the order from P to A and from P to B respectively, let X and Y respectively denote the first points which belong to t. The simple closed curve formed by the arc XY of t plus the arc XPY of APB contains the point P. Hence, P belongs to N, and it follows that $K+H+N=M$.

THEOREM 23. *If N denotes the point set consisting of the sum of all the simple closed curves contained in a continuous curve M, then every connected subset of $M-N$ is arcwise connected.*

Proof. Let L denote any definite connected subset of $M-N$. It follows from Theorem 22 that every point of L is either a cut point or an end point of M. And since, by Theorem 21, the set of all the end points of M is totally disconnected, L must contain at least one point P which is a cut point of M. By the part of Theorem 19 established by R. L. Moore, P belongs to the boundary B of some complementary domain D of M. I will show that L is a subset of B. Suppose, on the contrary, that L contains a point Q which does not belong to B. Then Q lies in a complementary domain R of B. By R. L. Moore's theorem, the boundary J of R is a simple closed curve which belongs to B. Since L contains no point of N, J contains neither P nor Q. Now R is either the interior or the exterior of J. And if R is the interior [exterior] of J, then Q belongs to the interior [exterior] of J, and P belongs to the exterior [interior] of J. Hence, in any case P and Q are separated by J. Therefore, L contains a point of J, contrary to hypothesis. Thus the supposition that L contains a point which does not belong to B leads to a contradiction. Hence, L is a subset of B, and by a theorem of R. L. Wilder's* it follows that L is arcwise connected.

* Loc. cit.

THEOREM 24. *Under the same hypothesis as in Theorem 23, if L is any connected subset of $M-N$, then $\bar{L}$ is an acyclic continuous curve which belongs to the boundary of some single complementary domain of M, and every point of $\bar{L}$ is either a cut point or an end point of M.*

Proof. From the proof of Theorem 23 it follows that $\bar{L}$ belongs to the boundary B of some complementary domain D of M. Now since, by R. L. Wilder's theorem, every connected subset of B is arcwise connected, and since every point of $\bar{L}-L$ is a limit point of L by definition, it can easily be shown by methods identical with those used in the proof of Theorem 16 that every point of $\bar{L}-L$ is either a cut point or an end point of B. Now, by Theorem 19, every cut point of B is a cut point also of M; and by Theorem 20, every end point of B is an end point also of M. Hence, since by Theorem 22, every point of L is either a cut point or an end point of M, every point of $\bar{L}$ is either a cut point or an end point of M. By Theorem 9 and the above argument, it follows that $\bar{L}$ is an acyclic continuous curve which satisfies all the conditions of Theorem 24.

THEOREM 25. *If M is the complete boundary of two mutually exclusive domains D_1 and D_2, then no point of M is an end point of any continuum which contains M.*

Proof. It is sufficient to show that M contains no end point of itself. Suppose, on the contrary, that there exists a point P which is an end point of M. Then P belongs to no continuum of condensation of M. For let H be any subcontinuum of M which contains P. R. L. Moore has shown* that $M-H$ is connected. Hence, since, by supposition, P is an end point of M, P is not a limit point of $M-H$. Therefore P belongs to no continuum of condensation of M. By a theorem of R. L. Wilder's† it follows that P is accessible from each of the domains D_1 and D_2. Hence, if A and B are points of D_1 and D_2 respectively, there exist arcs AP and BP such that $AP-P$ and $BP-P$ are subsets of D_1 and D_2 respectively. Since for any continuum H of M which contains P, P is not a limit point of $M-H$, it can easily be shown that there exists an arc EFG from a point E of $AP-P$ to a point G of $BP-P$ which contains no point whatever of M. This is impossible, because E belongs to D_1, and G belongs to D_2, and D_1 and D_2 are mutually exclusive domains by hypothesis. Thus the supposition that M contains an end point leads to a contradiction, and the theorem is proved.

THEOREM 26. *No end point of a continuum M can be a boundary point of more than one complementary domain of M.*

* *Concerning the common boundary of two domains*, loc. cit., Theorem 2.

† Loc. cit., Theorem 2.

Proof. Suppose, on the contrary, that an end point P of M belongs to the boundary of each of two mutually exclusive domains D_1 and D_2 which are complementary to M. Let N denote the outer boundary of D_2 with respect to D_1. By a theorem of R. L. Moore's* N is the complete boundary of each of two mutually exclusive domains R_1 and R_2 which contain D_1 and D_2 respectively. And since P is a limit point both of R_1 and R_2, P must belong to N. But P is an end point of M, and by Theorem 25, it cannot belong to any point set which belongs to M and is the complete boundary of two mutually exclusive domains. Thus the supposition that P belongs to the boundary of more than one complementary domain of M leads to a contradiction.

THEOREM 27. *The collection G of all the continua (X) contained in the boundary M of a simply connected bounded domain D such that X is the complete boundary of some two mutually exclusive domains, is countable.*

Proof. Let K denote the unbounded complementary domain of M, and let B denote its boundary. For every element X of G, I shall define a domain R_x as follows. (1) When $X=B$, let $R_x=K$. (2) For every element (X) of G such that B is not a subset of X, the unbounded complementary domain of X contains D. For every such element X of G let R_x denote one *bounded* domain which has X as its boundary. (3) For every element (X) of G such that $B \neq X$ but such that B is a subset of X, it is true that X is the complete boundary of at least two *bounded* mutually exclusive domains, because for every such element X, the unbounded complementary domain of X is identical with K, and X is not the complete boundary of K. Not both of these bounded domains can contain points of D. Then for every such element X of G let R_x denote one of the bounded domains of which X is the boundary and which contains no point whatever of D.

Clearly, for every element X of G there corresponds a domain R_x as above defined. It is evident that for every element X, R_x is a complementary domain of M. It is well known that the collection T of all such domains (R_x) is countable. Since every element of G is the boundary of at least one domain of the collection T, it follows that G is countable.

I will remark here that Theorem 27 is a generalization of a theorem of R. L. Wilder's to the effect that the collection of all the simple closed curves contained in the boundary of a complementary domain of a continuous curve is countable.

* Loc. cit., Theorem 1.

THEOREM 28. *If K denotes the set of all the cut points of a bounded continuum M, G denotes the collection of all the continua (X) of M such that X is the complete boundary of two mutually exclusive domains, and T denotes the point set obtained by adding together all the point sets of the collection G, then the set of points common to K and T is countable.*

Proof. Let H denote the set of points common to K and T. Let the complementary domains of M be ordered $D_1, D_2, D_3, \cdots$, and their boundaries denoted by $B_1, B_2, B_3, \cdots$, respectively. Now by the part of Theorem 19 established by R. L. Moore, K is a subset of the point set $B_1+B_2+B_3+\cdots$. Hence, if for every i, A_i denotes the set of points common to H and B_i, then $H=A_1+A_2+A_3+\cdots$. I will show that for every positive integer i, A_i is a countable set of points. Let P denote a point of A_i. Then P belongs to some element X of G, and X is the complete boundary of two domains R_1 and R_2. One of these domains, say R_1, contains no point whatever of D_i. Let Y denote the outer boundary of D_i with respect to R_1. Then Y is an element of G which contains P and is a subset of B_i. Let G_i denote the collection of all those elements of G which are subsets of B_i. Then by Theorem 27, G_i is countable. It was just shown that every point of A_i belongs to some element of G_i. Since by R. L. Moore's theorem, no element of G contains any cut point of itself, it follows that no element of G_i contains more than a countable number of points of K. It follows, then, that A_i is countable, and therefore H is countable.

THEOREM 29. *If K denotes the set of all the cut points and N denotes the point set consisting of the sum of all the simple closed curves of a continuous curve, then the set of points common to K and N is countable.*

Theorem 29 is a corollary to Theorem 28.

THEOREM 30. *Every continuum M in a plane S is connected im kleinen at every one of its end points which is accessible from some point of $S-M$.*

Proof. Suppose P is any end point of M which is accessible from $S-M$. There exists an arc t having P as one of its extremities and such that $t-P$ is a subset of $S-M$. Suppose, contrary to this theorem, that M is not connected im kleinen at P. Then there exists a circle C' having center at P and such that every circle which is concentric with C' encloses a point X which belongs to M but which lies in no connected subset of M which contains P and is enclosed by C'. Let C be a circle concentric with C' and of diameter less than $\frac{1}{2}$ the diameter of C' and also less than $\frac{1}{4}$ the diameter of t. Then M contains a countable infinity of continua $M_\infty, M_1, M_2, M_3, \cdots$

such that (1) each of these continua has at least one point on C and is contained in C plus its interior, (2) no two of these continua have a point in common, and, indeed, no one of them, save possibly M_∞, is a proper subset of any connected point set common to M and to C plus its interior, (3) no point of the set $M_1+M_2+M_3+\cdots$ lies together with P in any connected subset of M which is enclosed by C', and (4) M_∞ contains the point P and is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$.* Let I denote M_∞ plus all the bounded complementary domains of M_∞. It is clear that I is a maximal connected subset of the closed point set $I+M_1+M_2+M_3+\cdots$, and that I neither separates the plane nor contains any point of $t-P$. Hence, by a theorem of R. L. Moore's† there exists a simple closed curve J which encloses I, contains no point of the point set $M_1+M_2+M_3+\cdots$, is a subset of the interior of C', and is such that its exterior contains at least one point A_0 of t. Let B (see Fig. 2) denote a point

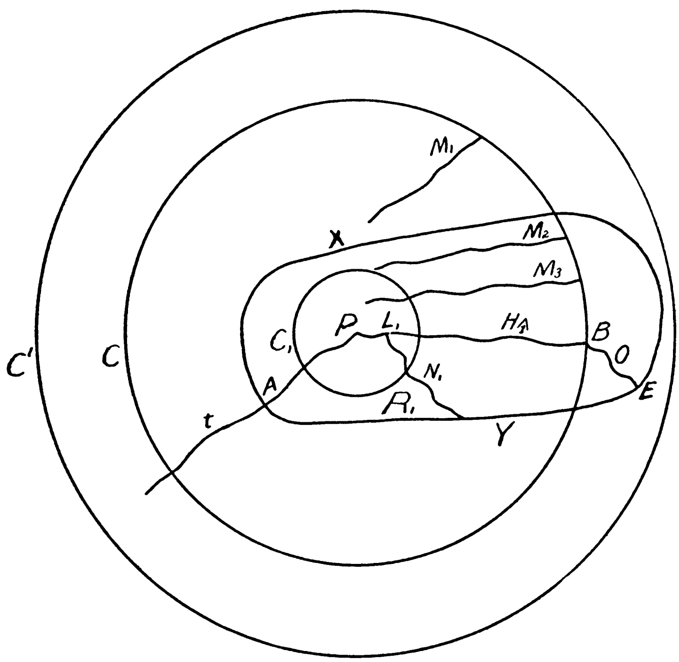

Fig. 2

* For indications of the proof of this statement see papers by R. L. Moore: *Continuous sets which have no continuous sets of condensation*, Bulletin of the American Mathematical Society, vol. 25 (1919), pp. 174–176, and *A characterization of Jordan regions by properties having no reference to their boundaries*, loc. cit.

† *Concerning the separation of point sets by curves*, loc. cit.

which is common to M_∞ and C. The point set M_∞ contains* a continuum H which is irreducible between P and B. Let H' denote the point set obtained by adding to H all of its bounded complementary domains. Now in the order from P to A_0 on t, let A denote the first point belonging to J. It is readily shown that there exists an arc BOE from B to a point E of J such that (BOE) is common to the interior of J and to the exterior of C. Let AXE and AYE respectively denote the two arcs of J from A to E. The continuum consisting of H' plus the arc PA of t plus the arc BOE divides the interior of J into just two domains D_1 and D_2. One of these domains, say D_1, has AXE in its boundary, and the other, D_2, has AYE in its boundary. It follows that one of these domains, say D_1, contains infinitely many of the continua $M_1, M_2, M_3, M_4, \cdots$.

Now let us consider the maximal connected subsets of $M-H$. It is evident that each of the continua $M_1, M_2, M_3, \cdots$ must belong to one such subset of $M-H$. And since P is an end point of M, it follows that no maximal connected subset of $M-H$ can contain more than a finite number of these continua. Hence, it is true that there exists an infinite sequence of distinct maximal connected subsets of $M-H$, each of which contains at least one of the continua $M_1, M_2, M_3, \cdots$. Let one such sequence be ordered $K_1, K_2, K_3, \cdots$. For every positive integer i, H contains at least one limit point of K_i. Let C_1 be a circle having P as center which lies entirely within J and is of diameter less than $\frac{1}{2}$ the diameter of C. From a theorem of Janiszewski's† it follows that H contains a continuum L_1 which contains P and a point of C_1 and which is the maximal connected subset of H which contains P and which belongs to C_1 plus its interior. By a theorem of Miss Mullikin's,‡ the continuum H contains a connected set Q which contains neither the point B nor any point of L_1 but which has B for a limit point and has at least one limit point in L_1. Now since H is irreducible between P and B, it readily follows that if H_1 denotes the point set $Q+B$, then $H=\overline{H}_1+L_1$. Since P is an end point of M, it follows (1) that P is not a limit point of H_1, and (2) that for not more than a finite number of positive integers (i) does $\overline{H}_1$ contain a limit point of K_i. Hence, there exists a positive integer n_1 such that $\overline{H}_1$ contains no limit point of K_{n_1}. Hence since H contains at least one limit point of K_{n_1}, L_1 must contain a point X_1 which is a limit point of K_{n_1}. Now from condition (3), above, which the sequence of sets $M_1, M_2, M_3, \cdots$ satisfies, it follows that for every positive integer i, K_i contains at least one point in common with J. Hence, by Miss Mullikin's

* Cf. Janiszewski, loc. cit.

† Loc. cit.

‡ These Transactions, vol. 24 (1922), pp. 144–162.

theorem mentioned above, $\overline{K}_{n_1}$ contains a connected set N_1^0 which contains no point of either of the continua L_1 and $J+BOE$ but is such that each of these continua contains at least one limit point of N_1^0. Clearly, N_1^0 is a subset either of D_1 or of D_2. And since D_1 contains infinitely many of the continua $M_1, M_2, M_3, \cdots$, of which only a finite number can contain points in common with N_1^0, it can be shown that N_1^0 cannot belong to D_1, and therefore, must belong to D_2. Let N_1 denote the point set obtained by adding to N_1^0 all of its limit points. It has already been shown that N_1 must be a subset of $D_2+L_1+$the arc $AYEOB$. It is evident that N_1 divides D_2 into at least two domains, one of which must have the arc AP of t in its boundary. Let R_1 denote the one which has AP in its boundary. It is clear, then, that no point of $\overline{H}_1-L_1\cdot\overline{H}_1$ is a limit point of R_1 and, therefore, that the boundary of R_1 is a subset of $N_1+L_1+$the arc $PAYEOB$.

Let C_2 be a circle concentric with C_1 which encloses and contains no point of either N_1 or H_1 and which is of diameter less than $\frac{1}{2}$ the diameter of C_1. Let L_2 be a subcontinuum of H which bears the same correspondence to C_2 as L_1 bears to C_1. Let sets H_2, K_{n_2}, N_2^0, and N_2 be selected and defined with respect to C_2 and L_2 just as the corresponding sets H_1, K_{n_1}, N_1^0, and N_1 were defined with respect to C_1 and L_1. Again, N_2^0 must be a subset of D_2. Hence, N_2 contains a point A_2 on the arc $AYEOB$. And since N_1 and N_2 can have no point in common, it can easily be shown that on $AYEOB$, in the order from A to B, A_2 precedes every point which belongs to N_1. Hence, N_2^0 is a subset of R_1. Let R_2 denote that complementary domain of the continuum $L_2+N_2+$the arc $PAYEOB$ which is a subset of R_1 and has the arc PA of t in its boundary. Again, $\overline{H}_2-L_2\cdot\overline{H}_2$ contains no limit point of R_2. This process may be continued indefinitely, and it follows that there exists an infinite sequence of continua $N_1, N_2, N_3, \cdots$, having the properties as above indicated. Also there exists a sequence of domains $R_1, R_2, R_3, \cdots$, such that for every positive integer n, R_n has the arc PA of t in its boundary, contains R_{n+1}, and contains $N_n+N_{n+1}+\cdots$. And there exist two sequences of connected point sets $L_1, L_2, L_3, \cdots$ and $H_1, H_2, H_3, \cdots$, such that for every positive integer n, $L_n+\overline{H}_n=H$, such that if r denotes the radius of C, then L_n contains P and is of diameter less than $3r/n$, and such that $\overline{H}_n-L_n\cdot\overline{H}_n$ contains no limit point whatever of R_n.

Let N denote the limiting set of the sequence of continua $N_1, N_2, N_3, \cdots$. It readily follows from the above properties of this sequence that N contains P but contains no other point whatever of H. The set N contains at least one point U of J. Now N is a continuum. Let N_u denote the maximal connected subset of $N-P$ which contains U. Then clearly P is a limit point of N_u. But P is an end point of M, and is, therefore, not a limit point of any connected subset of $M-H$. Thus the supposition that M is not connected im kleinen at P leads to a contradiction, and the theorem is proved.

The following example demonstrates that the conclusion of Theorem 30 does not *necessarily* remain valid if the restriction that the end point of M in question shall be accessible from $S-M$ is removed. Let I be the straight line interval from (0,0) to (1,0). And for every integer n such that $n=(2)^i$, where i takes on all positive integral values from 1 to ∞, let L_i denote the broken line through the points $(1/n,0)$, $(1/n,-1/n)$, $(-1/n,-1/n)$, $(-1/n, 1/n)$, $(1,1/n)$, $(1,3/4n)$, and $(0,3/4n)$ in the order named. (See Fig. 3.)

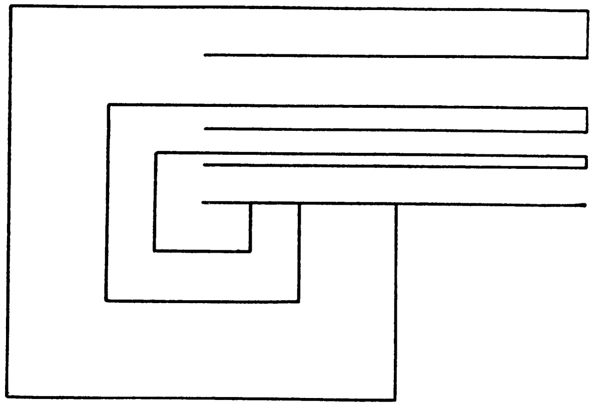

Fig. 3

If M denotes the continuum $I+L_1+L_2+L_3+\cdots$, and P denotes the point (0,0), then P is an end point of M, but M is not connected im kleinen at P.

THEOREM 31. *If a continuum M is irreducible between some pair of points A, B, then M is connected im kleinen at every one of its end points.*

Proof. Let P denote an end point of M. Let us first suppose that either $P\equiv A$ or $P\equiv B$, say $P\equiv B$. Then by Janiszewski's theorem mentioned above it follows that if C denotes any circle having P as center, then C encloses a subcontinuum H of M which consists of more than one point and which contains B but not A. From Miss Mullikin's theorem it follows immediately that $M-H$ contains a connected set N which contains A and which has at least one limit point in H. Since M is irreducible between A and P, clearly $M=H+\overline{N}$. And since P is an end point of M, P is not a limit point of $M-H$. Hence, there exists a circle K concentric with and within C which encloses no point of $M-H$. Any point of M which is interior to K lies together with P in a closed and connected subset of M which is enclosed by C, namely, in H itself. Hence, M is connected im kleinen at P.

Now in case neither $P\equiv A$ nor $P\equiv B$, then M is the sum of two continua K_a and K_b, irreducible between A and P, and B and P respectively. By the above argument, both K_a and K_b are connected im kleinen at P. It follows that their sum, M, is connected im kleinen at P.

In his paper *Concerning the cut points of continuous curves and of other closed and connected point sets*, R. L. Moore proves the following results.

I. In order that a bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that every subcontinuum of M should contain uncountably many points each of which is a cut point of M.

II. In order that the *continuous curve* M should contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points of M that are not cut points of M, then no subset of K disconnects M even in the weak sense.

In Theorem 32, below, I shall establish a generalization of R. L. Moore's result (II) quoted here.

THEOREM 32. *In order that the bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that if K denotes the set of all those points of M which are not cut points of M, then no subset of K disconnects M even in the weak sense.*

Proof. The condition is sufficient. For suppose a bounded continuum M satisfies the condition but is not an acyclic continuous curve. Then by result (I), above, of R. L. Moore's, it follows that M contains a subcontinuum N which does not contain more than a countable number of cut points of M. Let A and B denote two points of N. By hypothesis, $M-[K-(A+B)]$ is connected in the strong sense. Hence, it contains a continuum H which contains A and B. Since every point of H, except possibly the points A and B, is a cut point of M, it follows by Theorem 9 that H is a continuous curve. Therefore, H contains an arc t from A to B. Since N contains not more than a countable number of cut points of M, there exist points E and F on t in the order A, E, F, B such that the interval EF of t contains no point whatever of N. Since N contains both A and B, it follows by Miss Mullikin's theorem that N contains a connected set Q containing no point of t and such that each of the intervals AE and FB of t contains at least one limit point of Q. But t is a continuum every point of which, save possibly two, is a cut point of M, and Q is a connected subset of $M-t$. Hence, by Theorem 11, t can contain at most one limit point of Q. Thus the supposition that M is not an acyclic continuous curve leads to a contradiction.

It follows by R. L. Moore's theorem II quoted above that the condition is necessary.

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