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LOCAL THEORY OF COMPLEX FUNCTIONAL DIFFERENTIAL EQUATIONS(1)

BY
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Abstract. We consider the equation (*) $f'(z)=F(z,f(z),f(g(z)))$ where $F(z, u, w)$ and $g(z)$ are given analytic functions and $f(z)$ is an unknown function. The question of local existence of a solution of (*) about a point z_0 is natural only if $g(z_0)=z_0$. We classify fixed points z_0 of g as *attractive* if $|g'(z_0)|<1$, *indifferent* if $|g'(z_0)|=1$, and *repulsive* if $|g'(z_0)|>1$. In the attractive case (*) has a unique analytic solution satisfying an initial condition $f(z_0)=w_0$. This solution depends continuously on w_0 and on the functions F and g. For "most" indifferent fixed points the initial-value problem has a unique solution. Around a repulsive fixed point a solution in general does not exist, though in exceptional cases there may exist a singular solution which disappears if the equation is subjected to a suitable small perturbation.

Introduction. A natural way of generalizing ordinary differential equations is to permit the argument of the unknown function to appear, in at least one term, as a function of the independent variable. A fairly wide class of such "functional differential equations" can be written in the form

$$f'(z) = F(z,f(z),f(g(z))) \tag{*}$$

where F and g are given functions and f is an unknown function. The case of systems (and also higher order equations reducible to first order systems) can be included if we allow the functions F and f to be vector-valued.

In the real case functional differential equations have been extensively studied, but for the complex case the literature is quite sparse. Flamant [2] studied the linear equation $f'(z)=a(z)f(g(z))+b(z)$ in the case where $g(z)$ is a fractional linear transformation. Izumi [3] established existence theorems for the same equation where $g(z)$ is an analytic function mapping the unit disc into itself. In a series of papers Robinson ([6]–[9] and others) obtained various results on certain special FDE. Leont′ev [5] considered differential-difference equations with constant coefficients in the complex case. But none of these papers seem really concerned

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with a systematic general study of FDE over the complex plane. In this paper we attempt to begin such a study.

Let $F(z, u, w)$ and $g(z)$ be analytic functions. Our basic question concerns the existence and behavior of analytic solutions of (*). The natural starting point of a theory is the question of *local existence* of solutions, and this is the topic of the present paper. Note that the problem of the local existence of a solution $f(z)$ about a point z_0 is interesting only if z_0 is a fixed point of $g(z)$. For if $g(z_0) \neq z_0$, we can specify $f(z)$ to be an arbitrary analytic function in a small neighborhood of $g(z_0)$ and then determine $f(z)$ about z_0 by solving an ordinary differential equation. Fixed points of g will play a crucial role throughout our investigations.

After a brief introductory section on power series solutions, we take up systematically the study of local solutions about fixed points of g. We shall find that there are three cases, which we investigate individually. We conclude with a theorem on the local existence of a solution about a "cycle" of g.

1. **Power series solutions.** Let $g(z)$ be analytic in a neighborhood of the fixed point z_0, and let $F(z, u, w)$ be analytic in a neighborhood of (z_0, w_0, w_0). We then seek a solution of

$$f'(z) = F(z, f(z), f(g(z))) \tag{1.1}$$

satisfying the initial condition

$$f(z_0) = w_0. \tag{1.2}$$

It is natural to try to find a power series solution. We have the expansions

$$g(z) - z_0 = b_1(z-z_0) + b_2(z-z_0)^2 + \cdots, \tag{1.3}$$

$$F(z, u, w) = \sum_{i,j,k} A_{ijk}(z-z_0)^i(u-w_0)^j(w-w_0)^k, \tag{1.4}$$

and we seek a solution

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots. \tag{1.5}$$

First observe that $f(z_0) = w_0$ implies $a_0 = w_0$. Next substitute the series expansions into (1.1). We find that on the left-hand side the coefficient of $(z-z_0)^n$ is $(n+1)a_{n+1}$, while the coefficient on the right-hand side involves no a_k with $k > n$. Thus we have recursion relations

$$a_{n+1} = \Phi_n(a_0, a_1, \ldots, a_n),$$

and so the a_n are uniquely determined. Conversely, if the a_n are defined recursively by the above formula, then it is clear that the series (1.5) will formally satisfy (1.1). We have proved

THEOREM 1.1. *Let $g(z)$ and $F(z, u, w)$ have formal power series expansions* (1.3) *and* (1.4). *Then there is a unique formal power series* (1.5), *with $a_0 = w_0$, that formally satisfies equation* (1.1).

An immediate corollary is the following uniqueness theorem:

THEOREM 1.2. *Let $g(z)$ be analytic about z_0 and $F(z, u, w)$ be analytic about (z_0, w_0, w_0). Suppose $g(z_0)=z_0$. Then the FDE* (1.1) *has at most one analytic solution about z_0 that satisfies the initial condition $f(z_0)=w_0$.*

To establish the existence of an analytic solution to the FDE (1.1) it would suffice to prove the convergence of the formal power series solution. Izumi [3] derived an existence theorem for the linear equation $f'(z)=a(z)f(g(z))+b(z)$, where $g(z)$ maps the unit disc into itself, by estimating the coefficients. But in general proving convergence is quite complicated, and we will establish our basic local existence theorem by another method. However, power series are a very useful tool for investigating particular FDE.

For example, consider the equation

$$f'(z) = f(z^2). \tag{1.6}$$

The function $g(z)=z^2$ has fixed points 0 and 1. We find about 0 and 1 respectively the two formal power series solutions

$$f(z) = 1+z+\frac{z^3}{3}+\frac{z^7}{3\cdot 7}+\frac{z^{15}}{3\cdot 7\cdot 15}+\cdots \tag{1.7}$$

and

$$f(z) = 1+b_1(z-1)+b_2(z-1)^2+\cdots \tag{1.8}$$

where the b_j are determined recursively by

$$b_{n+1} = \frac{1}{n+1}\left[2^n b_n+\binom{n-1}{1} 2^{n-1}b_{n-1}+\cdots\right].$$

It is easy to see that the power series (1.7) has radius of convergence 1, so (1.7) is an analytic solution of (1.6) about $z=0$. On the other hand the coefficients b_n in (1.8) satisfy the inequality $b_n \geqq 2^{n(n-1)/2}/n!$ as can be verified by induction, so the series (1.8) has radius of convergence 0. Thus there exists no nontrivial analytic solution of (1.6) about $z=1$.

This example shows that the existence of an analytic solution about a fixed point depends on the nature of the fixed point. The following classification of fixed points plays a crucial role in our subsequent investigations.

DEFINITION. The fixed point z_0 of the analytic function $g(z)$ is *attractive* if $|g'(z_0)|<1$, *repulsive* if $|g'(z_0)|>1$, and *indifferent* if $|g'(z_0)|=1$.

2. **Attractive fixed point.** We shall study first the case of an attractive point, where the situation is as nice as could be desired. The initial-value problem possesses a unique analytic solution which depends continuously on the initial-value and on the functions F and g. The local existence theorem is proved by a straightforward adaptation of the proof for ordinary differential equations that uses the contraction

mapping theorem applied to an equivalent integral equation. What permits the proof to go through is the fact that g maps small discs about the fixed point into themselves. We consider the vector equation

$$f'(z) = F(z, f(z), f(g(z))) \tag{2.1}$$

where $F(z, u, w)$ is an analytic mapping of an open set in $\mathbf{C}\times\mathbf{C}^n\times\mathbf{C}^n$ into $\mathbf{C}^n$. We seek a solution $f(z)$ that maps a subset of $\mathbf{C}$ into $\mathbf{C}^n$.

THEOREM 2.1. *Let $g(z)$ be a function analytic in a neighborhood of an attractive fixed point z_0. Let the function $F(z, u, w)$ be defined and analytic on an open region $D\subset\mathbf{C}\times\mathbf{C}^n\times\mathbf{C}^n$ with range of F contained in $\mathbf{C}^n$. Let w_0 be any vector in $\mathbf{C}^n$ such that $(z_0, w_0, w_0)\in D$. Then there is a unique analytic function $f(z)$ mapping a neighborhood of z_0 into $\mathbf{C}^n$ which satisfies* (2.1) *and the initial condition $f(z_0)=w_0$.*

Proof. Choose $a, b>0$ so that $D_0=\{(z, u, w) : |z-z_0|\leqq a,\ |u-w_0|\leqq b,\ |w-w_0|\leqq b\}\subset D$ and g is analytic on the ball $B_a(z_0)=\{z : |z-z_0|<a\}$. Choose $\delta_1>0$ so small that $g(B_r(z_0))\subset B_r(z_0)$ for $0<r\leqq\delta_1$. (This is possible because z_0 is an attractive fixed point of g.) Let L be a Lipschitz constant for F on D_0, i.e.

$$|F(z, u, w)-F(z, u^*, w^*)| \leqq L \max(|u-u^*|, |w-w^*|).$$

Let M be the supremum of $|F|$ over D_0. Finally, choose δ so that

$$0 < \delta < \min(\delta_1, a, b/M, 1/L) \tag{2.2}$$

and let $E=B_\delta(z_0)$.

Consider the space S of functions analytic on E, continuous on $\bar{E}$, with range in $\mathbf{C}^n$. Define a norm on S by $\|f\|=\sup_{z\in E}|f(z)|$. Let $S_0=\{f\in S : \|f-w_0\|\leqq b\}$, and define an operator U on S_0 by

$$Uf(z) = w_0+\int_{z_0}^{z} F(\zeta, f(\zeta), f(g(\zeta)))\, d\zeta \tag{2.3}$$

where the integration path is the ray from z_0 to z.

Since F is analytic on D_0 and $g(E)\subset E$, it is clear that $Uf(z)$ is well defined if $f\in S_0$. Also $f\in S_0$ implies $Uf\in S_0$, because $|Uf(z)-w_0|\leqq M\cdot\delta\leqq b$ for $z\in E$. Finally, it is easy to see that $|Uf(z)-Uf^*(z)|\leqq\delta L\|f-f^*\|$, $\delta L<1$, so that $U: S_0\to S_0$ is a contraction operator. S_0 is a complete metric space, so we conclude that U has a unique fixed point, which is a solution of (2.1) satisfying $f(z_0)=w_0$.

Continuous dependence. The solution $f(z)$ of (2.1) depends on the initial value w_0 and on the functions g and F. We shall see that this dependence is continuous. The proof will actually establish a little more—if w_0, g, and F are all perturbed by a sufficiently small amount, the corresponding equation will still have a solution in the neighborhood $|z-z_0|<\delta/2$ (where δ is the same number defined in (2.2)), even though z_0 may not be a fixed point of the perturbed function g. Our proof depends on the following simple lemma about contraction operators in metric spaces.

LEMMA 2.2. *Let (X, d) be a complete metric space. Let U_0 be an arbitrary operator on X. Let U_1 be a contraction operator with contraction constant $\lambda<1$. Suppose U_0 has a fixed point x_0. Let x_1 be the fixed point of U_1 (which we know must exist and be unique). Then $d(x_0, x_1) \leqq d(U_0x_0, U_1x_0)/(1-\lambda)$.*

COROLLARY. *If $U_1 \to U_0$ strongly in such a way that $\lambda \leqq k<1$, then $x_1 \to x_0$.*

Proof of lemma. Since U_1 is a contraction operator, its fixed point is given by $x_1 = \lim_{n\to\infty} U_1^n x_0$. Hence

$$\begin{aligned} d(x_0, x_1) &= \lim_{n\to\infty} d(x_0, U_1^n x_0) \\ &\leqq \limsup_{n\to\infty} [d(x_0, U_1x_0)+d(U_1x_0, U_1^2x_0)+\cdots+d(U_1^{n-1}x_0, U_1^n x_0)] \\ &\leqq \limsup_{n\to\infty} [(1+\lambda+\cdots+\lambda^{n-1})\, d(x_0, U_1x_0)] \\ &\leqq d(x_0, U_1x_0)/(1-\lambda) = d(U_0x_0, U_1x_0)/(1-\lambda). \end{aligned}$$

We now introduce topologies (via norms) in order to make sense out of the statement that the function F^* and g^* are "close" to the functions F and g. Let D_0 and E be the sets defined in the proof of Theorem 2.1. Then for F analytic on D_0 define $\|F\| = \sup_{D_0} |F|$, and for g analytic on E define $\|g\| = \sup_{z\in E} |g(z)|$.

THEOREM 2.3. *If (w_0^*, g^*, F^*) is sufficiently close to (w_0, g, F), then the corresponding equation*

$$f^{*\prime}(z) = F^*(z, f^*(z), f^*(g^*(z))) \tag{2.1*}$$

with initial condition $f^*(z_0) = w_0^*$ has a unique solution $f^*(z) = f(z; w_0^*, g^*, F^*)$ defined for $|z-z_0| < \delta/2$ (where δ is the number defined in (2.2)). Also, there exist positive constants A, B, C such that

$$|f(z; w_0, g, F) - f(z; w_0^*, g^*, F^*)| \leqq A|w-w_0^*| + B\|g-g^*\| + C\|F-F^*\| \tag{2.4}$$

for $|z-z_0| < \delta/2$.

Proof. Let $\delta^* = \delta/2$. Define the sets E^*, S^*, S_0^* in the same way we defined the sets E, S, S_0 in the proof of Theorem 2.1, only in terms of δ^* in place of δ. Require g^* to be so close to g that $g^*(E^*) \subset E^*$, and require F^* to be so close to F that $\|F^*\| \leqq (3/2)\|F\| = (3/2)M$. Finally, require $|w_0^* - w_0| < b/4$. Then $U^* = U(w_0^*, g^*, F^*)$ defined by

$$U^*f(z) = w_0^* + \int_{z_0}^{z} F^*(\zeta, f(\zeta), f(g(\zeta)))\, d\zeta \tag{2.3*}$$

maps S_0^* into itself.

Indeed, U_0^* is defined on S_0^* because g^* maps E^* into itself. And $f \in S_0^*$ implies $U^*f \in S_0^*$ because

$$|U^*f(z)-w_0| \leqq |U^*f(z)-w_0^*|+|w_0^*-w_0| \leqq \frac{\delta}{2}\cdot\frac{3}{2}M+\frac{b}{4} \leqq \frac{b}{2M}\frac{3}{2}M+\frac{b}{4} = b$$

if $|z-z_0|<\delta/2$.

Next, by requiring F^* to be sufficiently close to F, we can make the Lipschitz constant $L^* \leqq 2L$. Then

$$|U^*f_1(z)-U^*f_2(z)| \leqq (\delta/2)\cdot 2L\|f_1-f_2\| = \delta L\|f_1-f_2\|,$$

so U^* is a contraction operator with contraction constant $\delta L<1$. It follows that for (w_0^*, g^*, F^*) sufficiently close to (w_0, g, F), (2.1*) has a unique solution f^*.

We now apply the lemma to conclude that

$$\begin{aligned}\|f-f^*\| &\leqq \frac{1}{1-\delta L}\|Uf-U^*f\| \\ &\leqq \frac{1}{1-\delta L}\sup_{z\in E^*}\Big\{|w_0-w_0^*| \\ &\qquad + \Big|\int_{z_0}^{z} F(\zeta,f(\zeta),f(g(\zeta)))\,d\zeta-\int_{z_0}^{z} F^*(\zeta,f(\zeta),f(g^*(\zeta)))\,d\zeta\Big|\Big\}.\end{aligned}$$

Adding and subtracting $\int_{z_0}^{z} F(\zeta,f(\zeta),f(g^*(\zeta)))\,d\zeta$, we find that the second term in the brace is majorized by

$$|z-z_0|\cdot L \sup_{|\zeta-z_0|<\delta/2}|f(g(\zeta))-f(g^*(\zeta))|+|z-z_0|\cdot\|F-F^*\|. \tag{2.5}$$

Now

$$|f(g(\zeta))-f(g^*(\zeta))| = \Big|\int_{g^*(\zeta)}^{g(\zeta)} f'(t)\,dt\Big| \leqq K|g(\zeta)-g^*(\zeta)| \leqq K\|g-g^*\|,$$

where $K=\sup_{|t-z_0|<\delta/2}|f'(t)|<\infty$. Combining these inequalities we have

$$\|f-f^*\| \leqq \frac{1}{1-\delta L}|w_0-w_0^*|+\frac{\delta/2}{1-\delta L}\|g-g^*\|+\frac{\delta/2}{1-\delta L}\|F-F^*\|,$$

establishing (2.4) and completing the proof of the theorem.

Analytic dependence. Suppose that the function F also depends on a complex parameter λ, and that this dependence is analytic for $\lambda\in\Omega$. Then the solution $f_\lambda(z)$ of

$$f'(z) = F(z,f(z),f(g(z));\lambda), \qquad f(z_0) = w \tag{2.6}$$

is analytic in λ for $\lambda\in\Omega$. Also the solution, considered as a function of the initial value w, is an analytic function of w. To prove these facts we need another lemma about contraction operators.

LEMMA 2.4. *Let X be a Banach space. Let $\{U_\lambda\}$ be a family of contraction operators on X, defined for the complex parameter λ in a neighborhood of λ_0. Suppose that the operator-valued mapping $\lambda \to U_\lambda$ is analytic in λ and that the contraction constants α_λ are all $\leqq \alpha < 1$ for λ in a neighborhood of λ_0. Then the fixed point f_λ depends analytically on λ.*

Proof. Let h be an arbitrary element in X. Then the fixed point f_λ is given by

$$f_\lambda = h + \sum_{n=0}^{\infty} (U_\lambda^{n+1} - U_\lambda^n)h = h + \sum_{n=0}^{\infty} h_n(\lambda). \tag{2.7}$$

Now for λ close to λ_0, $h_n(\lambda)$ is analytic, and the contraction constant α_λ of U_λ is $\leqq \alpha < 1$, so $\|h_n(\lambda)\| \leqq \|U_\lambda^n(U_\lambda h - h)\| \leqq \alpha^n \|U_\lambda h - h\|$. It follows by the Weierstrass M test that the series (2.7) is uniformly convergent, and so f_λ is an analytic function of λ.

THEOREM 2.5. *Suppose that the function $F(z, u, w; \lambda)$ depends analytically on λ for $\lambda \in \Omega$. Then the solution $f_\lambda(z)$ of* (2.6) *depends analytically on $\lambda \in \Omega$ for $|z - z_0| < \delta/2$. Also for $|z - z_0| < \delta/2$ the solutions $f(z, w, \lambda)$ is analytic in w.*

Proof. Consider the operator U_λ defined on S_0^* by

$$U_\lambda f(z) = w + \int_{z_0}^{z} F(\zeta, f(\zeta), f(g(\zeta)); \lambda)\, d\zeta \tag{2.8}$$

where $F(z, u, w; \lambda_0) = F(z, u, w)$. Then if λ is sufficiently close to λ_0, F_λ is sufficiently close to F that the Lipschitz constant $L_\lambda \leqq 2L$. It follows as in the proof of Theorem 2.3 that the contraction constant $\alpha_\lambda \leqq \delta L < 1$. Finally, U_λ is analytic in λ, for by (2.8) it is evident that $U_\lambda f$ is analytic in λ for every $f \in S_0^*$. Now we merely have to apply the lemma. A very similar argument proves that f is analytic in w.

3. **Indifferent fixed point.** In this section we generalize Theorem 2.1 by weakening the hypothesis concerning the fixed point of g. The only place in the proof of Theorem 2.1 where we required the fixed point to be attractive was at the very beginning, where we inferred the existence of a $\delta_1 > 0$ such that $g(B_r(z_0)) \subset B_r(z_0)$ for $0 \leqq r < \delta_1$. Now by a result of Siegel, for all attractive fixed points and "most" indifferent fixed points, g is conformally equivalent to a function which does map small discs about z_0 into themselves, and the local existence proof will go through.

DEFINITION. A fixed point z_0 of the analytic function g is said to be *stable* if it lies in the interior of a proper simply connected open set U that is mapped into itself by g.

Let z_0 be a stable fixed point. Let ϕ be a conformal map of U onto the unit disc Δ which takes z_0 to the origin. Let $G = \phi \circ g \circ \phi^{-1}$ be the map of Δ into itself induced by g. Now $G(0) = 0$, and by Schwarz' lemma either $|G(w)| < |w|$ for $w \in \Delta$ or else $G(w) = \alpha w$, $|\alpha| = 1$. In the first case 0 is an attractive fixed point of G, and in the second case 0 is an indifferent fixed point. In either case G maps all small discs about the origin into themselves.

THEOREM 3.1. *The hypotheses of Theorem* 2.1 *may be relaxed to permit the fixed point of g to be merely stable, and the conclusion will remain valid.*

For we may transform the FDE into an equivalent equation on the unit disc, and here small discs about the fixed point are mapped into themselves by g. Hence the proof of Theorem 2.1 will go through. We then transform back to get a solution of the original equation.

What happens in the neighborhood of a nonstable indifferent fixed point? An analytic solution may well fail to exist, as the following example shows. Let $g(z)=z+z^2$ and consider the FDE

$$f'(z) = f(z+z^2). \tag{3.1}$$

Then $z=0$ is a nonstable indifferent fixed point, and any formal power series solution about 0 is, up to a constant multiple,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.2}$$

where $a_0=1$ and

$$\begin{aligned} a_n &= \frac{1}{n}\left[a_{n-1}+\binom{n-2}{1}a_{n-2}+\binom{n-3}{2}a_{n-3}+\cdots\right] \\ &= \frac{1}{n}a_{n-1}+\frac{n-2}{n}a_{n-2}+\frac{(n-3)(n-4)}{2n}a_{n-3}+\cdots \\ &> \frac{(n-3)(n-4)}{2n}a_{n-3} \qquad (n>3). \end{aligned}$$

Therefore $a_n/a_{n-3}\to\infty$ as $n\to\infty$ and so $\sum a_{3n}z^{3n}$ has radius of convergence 0. Hence also $\sum a_n z^n$ has radius of convergence 0, and no nontrivial analytic solution of (3.1) exists.

We close this section by mentioning some criteria for an indifferent fixed point to be stable. We have shown that a function about a stable indifferent fixed point is conformally equivalent to a rotation, and the converse is obvious. Let z_0 be an indifferent fixed point of the function $g(z)$. Set $\alpha=g'(z_0)$, so $|\alpha|=1$. If $\alpha^n=1$ the fixed point is stable iff $g_n(z)=z$. (See Cremer [1]; here the iterates g_k of g are defined by $g_1(z)=g(z)$ and $g_{k+1}(z)=g(g_k(z))$.) If α is not a root of unity, stability depends on quite delicate arithmetic properties of α, involving how "nearly" α is a root of unity.

In particular there are the following two results. Siegel [10] showed that $\log|\alpha^n-1|=O(\log n)$ implies stability. This condition holds for almost all points α on the unit circle, and thus "most" indifferent fixed points are stable. On the other hand, if α satisfies $\liminf_{n\to\infty}|\alpha^n-1|^{1/n}=0$, then there exists an analytic function g with $g(z_0)=z_0$, $g'(z_0)=\alpha$, such that z_0 is *not* a stable fixed point (Cremer [1]).

4. Repulsive fixed point. Perhaps the most interesting aspect of the local theory is the question of the existence of solutions of (2.1) in a neighborhood of a repulsive fixed point of $g(z)$. At first glance we might hope for a nice symmetrical result when a solution exists: around an attractive fixed point, always; indifferent fixed point, sometimes; repulsive fixed point, never. This conjecture is, however, quickly refuted by a simple example. The FDE

$$f'(z) = f(2z-z^2) \tag{4.1}$$

has a solution $f(z)=1/(1-z)$ which is analytic around the repulsive fixed point $z=0$ of the function $g(z)=2z-z^2$.

There is one notable difference between the above solution to (4.1) and a solution to (2.1) around an attractive fixed point. The solution to (2.1) depends continuously on the functions F and g; if we perturb either slightly the resulting equation will still have a solution, which is close to the solution of the original equation. But for (4.1) neither fact is true. We can perturb either F or g by an arbitrarily small amount in such a way that the solution disappears altogether. Indeed, we shall see that the equations

$$f'(z) = (1+\varepsilon)f(2z-z^2) \tag{4.2}$$

and

$$f'(z) = f(2z-(1+\varepsilon)z^2) \tag{4.3}$$

do *not* have analytic solutions about 0 if ε is any sufficiently small nonzero number.

We consider the linear equation

$$f'(z) = a(z)f(g(z)) \tag{4.4}$$

where $a(z)$ is analytic about the repulsive fixed point z_0 and $a(z_0)\neq 0$. We will show that if (4.4) has an analytic solution about z_0, then the perturbed equation

$$f'(z) = (1+\varepsilon)a(z)f(g(z)) \tag{4.5}$$

does not, for ε a small nonzero number. We prove this by introducing the eigenvalue problem

$$f'(z) = \lambda a(z)f(g(z)) \tag{4.6}$$

where we are to find $\lambda\neq 0$ such that (4.6) has a nontrivial solution analytic about z_0. We show that the nonzero eigenvalues are all isolated. Once we know this we can apply perturbation theory to get out a number of other results.

An eigenvalue $\lambda=\lambda(g)$ depends continuously on the function g. In fact, if $g=g_s$ depends analytically on some complex parameter s, then λ will be an analytic function of s. Thus if 1 is an eigenvalue of (4.6) for $s=s_0$, then 1 will not be an eigenvalue for $0<|s-s_0|<\varepsilon$, unless $\lambda(s)$ is constant. Thus if $\lambda(s)$ is not a constant we can say that perturbing g in (4.4) does indeed make the solution disappear. For example, the eigenvalue $\lambda(s)$ in $f'(z)=\lambda(s)f(2z-sz^2)$ is not constant; it is $\lambda(s)=s$, with eigenfunction $f(z)=1/(1-sz)$. This shows that 1 is not an eigenvalue for s close to but unequal to 1; hence (4.3) has no solution for ε a small nonzero number.

Now the fact that the eigenvalues of (4.6) are isolated establishes at once that suitably perturbing the coefficient $a(z)$ annihilates a solution. If we could show that for some analytic perturbations of g the eigenvalue does not remain constant, then it would follow that suitably perturbing g annihilates a solution. This may well be true, but we have not been able to prove it. However, what really seems significant is not the negative assertion that any solution of (4.4) is "singular" in the sense that perturbing the equation destroys the solution, but rather the positive assertion that, at least for some functions g, equation (4.6) does have a nonzero eigenvalue. It is somewhat remarkable that a functional differential equation such as (4.6) does determine an eigenvalue problem with discrete spectrum, *without any boundary condition being imposed.*

After this general discussion we now state the main theorem of this section.

THEOREM 4.1. *Let z_0 be a repulsive fixed point of the analytic function $g(z)$. Let $a(z)$ be analytic in a neighborhood of z_0, and suppose $a(z_0)\neq 0$. Then the set of complex numbers λ for which* (4.6) *has a nontrivial solution is a set having no accumulation point, except possibly $\lambda=0$. The eigenspaces corresponding to these λ are all one dimensional.*

Proof. Since $|g'(z_0)|>1$, in a neighborhood of z_0 the function $g(z)$ has an inverse $h(z)$, which has z_0 as an attractive fixed point. Let $p(z)=a(z)f(g(z))$. Then (4.6) is equivalent to

$$\left(\frac{p}{a}\circ h\right)'(z) = \lambda p(z). \tag{4.7}$$

Pick numbers $R>r>0$ so small that $h(z)$ and $a(z)$ are analytic on $B_R(z_0)$, $\mathrm{Cl}\,(h(B_R(z_0)))\subset B_r(z_0)$ and $a(z)\neq 0$ on $\mathrm{Cl}\,(B_r(z_0))$. Let X be the Banach space of functions analytic on $B_r(z_0)$, continuous on $\mathrm{Cl}\,(B_r(z_0))$, with sup norm. Define an operator $T=T_2T_1\colon X\to X$ by

$$p \xrightarrow[T_1]{} \frac{p}{a} \xrightarrow[T_2]{} \left(\frac{p}{a}\circ h\right)'. \tag{4.8}$$

We show that the operator T is compact. This will prove the theorem, since a compact operator has discrete spectrum, and the eigenvalues of T are precisely those λ for which (4.6) has a nontrivial solution. The operator T_1 is continuous, so it will suffice to prove that T_2 is compact.

Let $\{q_n\}$ be a bounded family of functions in X, i.e. $\|q_n\|\leqq M$. Then $\{T_2q_n\}$ is a compact family. Indeed it is uniformly bounded:

$$|(q_n\circ h)'(z)| = \left|\frac{1}{2\pi i}\int_{|\zeta|=R}\frac{q_n(h(\zeta))}{(\zeta-z)^2}\,d\zeta\right| \leqq \frac{MR}{(R-r)^2}$$

for all z in $\mathrm{Cl}\,(B_r(z_0))$. And also equicontinuous:

$$|(q_n \circ h)'(z_1)-(q_n \circ h)'(z_2)| = \left|\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{(q_n(h(\zeta)))(2\zeta+z_2-z_1)(z_1-z_2)}{(\zeta-z_1)^2(\zeta-z_2)^2}\,d\zeta\right|$$
$$\leqq \frac{MR(2R+2r)}{(R-r)^4}\,|z_1-z_2|$$

for all $z_1, z_2 \in \mathrm{Cl}\,(B_r(z_0))$. Hence by the Arzela-Ascoli Theorem the family $\{T_2 q_n\}$ is compact.

Finally note that the eigenspaces are one dimensional. For the solution of (4.6) for a given λ is unique up to a constant multiple (if it exists), by Theorem 1.2.

Now that we have shown that the eigenvalues of (4.6) are actually the eigenvalues of a certain compact operator, all the facts we mentioned above about continuous and analytic dependence of a given eigenvector follow immediately from the theory of perturbation of a linear operator. (See [4, Chapters IV and VII].) Indeed, if g varies continuously or analytically, it is easy to see that T—and hence the eigenvalues of T—varies continuously or analytically. In particular we have the following result. Suppose (4.6) has a nontrivial solution for some $\lambda \neq 0$. Then if we perturb g slightly the eigenvalue $\lambda(g)$ will move only slightly, and so the perturbed equation will still have a nonzero eigenvalue.

5. **Attractive cycles.** There is a situation besides that of a fixed point for which there is a meaningful local existence problem for functional differential equations. That is the case of a *cycle*, i.e. a set of points $z_0, z_1, \ldots, z_{n-1}$ such that $z_i = g(z_{i-1})$ and $z_n = z_0$. If we specify $f(z)$ arbitrarily in a neighborhood of z_0, then $f(z)$ is determined by the FDE successively over neighborhoods of $g(z_0), g_2(z_0), \ldots$ and finally over a neighborhood of $g_n(z_0) = z_0$. The new definition of $f(z)$ around z_0 must agree with the old, and so there is a relation that must be satisfied.

Let g be analytic at the point z_0 and at its image points z_n. If n is the smallest integer such that $z_n = z_0$ we say that z_0 is *periodic* of order n. The points $(z_0, z_1, \ldots, z_{n-1})$ are then said to form a cycle of order n. The number $s = g_n'(z_0) = g'(z_0)g'(z_1)\cdots g'(z_{n-1})$ is the *multiplier* of the cycle. The cycle is *attractive* if $|s| < 1$, *repulsive* if $|s| > 1$, and *indifferent* if $|s| = 1$. If $A = A_0$ is a domain containing z_0 we define $A_1 = g(A_0), \ldots, A_i = g(A_{i-1}) = g_i(A_0)$. A neighborhood of the cycle $(z_0, z_1, \ldots, z_{n-1})$ is an open set (in general disconnected) containing each of the points z_i.

THEOREM 5.1. *Let $(z_0, z_1, \ldots, z_{n-1})$ be an attractive cycle of order n of the analytic function g. Suppose that $F(z, w)$ is analytic near*

$$\{(z_0, c_0), (z_1, c_1), \ldots, (z_{n-1}, c_{n-1})\},$$

where c_i $(i=0, 1, \ldots, n-1)$ are complex numbers. Then there exists a neighborhood E_0 of z_0 such that for $z \in E^* = E_0 \cup \cdots \cup E_{n-1}$ the equation

$$f'(z) = F(z, f(g(z))) \tag{5.1}$$

has a unique solution satisfying

$$f(z_i) = c_i \qquad (i = 0, 1, \ldots, n-1). \tag{5.2}$$

Proof. We require the following notation. Let D be a domain (open connected set) containing the point z_0. For $z \in D$ define

$$\Sigma(z, z_0) = \inf\{|\sigma| : \sigma \text{ is an arc connecting } z_0 \text{ to } z\}$$

where $|\sigma|$ is the length of the arc σ, and set $\Sigma(D, z_0) = \sup_{z \in D} \Sigma(z, z_0)$.

LEMMA 5.2. *Let g be analytic on D. Suppose that $\Sigma(D, z_0) < \infty$ and $K = \sup_{z\in D} |g'(z)| < \infty$. Then $\Sigma(g(D), g(z_0)) \leqq K\Sigma(D, z_0)$.*

Proof. Let $w = g(z)$. Then

$$g(\sigma) = \int_{g(\sigma)} |dw| = \int_{\sigma} |g'(z)|\,|dz| \leqq K|\sigma|.$$

Therefore $\Sigma(g(z), g(z_0)) \leqq |g(\sigma)| \leqq K|\sigma|$ for all arcs σ connecting z_0 to z. Hence

$$\Sigma(g(z), g(z_0)) \leqq K\Sigma(z, z_0) \leqq K\Sigma(D, z_0) \quad \text{for all } z \in D,$$

and so

$$\Sigma(g(D), g(z_0)) \leqq K\Sigma(D, z_0).$$

Proof of theorem. Choose numbers $a > 0$ and $b > 0$ so that $F(z, w)$ is analytic for $(z, w) \in \bigcup_i \mathrm{Cl}\,(B_a(z_i)) \times \mathrm{Cl}\,(B_b(c_i))$. Let M be the supremum of $|F(z, w)|$ taken over this set. Let L be a Lipschitz constant for F valid over this set. Choose a neighborhood D of z_0 so small that $D_i \subset B_a(z_i)$ $(i = 0, 1, \ldots, n-1)$ and $\Delta_n \subset \Delta$ for all discs Δ about z_0, $\Delta \subset D$. (The second condition can be imposed because the cycle is attractive. Recall that $D_i = g_i(D)$.) Set $D^* = D \cup D_1 \cup \cdots \cup D_{n-1}$, $K = \sup_{z \in D^*} |g'(z)|$. Now pick $r > 0$ so small that

(i) $E_0 = B_r(z_0) \subset D$,

(ii) $r < b/K^{n-1}M$,

(iii) $r < 1/LK^{(n-1)/2}$.

Let X be the space of functions f that are analytic on E_0, continuous on $\bar{E}_0$, and satisfy $|f(z) - c_0| \leqq b$ for $z \in E_0$. Give X the sup norm. Define an operator U of X into itself as follows.

For f analytic on E_0 define f_{n-1} on E_{n-1} by

$$f_{n-1}(z) = c_{n-1} + \int_{z_{n-1}}^{z} F(\zeta, f(g(\zeta)))\, d\zeta$$

and inductively define f_i on E_i by

$$f_i(z) = c_i + \int_{z_i}^{z} F(\zeta, f_{i+1}(g(\zeta)))\, d\zeta.$$

Doing this successively we finally get $f_0(z)$ defined on E_0. To make sure that this definition makes sense we must verify that the functions f_i stay in the proper range $|f_i(z)-c_i|\leqq b$. Indeed, given that $|f(z)-c_0|\leqq b$ we have

$$\begin{aligned}|f_{n-1}(z)-c_{n-1}| &\leqq \left|\int_{z_{n-1}}^{z} F(\zeta,f(g(\zeta)))\,d\zeta\right| \\ &\leqq \Sigma(E_{n-1},z_{n-1})M \leqq K^{n-1}\Sigma(E_0,z_0)M \quad \text{by the lemma.}\end{aligned}$$

Now E_0 is the disc $B_r(z_0)$, so $\Sigma(E_0,z_0)=r$. Hence $|f_{n-1}(z)-c_{n-1}|\leqq K^{n-1}rM\leqq b$ by (ii). We find by induction $|f_i(z)-c_i|\leqq K^i rM\leqq b$ $(i=0,1,\ldots,n-1)$.

Thus all the functions f_i are well defined. In particular f_0, which is analytic on E_0, is defined. The mapping $f\to f_0$ determines a mapping U of the space X into itself. We now show that U is a contraction operator. For $|z-z_0|<r$,

$$\begin{aligned}|f_{n-1}(z)-f^*_{n-1}(z)| &\leqq \Sigma(E_{n-1},z_{n-1})L\|f-f^*\| \leqq K^{n-1}rL\|f-f^*\|;\\ |f_{n-2}(z)-f^*_{n-2}(z)| &\leqq \Sigma(E_{n-2},z_{n-2})L\|f_{n-1}-f^*_{n-1}\|\\ &\leqq K^{n-2}rLK^{n-1}rL\|f-f^*\| = K^{n-2}K^{n-1}(rL)^2\|f-f^*\|.\end{aligned}$$

Continuing, we find by induction that

$$|f_0(z)-f_0^*(z)| \leqq K^{(n-1)/2}(rL)^n\|f-f^*\| = c\|f-f^*\|$$

where $c<1$ by (iii).

Thus U is a contraction mapping of the complete metric space X into itself. It follows that U has a unique fixed point, which is the solution of our equation.

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