TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 363, Number 5, May 2011, Pages 2467–2479
S 0002-9947(2010)04994-3
Article electronically published on December 7, 2010

ON QUASICONFORMAL MAPS WITH IDENTITY BOUNDARY VALUES

V. MANOJLOVIĆ AND M. VUORINEN

Abstract. Quasiconformal homeomorphisms of the unit ball B^n of $\mathbb{R}^n, n \geq 3$, onto itself with identity boundary values are studied. A spatial analogue of Teichmüller's theorem is proved.

1. Introduction

For a domain $G \subset \mathbb{R}^n$, $n \geqslant 2$, let

$$Id(\partial G) = \{f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n} \text{ homeomorphism } : f(x) = x, \quad \forall x \in \overline{\mathbb{R}^n} \setminus G\}.$$

Here $\overline{\mathbb{R}^n}$ stands for the Möbius space $\mathbb{R}^n \cup \{\infty\}$. We shall always assume that $card\{\overline{\mathbb{R}^n} \setminus G\} \geq 3$. If $K \geqslant 1$, then the class of K-quasiconformal maps in $Id(\partial G)$ is denoted by $Id_K(\partial G)$. Throughout this paper we adopt the standard notation and terminology from Väisälä's book [V]. In particular, K-quasiconformal maps are defined in terms of the maximal dilatation as in [V, p. 42] if not otherwise stated. The maximal dilatation of a homeomorphism $f : G \to G'$ where $G, G' \subset \mathbb{R}^n$ are domains, is denoted by $K(f)$.

The subject of this research is to study the following well-known problem.

1.1. **Problem.** (1) Given $a, b \in G$ and $f \in Id(\partial G)$ with $f(a) = b$, find a lower bound for $K(f)$.

(2) Given $a, b \in G$, construct $f \in Id(\partial G)$ with $f(a) = b$ and give an upper bound for $K(f)$.

O. Teichmüller studied this problem in the case when G is a plane domain with $card(\overline{\mathbb{R}^2} \setminus G) = 3$ and solved it by proving the following theorem with a sharp bound for $K(f)$.

1.2. **Theorem.** *Let $G = \mathbb{R}^2 \setminus \{0, 1\}$, $a, b \in G$. Then there exists $f \in Id_K(\partial G)$ with $f(a) = b$ iff*

$$\log(K(f)) \geqslant s_G(a, b),$$

where $s_G(a, b)$ is the hyperbolic metric of G.

Received by the editors May 12, 2008 and, in revised form, December 29, 2008.
2010 *Mathematics Subject Classification.* Primary 30C65; Secondary 30C62.

Motivated by a question of F.W. Gehring, J. Krzyż [K, Theorem 1] proved the following theorem. See also Teichmüller [T] and Krushkal [Kr, p. 59]. Write $B^n(r) = \{x \in \mathbb{R}^n : |x| < r\}$ and $B^n = B^n(1)$.

1.3. **Theorem** (Krzyż [K, Theorem 1])**.** *For $f \in Id_K(\partial B^2)$ the sharp bounds are*

$$|f(0)| \leqslant \mu^{-1}\left(\log \frac{\sqrt{K}+1}{\sqrt{K}-1}\right) \equiv c_1, \tag{1.4}$$

where μ is the function defined in (2.4) *and*

$$\tanh \frac{\rho_{B^2}(f(z),z)}{2} \leqslant c_1 \tag{1.5}$$

for every $z \in B^2$, where ρ_{B^2} is the hyperbolic metric defined in Lemma 2.1*.*

The constant c_1 in (1.4) is quite involved. It is hard to see how it behaves in the crucial passage to limit $K \to 1$. Therefore we give an explicit bound for this constant.

1.6. **Lemma.** *The constant c_1 in* (1.4) *satisfies for $K > 1$,*

$$\frac{K-1}{K+1} < c_1 < 2\frac{K-1}{\sqrt{K}+1}.$$

Later studies of this topic include the paper of G. Martin [M]. He formulated a question of the same type as Gehring did, but for general plane domains. This question was solved in the negative, at the same time by A. Solynin and M. Vuorinen [SV] and H. Xinzhong and N.E. Cho [XC].

Our goal here is to study the n-dimensional case.

For any proper domain $G \subset \mathbb{R}^n$ we consider the density $\rho(x) = \frac{1}{d(x,\partial G)}$, $x \in G$. The corresponding metric, denoted by k_G [GP], is called the quasihyperbolic metric in G. Thus, for $x, y \in G$,

$$k_G(x,y) = \inf_{\gamma} \int_{\gamma} \rho \, ds,$$

where the infimum is taken over the family of all rectifiable curves γ in G joining x to y.

Gehring and Palka [GP] proved the following upper bound for Problem 1.1. Presumably this bound could be improved.

1.7. **Theorem** ([GP, Lemma 3.1])**.** *In Problem* 1.1 (2) *we can choose $K(f) \leqslant \exp(c_2 k_G(a,b))$, where $c_2 > 0$ only depends on the dimension n.*

In the case of uniform domains with connected boundary, a lower bound was given by the second author in [VU1]; see Theorem 3.2 below. For the case of the unit ball this problem was studied by G.D. Anderson and M.K. Vamanamurthy [AV], who found the following counterpart for Theorem 1.3 for dimensions $n \geq 3$. Note, in particular, that here they use the linear dilatation and that an additional symmetry hypothesis is required. They conjectured on p. 2 of [AV] that the result also holds without this additional hypothesis.

1.8. **Theorem** ([AV])**.** *For $f \in Id(\partial B^n)$ with the linear dilatation $H(f) = K$ (cf.* [V, p. 78]*) we have*

$$|f(0)| \leqslant c_1,$$

where c_1 is as in (1.4) *provided that f satisfies a certain symmetry hypothesis.*

The goal of this paper is to prove the following theorem where no extra symmetry hypotheses are required.

1.9. **Theorem.** *If $f \in Id_K(\partial B^n)$, then for all $x \in B^n$,*

$$\rho_{B^n}(f(x), x) \leqslant \log \frac{1-a}{a}, \quad a = \varphi_{1/K,n}(1/\sqrt{2})^2,$$

where ρ_{B^n} is the hyperbolic metric defined in Lemma 2.1 and $\varphi_{K,n}$ is as in (2.10).

1.10. **Theorem.** *If $f \in Id_K(\partial B^n)$, then for all $x \in B^n, n \geq 2$, and $K \in [1, 17]$*

$$|f(x) - x| \leq \frac{9}{2}(K-1)\,. \tag{1.11}$$

For $n = 2$ and $K > 1$ we have

$$|f(x) - x| \leqslant \frac{b}{2}(K-1), \quad b \leqslant 4.38. \tag{1.12}$$

The theory of K-quasiregular mappings in $\mathbb{R}^n, n \geq 3$, with maximal dilatation K close to 1 has been extensively studied by Yu.G. Reshetnyak [R] under the name "stability theory". By Liouville's theorem we expect that when $n \geq 3$ is fixed and $K \to 1$ the K-quasiregular maps "stabilize", become more and more like Möbius transformations, and this is the content of the deep main results of [R] (see p. 286). We have been unable to decide whether Theorem 1.9 follows from Reshetnyak's stability theory in a simple way. V. I. Semenov [S] has also made significant contributions to this theory. For the plane case, P. P. Belinskii has found several sharp results in [Be].

Finally, it seems to be an open problem whether a new kind of stability behavior holds: If $K > 1$ is fixed, do maps in $Id_K(\partial B^n)$ approach identity when $n \to \infty$? Our results do not answer this question. This kind of behavior is anticipated in [AVV, Open problem 9, p. 478].

2. Preliminary results

We shall follow the terminology of [V], where, for instance, the moduli of curve families are discussed. For the hyperbolic metric ρ_{B^n} of the unit ball B^n our main reference is [B]. In the next lemma we give the useful estimate (2.3) for it. Some applications of (2.3) were given in [VU2, pp. 141-142]. Very recently, Earle and Harris [EH] have given several applications and extended this inequality to other metrics such as the Carathéodory metric.

2.1. **Lemma.** *For $x, y \in B^n$ let $t = \sqrt{(1-|x|^2)(1-|y|^2)}$. Then*

$$\tanh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + t^2}\,, \tag{2.2}$$

$$|x-y| \leqslant 2 \tanh \frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2}+t}\,, \tag{2.3}$$

where equality holds for $x = -y$.

Proof. For (2.2) see [B, p. 40]; for (2.3) see [VU2, (2.18), 2.27]. □

Next, we consider a decreasing homeomorphism $\mu : (0,1) \longrightarrow (0,\infty)$ defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \tag{2.4}$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r' = \sqrt{1-r^2}$, for all $r \in (0,1)$.

The Hersch-Pfluger distortion function is an increasing homeomorphism $\varphi_K : (0,1) \longrightarrow (0,1)$ defined by

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K) \tag{2.5}$$

for all $r \in (0,1)$, $K > 0$. By continuity we set $\varphi_K(0) = 0$, $\varphi_K(1) = 1$. From (2.4) we see that $\mu(r)\mu(r') = \left(\frac{\pi}{2}\right)^2$ and from this we are able to conclude a number of properties of φ_K. For instance, by [AVV, Thm 10.5, p. 204],

$$\varphi_K(r)^2 + \varphi_{1/K}(r')^2 = 1, \quad r' = \sqrt{1-r^2}, \tag{2.6}$$

holds for all $K > 0$, $r \in (0,1)$.

2.7. *Proof of Lemma* 1.6. By [AVV, (5.27)] we have for $y > 0$,

$$\sqrt{1-\tanh^2 y} < \sqrt{1-\tanh^8 y} < \mu^{-1}(y) < 4e^{-y}\,.$$

With

$$y = \log\frac{\sqrt{K}+1}{\sqrt{K}-1} = 2\mathrm{artanh}(1/\sqrt{K})$$

this inequality yields

$$\frac{\sqrt{K}-1}{K+1} < c_1 = \mu^{-1}(y) < 4\frac{\sqrt{K}-1}{\sqrt{K}+1} < 2\frac{K-1}{\sqrt{K}+1}\,. \qquad \square$$

2.8. The Grötzsch and Teichmüller rings. The Grötzsch and Teichmüller ring domains $R_G(s), s > 1$, and $R_T(t), t > 0$, are doubly connected domains with complementary components $(\overline{B}^n, [se_1,\infty))$ and $([-e_1,0],[te_1,\infty))$, respectively. Their capacities $\mathrm{cap}R_G(s)$ and $\mathrm{cap}R_T(t)$ are often used below. The Grötzsch capacity $\gamma_n(s) = \mathrm{cap}R_G(s)$ is a decreasing homeomorphism $\gamma_n : (1,\infty) \longrightarrow (0,\infty)$; see [VU2, p. 66] and [AVV, Section 8]. The Teichmüller capacity $\tau_n(t) = \mathrm{cap}R_T(t)$, is a decreasing homeomorphism $\tau_n : (0,\infty) \to (0,\infty)$ connected with γ_n by the identity

$$\tau_n(t) = 2^{1-n}\gamma_n(\sqrt{1+t}),\ t > 0. \tag{2.9}$$

Given $E, F, G \subset \mathbb{R}^n$ we use the notation $\Delta(E,F;G)$ for the family of all curves that join the sets E and F in G and $M(\Delta(E,F;G))$ for its modulus; see [V, Chapter I]. Then $\tau_n(t) = M(\Delta(E,F;\mathbb{R}^n))$ where E and F are the complementary components of the Teichmüller ring and a similar relation also holds for $\gamma_n(s)$.

We use the standard notation

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}. \tag{2.10}$$

Then $\varphi_{K,n} : (0,1) \longrightarrow (0,1)$ is an increasing homeomorphism; see [VU2, (7.44)]. Because $\gamma_2(1/r) = 2\pi/\mu(r)$ by [VU2, (5.56)] and [LV], it follows that $\varphi_{K,2}(r)$ is the same as the function $\varphi_K(r)$ in (2.5).

2.11. The key constant. The special functions introduced above will have a crucial role in what follows. For the sake of easy reference we give here some well-known identities between them that can be found in [AVV]. First, the function

$$\eta_{K,n}(t) = \tau_n^{-1}(\tau_n(t)/K) = \frac{1-\varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2},\ K>0\,, \tag{2.12}$$

defines an increasing homeomorphism $\eta_{K,n} : (0,\infty) \to (0,\infty)$ (cf. [AVV, p. 193]). The constant $(1-a)/a, a = \varphi_{1/K,n}(1/\sqrt{2})^2$, in Theorem 1.9 can be expressed as follows for $K > 1$:

$$(1-a)/a = \eta_{K,n}(1) = \tau_n^{-1}(\tau_n(1)/K)\,. \tag{2.13}$$

Furthermore, by (2.6),

$$\eta_{K,2}(t) = \frac{s^2}{1-s^2}, \quad s = \varphi_{K,2}(\sqrt{t/(1+t)}) \tag{2.14}$$

and

$$\eta_{K,2}(1) \in (e^{\pi(K-1)}, e^{b(K-1)}) \tag{2.15}$$

where $b = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 = 4.376879....$ Note that the constant $\lambda(K)$ in [AVV, 10.33] is the same as $\eta_{K,2}(1)$. In passing we remark that P. P. Belinskii in [Be, Lemma 12, p. 80] gave the inequality

$$\eta_{K,2}(1) \equiv \lambda(K) < 1 + 12(K-1)$$

for K close to 1, however, with an incorrect proof as pointed out in [AQVu, (3.10)]. Because this inequality is one of the key technical estimates of [Be], it is fortunate that this error was detected and a correct proof was later found (see [AQVu, Corollary 3.7]).

For the proof of Lemma 2.24, we record a lower bound for $\varphi_{1/K,n}(r)$. The constant $\lambda_n \in [4, 2e^{n-1})$ is the so-called Grötzsch ring constant; see [AVV].

2.16. **Lemma** ([VU2, 7.47, 7.50])**.** *For $n \geq 2, K \geq 1$, and $0 \leq r \leq 1$,*

$$\varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^\beta,\ \beta = K^{1/(n-1)}, \tag{2.17}$$

$$\lambda_n^{1-\beta} \geq 2^{1-\beta}K^{-\beta} \geq 2^{1-K}K^{-K}\,. \tag{2.18}$$

In the next lemma we consider two strictly increasing continuous functions $p, q : [1,\infty) \to (0,\infty)$ such that $p(1) < q(1)$ and that the opposite inequality $p(x_1) > q(x_1)$ holds for some $x_1 > 1$. In the first part of the lemma we find, for the given functions, a concrete value $\varepsilon > 0$ such that $p(x) < q(x)$ for all $x \in [1, 1+\varepsilon)$. In the second part of the lemma we apply an iterative method with $1+\varepsilon$ as a starting value to find the largest number $a \in [1+\varepsilon, x_1)$ such that $p(x) < q(x)$ for all $x \in [1, a)$ and show that $a > 17$.

2.19. **Lemma.** (1) *For all $m, n \geqslant 1$ there is $M > 1$ such that the inequality*

$$\log(2^{mx-m+1}x^{nx} - 1) \leqslant (2m\log 2 + 2n)(x-1) \tag{2.20}$$

holds for $x \in [1, M]$ with equality only for $x = 1$. Moreover, with $t = (m\log 2 - n)/(2n)$, M can be chosen as

$$M = \sqrt{\frac{(m-1)\log 2 + \log\left(1 + \frac{(n+m\log 2)^2}{n}\right)}{n} + t^2} - t.$$

(2) *Let* $p(x) = \log(2^{mx-m+1}x^{nx} - 1)$, $q(x) = (2m \log 2 + 2n)(x - 1)$ *and let us use the above notation. Let* $a_0 = M$ *and* $a_{n+1} = p^{-1}(q(a_n))$ *for* $n \geqslant 1$. *Then the sequence* a_n *is increasing and bounded. If* $a = \lim_{n\to\infty} a_n$, *then the inequality* (2.20) *holds for* $x \in [1, a]$ *with equality iff* $x \in \{1, a\}$. *For* $m = 3$ *and* $n = 2$ *we have* $a > 17$.

Proof. Let

$$u(x) = (mx-m+1)\log 2+nx\log x, \quad v(x) = \log(e^{u(x)}-1) = \log(2^{mx-m+1}x^{nx}-1).$$

Then we have

$$\begin{aligned} v''(x) &= (\log(e^{u(x)} - 1))'' = \left(\frac{u'(x)\, e^{u(x)}}{e^{u(x)} - 1}\right)' \\ &= \frac{(u''(x)e^{u(x)} + (u'(x))^2 e^{u(x)})(e^{u(x)} - 1) - (u'(x)\, e^{u(x)})^2}{(e^{u(x)} - 1)^2} \\ &= \frac{e^{u(x)}}{(e^{u(x)} - 1)^2} \cdot ((u''(x) + (u'(x))^2)(e^{u(x)} - 1) - (u'(x))^2 e^{u(x)}) \\ &= \frac{e^{u(x)}}{(e^{u(x)} - 1)^2} \cdot (u''(x)(e^{u(x)} - 1) - (u'(x))^2). \end{aligned}$$

Thus,

$$v''(x) \leqslant 0 \quad \Leftrightarrow \quad u''(x)(e^{u(x)} - 1) \leqslant (u'(x))^2.$$

Since

$$e^{u(x)} = 2^{mx-m+1}x^{nx}, \quad u'(x) = n + m \log 2 + n \log x, \quad u''(x) = \frac{n}{x},$$

we have

$$v''(x) \leqslant 0 \quad \Leftrightarrow \quad \frac{n}{x}(2^{mx-m+1}x^{nx} - 1) \leqslant (n + m \log 2 + n \log x)^2.$$

Therefore we see that for $x \geqslant 1$ the condition $v''(x) \leqslant 0$ is equivalent to

$$2^{mx-m+1}x^{nx} - 1 \leqslant \frac{x}{n}(n + m \log 2 + n \log x)^2.$$

Let $f(x) = 2^{mx-m+1}x^{nx} - 1$ and $g(x) = \frac{x}{n}(n + m \log 2 + n \log x)^2$. Both functions f and g are increasing on $[1, +\infty)$ and $f(1) < g(1)$ because

$$f(1) = 1 \leqslant n = \frac{1}{n} \cdot n^2 < \frac{1}{n}(n + m \log 2)^2 = g(1).$$

By the continuity of f we can conclude that there is $M > 1$ such that $f(M) \leqslant g(1)$. For such M, we have

$$f(x) \leqslant f(M) \leqslant g(1) \leqslant g(x), \quad x \in [1, M].$$

This implies that v is concave on $[1, M]$ and therefore

$$v(x) \leqslant v(1) + v'(1)(x - 1), \quad x \in [1, M],$$

i.e.,

$$\log(2^{mx-m+1}x^{nx} - 1) \leqslant (2m \log 2 + 2n)(x - 1), \quad x \in [1, M].$$

The inequality $f(x) \leqslant g(1)$ is equivalent to

$$(mx - m + 1) \log 2 + nx \log x \leqslant \log\left(1 + \frac{(n + m \log 2)^2}{n}\right). \tag{2.21}$$

Because

$$(mx - m + 1)\log 2 + nx\log x \leqslant (mx - m + 1)\log 2 + nx(x-1), \tag{2.22}$$

the inequality (2.21) is a consequence of the inequality

$$(mx - m + 1)\log 2 + nx(x-1) \leqslant \log\left(1 + \frac{(n + m\log 2)^2}{n}\right). \tag{2.23}$$

In (2.22) the equality sign holds only for $x = 1$. Because

$$1 + \frac{(n + m\log 2)^2}{n} > 1 + \frac{n^2}{n} = 1 + n \geqslant 2,$$

the inequality (2.23) is a strict inequality for $x = 1$. By this reasoning, the greater root of the quadratic equation

$$(mx - m + 1)\log 2 + nx(x-1) = \log\left(1 + \frac{(n + m\log 2)^2}{n}\right)$$

is greater than 1. If we denote this root with M, the inequality (2.21) holds for $x \in [1, M]$ with equality only for $x = 1$. The first part of the lemma is proved.

Now we prove the second part of the inequality. Both of the functions $p(x)$ and $q(x)$ are continuous and increasing. Consequently, $r(x) = p^{-1}(x)$ is continuous and increasing. Because

$$p(a_1) = q(a_0) > p(a_0),$$

using monotonicity of $p(x)$, we can conclude that $a_1 > a_0$. Now, by induction and monotonicity of r, we can conclude that the sequence a_n is increasing. Now for $x \in [a_n, a_{n+1})$ we have

$$p(x) < p(a_{n+1}) = q(a_n) \leqslant q(x).$$

Therefore the inequality $p(x) < q(x)$ holds for $x \in \bigcup_{n=0}^{\infty}[a_n, a_{n+1}) = [a_0, a)$ and, using what was already proved, we see that the inequality $p(x) < q(x)$ holds for the whole interval $1 < x < a$. For $x \geqslant 1$ we see that $mx - m + 1 > 1$ and $x^{nx} \geqslant 1$ and, consequently,

$$p(x) = \log(2^{mx-m+1}x^{nx} - 1) > \log(2\,x^{nx} - 1) \geqslant nx\log x.$$

Because $p(x) > nx\log x \geqslant (n\log x)(x-1)$ the inequality $p(c) > q(c)$ holds for c such that $n\log c \geqslant 2m\log 2 + 2n$. It is easy to check that it is true for $c = 2^{\frac{2m}{n}}e^2$. It implies that a is finite (for example $a < 2^{\frac{2m}{n}}e^2$) and a_n is bounded. The relation $p(a_{n+1}) = q(a_n)$ and the continuity of both functions show that $\lim p(a_{n+1}) = p(a) = q(a) = \lim q(a_n)$. The lower bound for a follows because $a_{36} > 17$. □

2.24. **Lemma.** *If $a = \varphi_{1/K,n}(1/\sqrt{2})^2$ is as in Theorem 1.9, then for $M > 1$ and $\beta \in [1, M]$,*

$$\log\left(\frac{1-a}{a}\right) \leq \log(\lambda_n^{2(\beta-1)}2^{\beta} - 1) \leq V(n)(\beta - 1) \tag{2.25}$$

with $V(n) = (2\log(2\lambda_n^2))(2\lambda_n^2)^{M-1}$ and for $K \in [1, 17]$,

$$\log\left(\frac{1-a}{a}\right) \leqslant (K-1)(4 + 6\log 2) < 9(K-1), \tag{2.26}$$

with equality only for $K = 1$. For $n = 2$ and $K > 1$,

$$\log\left(\frac{1-a}{a}\right) = \log\left(\frac{\varphi_{K,2}(1/\sqrt{2})^2}{\varphi_{1/K,2}(1/\sqrt{2})^2}\right) \leqslant b(K-1), \tag{2.27}$$

where $b = (4/\pi)\mathcal{K}(1/\sqrt{2})^2 \leq 4.38$.

Proof. For $\beta \in [1, M]$ we have by (2.17),

$$\log\left(\frac{1-a}{a}\right) \leq \log(\lambda_n^{2(\beta-1)} 2^\beta - 1).$$

Furthermore, we have

$$\frac{\log(\lambda_n^{2(\beta-1)} 2^\beta - 1)}{\beta - 1} \leqslant 2\,\frac{(2\lambda_n^2)^{\beta-1} - 1}{\beta - 1} \leqslant (2\log(2\lambda_n^2))(2\lambda_n^2)^{M-1}.$$

The second inequality follows from the inequality $\log(t) \leqslant t - 1$ and the third one from Lagrange's theorem and the monotonicity of the function $(2\log(2\lambda_n^2))(2\lambda_n^2)^{x-1}$. This proves (2.25).

From (2.18) it follows that the constant a satisfies the inequality

$$a \geq 2^{2(1-K)} K^{-2K} (1/\sqrt{2})^{2K}$$

and also

$$1/a \leq 2^{3K-2} K^{2K}, \quad K > 1.$$

By Lemma 2.19 we have

$$\log(2^{3K-2} K^{2K} - 1) \leqslant (4 + 6\log 2)(K-1)$$

for $K \in [1, 17]$ with equality only for $K = 1$. Now, from

$$\frac{1-a}{a} < 2^{3K-2} K^{2K} - 1, \quad K > 1,$$

we conclude that

$$\log\left(\frac{1-a}{a}\right) \leqslant (4 + 6\log 2)(K-1) < 9(K-1).$$

For the case $n = 2$ we can apply the identity (2.14) and the inequality in (2.15). □

3. Proof of Theorem 1.9

Lemma 3.1 and Theorem 3.2 deal with the first part of Problem 1.1.

3.1. **Lemma** ([VU1])**.** *Let $f \in Id_K(\partial G)$, $a, b \in G$, $f(a) = b$ and let the boundary ∂G be connected. If $x \in \partial G$ is such that $d(a) = d(a, \partial G) = |a - x| \leqslant |b - x|$, then*

$$K(f) \geqslant \overline{d}_n \left(\log\frac{|b-x|}{|a-x|}\right)^n, \quad \overline{d}_n = \frac{c_n}{\omega_{n-1}}\,\frac{(n-1)^{n-1}}{n^n}.$$

The following result was proved in [VU1], however, under the condition that the points are far away from each other. The general case follows from the original result by reducing the constant. In [VU1], an example was given to the effect that Theorem 3.2 cannot be improved to the claim that $a, b \in G, k_G(a, b) > 0$ implies $K(f) > 1$.

3.2. **Theorem** ([VU1]). *Let $f \in Id_K(\partial G)$, $a, b \in G$ with $f(a) = b$. If G is a uniform domain with connected boundary ∂G, then*

$$K(f) \geqslant d_n\, k_G(a,b)^n$$

where d_n depends only on n and G.

3.3. *Proof of Theorem* 1.9. Fix $x \in B^n$ and let T_x denote a Möbius transformation of $\overline{\mathbb{R}^n}$ with $T_x(B^n) = B^n$ and $T_x(x) = 0$. Define $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by setting $g(z) = T_x \circ f \circ T_x^{-1}(z)$ for $z \in B^n$ and $g(z) = z$ for $z \in \mathbb{R}^n \setminus B^n$. Then $g \in Id_K(\partial B^n)$ with $g(0) = T_x(f(x))$. By the invariance of ρ_{B^n} under the group $\mathcal{GM}(B^n)$ of Möbius selfautomorphisms of B^n we see that for $x \in B^n$,

$$\rho_{B^n}(f(x), x) = \rho_{B^n}(T_x(f(x)), T_x(x)) = \rho_{B^n}(g(0), 0). \tag{3.4}$$

Choose $z \in \partial B^n$ such that $g(0) \in [0, z] = \{tz : 0 \leqslant t \leqslant 1\}$. Let $E' = \{-sz : s \geqslant 1\}$, $\Gamma' = \Delta([g(0), z], E'; \mathbb{R}^n)$ and $\Gamma = \Delta(g^{-1}[g(0), z], g^{-1}E'; \mathbb{R}^n)$. Observe that $E' = g^{-1}E'$ because $g \in Id_K(\partial B^n)$.

The spherical symmetrization with center at 0 yields, by [AVV, Thm 8.44],

$$M(\Gamma) \geqslant \tau_n(1) \quad (= 2^{1-n}\gamma_n(\sqrt{2}))$$

because $g(x) = x$ for $x \in \mathbb{R}^n \setminus B^n$. Next, we see by the choice of Γ' that

$$M(\Gamma') = \tau_n\left(\frac{1 + |g(0)|}{1 - |g(0)|}\right).$$

By K-quasiconformality we have $M(\Gamma) \leqslant K\, M(\Gamma')$ implying

$$\exp(\rho_{B^n}(0, g(0))) = \frac{1 + |g(0)|}{1 - |g(0)|} \leqslant \tau_n^{-1}(\tau_n(1)/K) = \frac{1-a}{a}. \tag{3.5}$$

The last equality follows from (2.13). Finally, (3.4) and (3.5) complete the proof. □

3.6. *Proof of Theorem* 1.10. We have

$$\begin{aligned} |f(x) - x| &\leqslant 2\tanh\left(\frac{\rho_{B^n}(f(x), x)}{4}\right) \leqslant 2\tanh\left(\frac{\log\left(\frac{1-a}{a}\right)}{4}\right) \\ &\leqslant 2\tanh\left(\frac{(K-1)(4 + 6\log 2)}{4}\right) \\ &\leqslant (K-1)(2 + 3\log 2) \leqslant \frac{9}{2}(K-1). \end{aligned}$$

The first inequality follows from (2.3), the second one from Theorem 1.9, the third one from Lemma 2.24 and the fourth one from the inequality $\tanh(t) \leqslant t$ for $t \geqslant 0$.

For $n = 2$ we use the same first two steps and the planar case of Lemma 2.24 to derive the inequality

$$|f(x) - x| \leqslant \frac{b}{2}(K-1). \qquad \square$$

A lower bound corresponding to the upper bound in (1.11) is given in the next lemma.

3.7. **Lemma.** *For $f \in Id(\partial G)$ let*

$$\delta(f) \equiv \sup\{|f(z)-z| : z \in G\}\,.$$

Then for $f \in Id_K(\partial B^n), K>1, \alpha = K^{1/(1-n)}$,

$$\delta(f) \ge (1-\alpha)\alpha^{\alpha/(1-\alpha)} > \frac{1}{e}(1-\alpha). \tag{3.8}$$

Proof. The radial stretching $f : B^n \to B^n, n \ge 2$, defined by $f(z) = |z|^{\alpha-1} z, z \in B^n$, $(0<\alpha<1)$, is K-quasiconformal with $\alpha = K^{1/(1-n)}$ [V, p. 49] and $f \in Id_K(\partial B^n)$. Now we have

$$|f(z)-z| = ||z|^{\alpha-1}z - z| = |r^\alpha - r|, \quad |z| = r.$$

Furthermore, we see that

$$\delta(f) = \sup_{0<r<1} (r^\alpha - r),$$

where the supremum is attained for $r = r_\alpha = \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha-1}}$, so

$$\delta(f) = (1-\alpha)\alpha^{\alpha/(1-\alpha)}\,.$$

A crude, but simple, estimate is

$$\delta(f) \ge (1/e)^\alpha - (1/e) = \frac{1}{e}\left(\frac{1}{e^{\alpha-1}} - 1\right) = \frac{1}{e}\left(e^{1-\alpha}-1\right) \geqslant \frac{1}{e}(1-\alpha)\,.$$

□

3.9. **Theorem.** *Let $f : \overline{\mathbb{R}^n} \longrightarrow \overline{\mathbb{R}^n}$ be a K-quasiconformal homeomorphism with $f(\infty) = \infty$ and $B^n(m) \subset f(B^n) \subset B^n(M)$, where $0 < m \le 1 \le M$. Then*

$$\eta_{1/K,n}\left(\frac{1+|x|}{1-|x|}\right) \leqslant \frac{M+|f(x)|}{m-|f(x)|}$$

and

$$\frac{m+|f(x)|}{M-|f(x)|} \leqslant \eta_{K,n}\left(\frac{1+|x|}{1-|x|}\right)$$

for all $x \in B^n$ where $\eta_{K,n}(t) = \tau_n^{-1}(\tau_n(t)/K)$.

In particular, if $m = 1 = M$, then we have

$$\eta_{1/K,n}\left(\frac{1+|x|}{1-|x|}\right) \leqslant \frac{1+|f(x)|}{1-|f(x)|} \leqslant \eta_{K,n}\left(\frac{1+|x|}{1-|x|}\right)\,.$$

Proof. The proof is similar to the proof of Theorem 1.9. Fix $x \in B^n$ and choose $z' \in \partial f(B^n)$ such that $f(x) \in [0,z']$ and $[f(x), z') \subset f(B^n)$ and fix $z'' \in \partial f(B^n)$ such that $z', 0, z''$ are on the same line, $0 \in [z', z'']$, and $\{-sz'' : s \geqslant 1\} \subset \mathbb{R}^n \setminus f(B^n)$. Let $\Gamma' = \Delta([f(x), z'], E'; \mathbb{R}^n)$, $E' = \{-sz'' : s \geqslant 1\}$ and $\Gamma = \Delta(f^{-1}[f(x), z'], f^{-1}E'; \mathbb{R}^n)$. Then

$$M(\Gamma') \le \tau_n\left(\frac{m+|f(x)|}{M-|f(x)|}\right)$$

while applying a spherical symmetrization with center at the origin gives

$$M(\Gamma) \geqslant \tau_n\left(\frac{1+|x|}{1-|x|}\right)$$

because $f^{-1}E'$ connects ∂B^n and ∞. Then the inequality $M(\Gamma) \leqslant K\, M(\Gamma')$ yields

$$\tau_n\left(\frac{1+|x|}{1-|x|}\right) \le K\tau_n\left(\frac{m+|f(x)|}{M-|f(x)|}\right),$$

$$\tau_n^{-1}(\frac{1}{K}\tau_n\left(\frac{1+|x|}{1-|x|}\right)) \ge \frac{m+|f(x)|}{M-|f(x)|}, \tag{3.10}$$

$$\frac{m+|f(x)|}{M-|f(x)|} \leqslant \eta_{K,n}\left(\frac{1+|x|}{1-|x|}\right).$$

The lower bound follows if we apply a similar argument to f^{-1} and the lower bound

$$M(\Gamma') \ge \tau_n\left(\frac{M+|f(x)|}{m-|f(x)|}\right).$$

□

3.11. *Remark.* Putting $x = 0, m = 1 = M$ in (3) we obtain by (2.13) for a K-quasiconformal homeomorphism $f : \overline{\mathbb{R}^n} \longrightarrow \overline{\mathbb{R}^n}$ with $f(\infty) = \infty$ and $f(B^n) = B^n$ that

$$|f(0)| \le 1 - 2a\,, a = \varphi_{1/K,n}(1/\sqrt{2})^2\,.$$

Furthermore, if we use the lower bound (2.18) from Lemma 2.16 we obtain

$$|f(0)| \le 1 - 2^{1-\beta}4^{1-K}K^{-2K}\,.$$

In the special case when $n = 2$ we have

$$|f(0)| \le 1 - 2^{3(1-K)}K^{-2K} \le (2 + 3\log 2)(K-1)\,.$$

Note that this last inequality does not suppose that $f \in Id_K(\partial B^n)$, only the hypotheses of Theorem 3.9 are needed.

3.12. **Corollary.** *Let $n = 2$ in addition to the hypotheses of Theorem* 3.9. *Then*

$$\eta_{K,2}(t) = \frac{u^2}{1-u^2} = \frac{u^2}{v^2}, \tag{3.13}$$

where $u = \varphi_{K,2}\left(\sqrt{\frac{t}{1+t}}\right)$, $v = \varphi_{1/K,2}\left(\frac{1}{\sqrt{1+t}}\right)$ and

$$|f(x)| \leqslant 2\,\varphi_{K,2}\left(\sqrt{\frac{1+|x|}{2}}\right)^2 - 1 \tag{3.14}$$

for all $x \in B^2$.

Proof. The identity (3.13) holds by (2.14). Next Theorem 3.9 together with (3.13) yield

$$\frac{1+|f(x)|}{1-|f(x)|} \leqslant \frac{w^2}{1-w^2}$$

where $w = \varphi_{K,2}\left(\sqrt{\frac{1+|x|}{2}}\right)$. Solving this for $|f(x)|$ yields (3.14). □

3.15. *Remark.* By the K-quasiconformal Schwarz lemma, if $f : B^2 \longrightarrow B^2$ is K-quasiconformal with $f(0) = 0$, then $|f(z)| \leqslant \varphi_{K,2}(|z|)$, for all $z \in B^2$, where the sharp bound is attained for a map with $f(B^2) = B^2$ ([LV]). Note that in Corollary 3.12 the condition $f(0) = 0$ is not required. We conclude that

$$\varphi_{K,2}(r) \leqslant 2\,\varphi_{K,2}(\sqrt{\frac{1+r}{2}})^2 - 1. \tag{3.16}$$

Writing $A(r,s) = \sqrt{\frac{r+s}{2}}$ (3.16) says that if $t = 1, r \in (0,1)$, then

$$A(\varphi_{K,2}(t), \varphi_{K,2}(r)) \leqslant \varphi_{K,2}(A(t,r)).$$

It seems natural to expect that this inequality holds for all $t, r \in (0,1)$.

Acknowledgement

The research of the second author was supported by the project "Quasiconformal Maps" nr 209539 of Matti Vuorinen funded by the Academy of Finland.

References

[AV] G.D. Anderson and M.K. Vamanamurthy: An extremal displacement mapping in n-space. Complex analysis Joensuu 1978 (Proc. Colloq., Univ. Joensuu, Joensuu, 1978), pp. 1–9, Lecture Notes in Math., 747, Springer, Berlin, 1979. MR553026 (81b:30030)

[AQVu] G.D. Anderson, S.-L. Qiu, and M. Vuorinen: Modular Equations and Distortion Functions. Ramanujan J. 18 (2009), no. 2, 147–169. MR2475934 (2010a:30032)

[AVV] G.D. Anderson, M.K. Vamanamurthy, and M.K. Vuorinen: Conformal invariants, inequalities, and quasiconformal maps. John Wiley & Sons, Inc., New York, 1997. MR1462077 (98h:30033)

[B] A.F. Beardon: The geometry of discrete groups. Springer-Verlag, New York, 1983. MR698777 (85d:22026)

[Be] P.P. Belinskii: General properties of quasiconformal mappings (Russian). Izd. Nauka, Novosibirsk, 1974. MR0407275 (53:11054)

[EH] C.J. Earle and L.A. Harris: Inequalities for the Carathéodory and Poincaré metrics in open unit balls, Manuscript, 2008.

[GP] F.W. Gehring and B.P. Palka: Quasiconformally homogeneous domains. J. Analyse Math. 30 (1976), 172–199. MR0437753 (55:10676)

[Kr] S.L. Krushkal: Variational principles in the theory of quasiconformal maps. Handbook of complex analysis: Geometric function theory. Edited by R. Kuhnau. Vol. 2, 31–98, Elsevier, Amsterdam, 2005. MR2121857 (2005k:30039a)

[K] J. Krzyż: On an extremal problem of F. W. Gehring. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16, 1968, 99–101. MR0226002 (37:1592)

[LV] O. Lehto and K.I. Virtanen: Quasiconformal mappings in the plane. Second edition. Translated from the German by K.W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York, Heidelberg, 1973. viii+258 pp. MR0344463 (49:9202)

[M] G.J. Martin: The distortion theorem for quasiconformal mappings, Schottky's theorem and holomorphic motions. Proc. Amer. Math. Soc. 125 (1997), no. 4, 1095–1103. MR1363178 (97g:30017)

[R] Yu.G. Reshetnyak: Stability theorems in geometry and analysis. Translated from the 1982 Russian original by N.S. Dairbekov and V.N. Dyatlov, and revised by the author. Translation edited and with a foreword by S.S. Kutateladze. Mathematics and its Applications, 304. Kluwer Academic Publishers Group, Dordrecht, 1994. xii+394 pp. ISBN: 0-7923-3118-4. MR1326375 (96i:30016)

[S] V.I. Semenov: Estimates of stability, distortion theorems, and topological properties of quasiregular mappings. Mat. Zametki 51 (1992), 109–113. MR1186539 (93h:30032)

[SV] A.Yu. Solynin and M. Vuorinen: Estimates for the hyperbolic metric of the punctured plane and applications. Israel J. Math. 124 (2001), 29–60. MR1856503 (2002j:30071)

[T] O. Teichmüller: Ein Verschiebungssatz der quasikonformen Abbildung. (German) Deutsche Math. 7, (1944). 336–343. see also Teichmüller, Oswald Gesammelte Abhandlungen. (German) [Collected papers] Edited and with a preface by Lars V. Ahlfors and Frederick W. Gehring. Springer-Verlag, Berlin, New York, 1982. viii+751 pp. MR0018761 (8:327b)

[V] J. Väisälä: Lectures on n-dimensional quasiconformal mappings. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin, New York, 1971. MR0454009 (56:12260)

[VU1] M. Vuorinen: A remark on the maximal dilatation of a quasiconformal mapping. Proc. Amer. Math. Soc. 92(1984), no. 4, 505–508. MR760934 (86a:30039)

[VU2] M. Vuorinen: Conformal geometry and quasiregular mappings. Lecture Notes in Mathematics, 1319. Springer-Verlag, Berlin, 1988. xx+209 pp. MR950174 (89k:30021)

[XC] H. Xinzhong and N.E. Cho: On the distortion theorem for quasiconformal mappings with fixed boundary values. (English summary) J. Math. Anal. Appl. 256 (2001), no. 2, 694–697. MR1821766 (2002a:30030)

Department of Mathematics, Faculty of Organizational Sciences, University of Belgrade, Jove Ilica 154, 11000 Belgrade, Serbia

E-mail address: vesnam@fon.rs

Department of Mathematics, University of Turku, FIN-20014 Turku, Finland

E-mail address: vuorinen@utu.fi