MATHEMATICS OF COMPUTATION
VOLUME 51, NUMBER 184
OCTOBER 1988, PAGES 507-534

Convergence of a Higher-Order Vortex Method for Two-Dimensional Euler Equations*

By C. Chiu and R. A. Nicolaides

Abstract. There has been considerable interest recently in the convergence properties of point vortex methods. In this paper, we define a vortex method using vortex multipoles and obtain error estimates for it. In the case of a nonuniform mesh, the rate of convergence of the dipolar algorithm is shown to be of higher order of accuracy than obtained with the simple vortex methods.

1. Introduction. Although vortex methods have been used for many years for approximation of the partial differential equations of incompressible inviscid fluid dynamics, [12], [13], [15], a precise mathematical analysis was not available until very recently. In fact, the first complete analysis of a two-dimensional vortex method was given by Hald [10] in 1979. Since then, many papers have appeared giving error estimates for two-dimensional and three-dimensional vortex methods, including [1], [2], [3], [4], [5] and [18]. These analyses mostly assume a uniform mesh for the initial vorticity discretization. As a result of the mesh uniformity, the resulting error estimates are of unexpectedly high order of accuracy, being limited essentially by the regularity of the initial vorticity distribution. In more realisitic situations, it is improbable that uniform meshes can be used, e.g., if there are irregular bodies in the flow. In this case, the accuracy of the standard vortex methods will drop to first or second order, regardless of the initial regularity. In order to deal with nonuniform meshes, [17] defines some new vortex schemes for the two-dimensional incompressible Euler equations. In this paper we shall give a complete error estimate for one of them. This method yields higher order of accuracy even on nonuniform meshes. This is achieved by using not only the usual δ function point vortices, but also derivatives of such distributions.

In the next section we will define the algorithm and give explicit formulas for its implementation. Then, a rigorous error estimate will be provided following the Sobolev space technique of [5] and [18].

2. The Construction of a Higher-Order Vortex Method.

2-D Euler Equations. Let $u(x,t) = (u_1(x,t), u_2(x,t))$, $x \in \mathbf{R}^2$ and $t \in [0,\infty)$, be the velocity field and $w = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$ be the vorticity. Assume that the exterior forces acting on the fluid are potential. Then the Euler equations are:

$$(2.1)\qquad \begin{cases} w_t + (u\cdot\nabla)w = Dw/Dt = 0, & w(x,0) = w_0(x),\ x \in \mathbf{R}^2,\\ \operatorname{div} u = 0, \\ u \to 0 \quad \text{as } |x| \to \infty. \end{cases}$$

Received August 8, 1986; revised October 13, 1987 and February 24, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 65N30.
*This work was supported by the U. S. Air Force, under grant AFOSR 84-0137.

Here, by definition, $D/Dt := \partial_t + (u \cdot \nabla)$ and is the usual material derivative. Concerning the existence and the uniqueness of solutions to the equation (2.1), see [14], [16], [19]. Smooth solutions are known to exist for all time in the two-dimensional case with smooth initial data. In this paper we assume that the initial vorticity $w_0(x)$ of (2.1) is smooth so that there exists a smooth solution of (2.1) on some space-time interval $\mathbf{R}^2 \times [0, T]$.

Let $X(x, t)$ be the path followed by a fluid particle which is at the position x when $t = 0$. Then the map $x \to X(x, t)$ satisfies the system of ordinary differential equations

$$\frac{dX(x,t)}{dt} = u(X(x,t),t), \qquad X(x,0) = x. \tag{2.2}$$

Then, (2.1) with the initial vorticity $w(x,0) = w_0(x)$ satisfies $w(X(x,t),t) = w_0(x)$. In this paper we only consider those flows with smooth vorticity $w_0(x)$ which have compact support. Under this assumption, there exists a bounded set Ω such that $\operatorname{supp} w(\cdot, t) \subset \Omega$, $\forall t \in [0, T]$.

Vortex Methods. Vortex methods are based upon the tracking of finite numbers of fluid particles and evaluating velocities by discretizing certain singular integrals. The basic idea of vortex methods is to approximate the initial vorticity by a linear combination of Dirac delta functions. For example, approximate w_0 by $w_0^h = \sum_{j \in J} \alpha_j \delta(x - x_j)$ where $\alpha_j \in \mathbf{R}$.

By following those particles whose positions at $t = 0$ are $\{x_j\}_{j \in J}$, using (2.2) with $x = x_j$, we get $w^h(x,t) = \sum_{j \in J} \alpha_j \delta(x - X(x_j, t))$.

To compute u, one uses the fact that $\operatorname{div} u = 0$ to introduce a stream function from which the velocity may be expressed as a singular integral. The singular kernel is then smoothed by a cutoff function, and quadrature rules are then needed to evaluate the integral. In order to get arbitrarily high order of accuracy by the above method, a uniform mesh has to be assumed. It can be obtained, for example, by subdividing the plane into squares of side h and letting $\{x_j\}_{j \in J}$ be the corner points of the squares [18]. We will now introduce our algorithm and some related theorems. This algorithm allows us to deal with nonuniform meshes and still obtain high-order accuracy.

A Higher-Order Vortex Method. Recall that if the initial vorticity function is smooth, then the classical solution of (2.1) is given by $w(X(x,t),t) = w_0(x)$. Now, let $\phi(\cdot) \in \mathscr{D}(\mathbf{R}^2)$ where $\mathscr{D}(\mathbf{R}^2) = \{\phi(\cdot) \in C^\infty(\mathbf{R}^2) \mid \phi(\cdot) \text{ has compact support}\}$; $\mathscr{D}'(\mathbf{R}^2)$ is the dual space of $\mathscr{D}(\mathbf{R}^2)$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. Then,

$$\langle w(\cdot,t), \phi(\cdot) \rangle = \int_{\mathbf{R}^2} w(x,t)\phi(x)\,dx = \int_{\mathbf{R}^2} w_0(x)\phi(X(x,t))\,dx = \langle w_0(\cdot), \phi(X(\cdot,t)) \rangle,$$

because the determinant of the Jacobian matrix of $x \to X$ is 1 since $\operatorname{div} u = 0$. Thus, we define a weak solution of the Euler equation as follows:

Definition 2.1. Assume that a unique solution of (2.2) exists. For $w(\cdot,t) \in \mathscr{D}'(\mathbf{R}^2)$ and $w_0(\cdot) \in \mathscr{D}'(\mathbf{R}^2)$, if

$$\langle w(\cdot,t), \phi(\cdot) \rangle = \langle w_0(\cdot), \phi(X(\cdot,t)) \rangle \quad \forall \phi(\cdot) \in \mathscr{D}(\mathbf{R}^2),$$

then, $w(\cdot, t)$ is said to be the weak vorticity of the Euler equation (2.1).

THEOREM 2.1. *Suppose that $X(x_0, t)$ exists.*

If $w_0(x) = a\delta(x - x_0) + b\delta_{x_1}(x - x_0) + c\delta_{x_2}(x - x_0)$, where a, b and c are constants, then the weak vorticity as defined above is

$$w(x,t) = a\delta(x - X(x_0,t)) + b(t)\delta_{x_1}(x - X(x_0,t)) + c(t)\delta_{x_2}(x - X(x_0,t)), \tag{2.3}$$

where δ_{x_1} and δ_{x_2} are derivatives of the Dirac Delta function δ and

$$\begin{pmatrix} b(t) \\ c(t) \end{pmatrix} = M(x_0,t)\begin{pmatrix} b \\ c \end{pmatrix}, \qquad M(x_0,t) = \left(\frac{dX_i}{dx_j}\right)$$

is the Jacobian matrix of $x \to X$ at x_0.

Remark. Note that

$$\frac{d}{dt}M(x_0,t) = \frac{d}{dt}\left(\frac{dX_i}{dx_j}\right) = \left(\frac{du_i}{dx_j}\right) = \left(\frac{du_i}{dX_k}\cdot\frac{dX_k}{dx_j}\right),$$

using the summation convention. So, $M(x_0, t)$ satisfies the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dM}{dt} = \nabla u \cdot M, \\ M(x_0, 0) = I, \end{cases}$$

where $M = M(x_0,t)$ and $u = u(X(x_0,t),t)$.

Proof. For all $\phi \in \mathscr{D}(\mathbf{R}^2)$ we have

$$\begin{aligned}
\langle w_0(\cdot), \phi(X(\cdot,t))\rangle &= a\phi(X(x_0,t)) - bd\phi(X(x_0,t))/dx_1 - cd\phi(X(x_0,t))/dx_2 \\
&= a\phi(X(x_0,t)) - b[(d\phi/dX_1)(dX_1/dx_1) + (d\phi/dX_2)(dX_2/dx_1)] \\
&\quad - c[(d\phi/dX_1)(dX_1/dx_2) + (d\phi/dX_2)(dX_2/dx_2)] \quad (\text{at } x_0) \\
&= a\phi(X(x_0,t)) - b(t)\, d\phi(X(x_0,t))/dX_1 - c(t)\, d\phi(X(x_0,t))/dX_2 \\
&= \langle a\delta(\cdot - X(x_0,t)) + b(t)\delta_{x_1}(\cdot - X(x_0,t)) + c(t)\delta_{x_2}(\cdot - X(x_0,t)), \phi(\cdot)\rangle \\
&= \langle w(\cdot,t), \phi(\cdot)\rangle.
\end{aligned}$$

Using Definition 2.1, $w(\cdot, t)$ is the weak vorticity. □

Now we will define our vortex method by specifying a, b and c over an initial vortex distribution. This can be done in many ways [17]. The method used below is based on direct numerical integration.

The Vorticity Field.

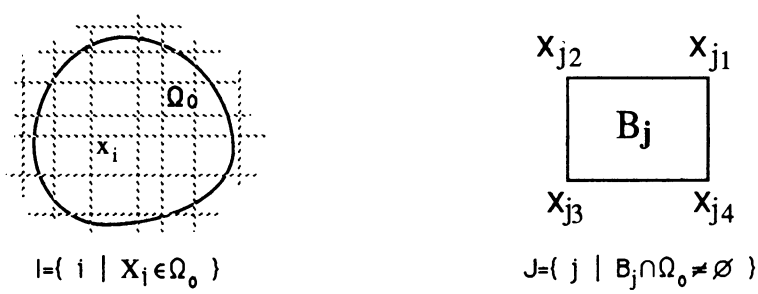

FIGURE 1

Take an arbitrarily spaced rectangular mesh on $\mathbf{R}^2$. Let $\operatorname{supp} w_0 \subset \Omega_0$ which is bounded. To each rectangle formed by adjacent coordinate lines, assign an index j and denote it by B_j. Denote the lengths of the edges of B_j by h_{j1} and h_{j2}. Let $\{x_{jk}\}_{k=1,4}$ be the four corners of B_j. To each corner of $B_j \cap \Omega_0 \neq \emptyset$, assign a global index i, as shown in Figure 1.

Interpreting the initial vorticity w_0 as a distribution, we shall approximate w_0 by another distribution w_0^h of the form

$$\begin{aligned} w_0^h(x) &= \sum_{j\in J}\sum_{k=1,4} [a_{jk}\delta(x-x_{jk}) + b_{1jk}\delta_{x_1}(x-x_{jk}) + b_{2jk}\delta_{x_2}(x-x_{jk})] \\ &= \sum_{i\in I}[c_i\delta(x-x_i) + d_{1i}\delta_{x_1}(x-x_i) + d_{2i}\delta_{x_2}(x-x_i)]. \end{aligned}$$

Then, based on Theorem 2.1, we expect that $w(\cdot,t)$ can be approximated by

$$\begin{aligned} w^h(x) &= \sum_{j\in J}\sum_{k=1,4} [a_{jk}\delta(x-X(x_{jk},t)) + b_{1jk}(t)\delta_{x_1}(x-X(x_{jk},t)) \\ &\qquad\qquad + b_{2jk}(t)\delta_{x_2}(x-X(x_{jk},t))] \\ &= \sum_{i\in I}[c_i\delta(x-X(x_i,t)) + d_{1i}(t)\delta_{x_1}(x-X(x_i,t)) + d_{2i}(t)\delta_{x_2}(x-X(x_i,t))], \end{aligned}$$

where

$$\begin{pmatrix} b_{1jk}(t) \\ b_{2jk}(t) \end{pmatrix} = M(x_{jk},t)\begin{pmatrix} b_{1jk} \\ b_{2jk} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_{1i}(t) \\ d_{2i}(t) \end{pmatrix} = M(x_i,t)\begin{pmatrix} d_{1i} \\ d_{2i} \end{pmatrix}.$$

Concerning the choice of the coefficients $\{a_{jk}, b_{1jk}, b_{2jk}\}$, observe that for $\phi(\cdot) \in \mathscr{D}(\mathbf{R}^2)$,

$$\langle w_0^h(\cdot), \phi(\cdot)\rangle = \sum_{j\in J}\sum_{k=1,4} [a_{jk}\phi(x_{jk}) - b_{1jk}(t)\phi_{x_1}(x_{jk}) - b_{2jk}(t)\phi_{x_2}(x_{jk})],$$

whereas

$$\langle w_0(\cdot), \phi(\cdot)\rangle = \int_{\mathbf{R}^2} w_0(x)\phi(x)\,dx = \sum_{j\in J}\int_{Bj} w_0(x)\phi(x)\,dx.$$

This suggests that approximating w_0 by w_0^h corresponds to approximating the integral

$$\int_{Bj} w_0(x)\phi(x)\,dx$$

by some numerical integration rule, where $\{a_{jk}, b_{1jk}, b_{2jk}\}_{j\in J, k=1,4}$ define this rule. For our algorithm, the following quadrature rule is used [17]. For a 2-dimensional rectangle B with corners $\{P_i\}_{i=1,4}$, sides h_1 and h_2, as shown in Figure 2, and $f \in C^1(B)$,

$$\begin{aligned} \int_B f(x)\,dx &\approx \frac{h_1h_2}{4}(f(P_1)+f(P_2)+f(P_3)+f(P_4)) \\ (*) \qquad &+ \frac{h_1^2h_2}{24}[-f_x(P_1)+f_x(P_2)+f_x(P_3)-f_x(P_4)] \\ &+ \frac{h_1h_2^2}{24}[-f_y(P_1)-f_y(P_2)+f_y(P_3)+f_y(P_4)]. \end{aligned}$$

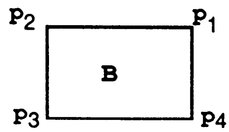

FIGURE 2

For this rule, a direct calculation shows:

THEOREM 2.2. *The quadrature rule* $(*)$ *is exact for all third-degree polynomials.* □

Denote the right-hand side of $(*)$ by $Q_2(B, f)$, i.e., $\int_B f(x)\,dx \approx Q_2(B, f)$. Thus, if $\{a_{jk}, b_{1jk}, b_{2jk}\}_{j\in J,\ k=1,4}$ are chosen by the above rule, then for $\phi(\cdot) \in \mathscr{D}(\mathbf{R}^2)$,

$$\begin{aligned}\langle w_0^h(\cdot), \phi(\cdot)\rangle &= \sum_{j\in J}\sum_{k=1,4}[a_{jk}\phi(x_{jk}) - b_{1jk}\phi_{x_1}(x_{jk}) - b_{2jk}\phi_{x_2}(x_{jk})]\\ &= \sum_{j\in J} Q_2(B_j, \phi).\end{aligned}$$

Then, when $h \to 0$, $\sum_{j\in J} Q_2(B_j, \phi) \to \langle w_0(\cdot), \phi(\cdot)\rangle$. More precisely, w_0^h converges in $\mathscr{D}'(\mathbf{R}^2)$ to w_0 as $h \to 0$. Correspondingly, $w^h \to w$ in $\mathscr{D}'(\mathbf{R}^2)$, by Definition 2.1. To see what the corresponding coefficients $\{c_i, d_{1i}, d_{2i}\}$ are, let us consider an example.

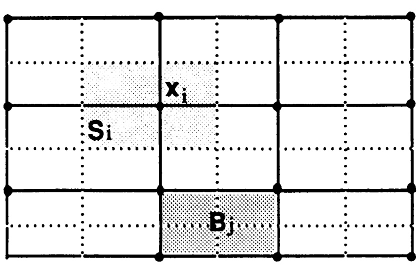

FIGURE 3

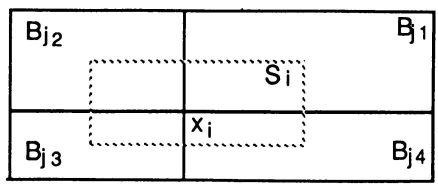

FIGURE 4

Suppose that the mesh is as shown in Figure 3. The points with closed circles are the nodal points of the mesh. Then B_j is some rectangle with nodal points as its four corners while S_i is a rectangle which contains the nodal point x_i and is bounded by dotted lines. Note that dotted lines equally divide sides of every B_j. B_{j1}, B_{j2}, B_{j3} and B_{j4}, as shown in Figure 4, are four adjacent rectangles. Let S_i be the rectangle shown and $|S_i|$ be its area. Since x_i is a common corner of B_{j1}, B_{j2}, B_{j3} and B_{j4}, by the quadrature rule $(*)$, we have the following:

$$\begin{aligned}&c_i\phi(x_i) - d_{1i}\phi_{x_1}(x_i) - d_{2i}\phi_{x_2}(x_i)\\ &\quad= (1/4)[h_{j_{11}}h_{j_{12}} + h_{j_{21}}h_{j_{22}} + h_{j_{31}}h_{j_{32}} + h_{j_{41}}h_{j_{42}}]w_0(x_i)\phi(x_i)\\ &\qquad+ (1/24)[h_{j_{11}}^2 h_{j_{12}} - h_{j_{21}}^2 h_{j_{22}} - h_{j_{31}}^2 h_{j_{32}} + h_{j_{41}}^2 h_{j_{42}}](w_0\phi)_{x_1}(x_i)\\ &\qquad+ (1/24)[h_{j_{11}}h_{j_{12}}^2 + h_{j_{21}}h_{j_{22}}^2 - h_{j_{31}}h_{j_{32}}^2 - h_{j_{41}}h_{j_{42}}^2](w_0\phi)_{x_2}(x_i)\\ &\quad= |S_i|\{w_0(x_i)\phi(x_i) + H_{i1}(w_0\phi)_{x_1}(x_i) + H_{i2}(w_0\phi)_{x_2}(x_i)\}.\end{aligned}$$

Assume that there exists a constant $C > 0$ such that

$$\frac{\max_{j\in J}(h_{j1}, h_{j2})}{\min_{j\in J}(h_{j1}, h_{j2})} \le C$$

and let $h = \max_{j\in J}(h_{i1}, h_{j2})$; then, $H_{i1} = O(h)$ and $H_{i2} = O(h)$ and

$$\begin{aligned} c_i &= |S_i|\{w_0(x_i) + H_{i1}w_{0x_1}(x_i) + H_{i2}w_{0x_2}(x_i)\}, \\ d_{1i} &= -|S_i|H_{i1}w_0(x_i), \qquad d_{2i} = -|S_i|H_{i2}w_0(x_i), \quad \forall i \in I. \end{aligned} \tag{2.4}$$

The Velocity Field. In order to obtain the velocity field from the vorticity field, we need the following result. Let $K\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$K = \frac{1}{2\pi|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

LEMMA 2.1. *The convolution operator $f \to K * f$ is a bounded linear mapping from $L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ into $B^0(\mathbf{R}^2)^2$* (set of bounded and continuous 2-D vector-valued functions). *Moreover, if $f \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ and satisfies $f(x) \to 0$ as $|x| \to \infty$, then the function $v = K * f$* (2-D vector) *is the unique solution of*

$$\begin{cases} \operatorname{div} v = 0, \\ \operatorname{curl} v = f, \\ v(x) \to 0 \text{ as } |x| \to \infty, x \in \mathbf{R}^2. \end{cases}$$

Proof. See [16]. □

It follows from Lemma 2.1 that $u(\cdot, t) = K * w(\cdot, t)$ in problem (2.1). After computing the approximate vorticity field w^h, we need to find the corresponding velocity field. It would seem natural to set $u^h(\cdot, t) = K * w^h(\cdot, t)$, but since the kernel K is a singular function, its convolution with delta functions is not defined in general. To avoid this problem, the now standard remedy is to regularize K as follows.

Let $f(x)\colon \mathbf{R}^2 \to \mathbf{R}$ satisfy $\int_{\mathbf{R}^2} f(x)\,dx = 1$ and let $f_\varepsilon(x) = (1/\varepsilon^2)f(x/\varepsilon)$. f and f_ε are referred to as "cutoff" functions. If $K_\varepsilon = K * f_\varepsilon$, then Lemma 2.1 implies that $K_\varepsilon \in B^1(\mathbf{R}^2)^2$ provided $f \in W^{1,\infty}(\mathbf{R}^2) \cap W^{1,1}(\mathbf{R}^2)$. Then, $u(\cdot, t)$ will be approximated by $u^h_\varepsilon = w^h * K_\varepsilon$, so that

$$\begin{aligned} u^h_\varepsilon(x) &= \sum_{j\in J}\sum_{k=1,4}[a_{jk}K_\varepsilon(x - X(x_{jk}, t)) - b_{1jk}(t)K_{\varepsilon x_1}(x - X(x_{jk}, t)) \\ &\qquad\qquad - b_{2jk}(t)K_{\varepsilon x_2}(x - X(x_{jk}, t))] \\ &= \sum_{i\in I}[c_iK_\varepsilon(x - X(x_i, t)) - d_{1i}(t)K_{\varepsilon x_1}(x - X(x_i, t)) \\ &\qquad\qquad - d_{2i}(t)K_{\varepsilon x_2}(x - X(x_i, t))]. \end{aligned}$$

Here, $K_{\varepsilon x_1} = dK_\varepsilon/dx_1$, $K_{\varepsilon x_2} = dK_\varepsilon/dx_2$ and $\{X(x_i, t)\}_{i\in I}$ are exact particle positions at t. But we can compute only approximate positions $\{X^h(x_i, t)\}_{i\in I}$ and corresponding approximate Jacobian matrices $\{M^h(x_i, t)\}_{i\in I}$. Therefore, only approximate coefficients $\{\mathscr{d}_{1i}(t), \mathscr{d}_{2i}(t)\}_{i\in I}$ can be obtained. So, the actual velocity field we compute is

$$\begin{aligned} \mathscr{U}^h_\varepsilon(x, t) = \sum_{i\in I}[c_iK_\varepsilon(x - X^h(x_i, t)) - \mathscr{d}_{1i}(t)K_{\varepsilon x_1}(x - X^h(x_i, t)) \\ - \mathscr{d}_{2i}(t)K_{\varepsilon x_2}(x - X^h(x_i, t))]. \end{aligned} \tag{2.5}$$

Here we use $\mathscr{U}$ to denote the numerical velocity field.

In summary, the 2-D algorithm is as follows:

$$\begin{cases} \dfrac{dX^h(x,t)}{dt} = \mathscr{U}_\varepsilon^h(X^h,t), & X^h(x,0) = x, \\ \dfrac{dM^h(x,t)}{dt} = \nabla\mathscr{U}_\varepsilon^h(X^h,t)M^h(x,t), & M^h(x,0) = I \end{cases}$$

(see the remark after Theorem 2.1), where $\mathscr{U}_\varepsilon^h$ is the numerical velocity field given by (2.5), $X^h(x,t)$ is the computed particle position at t with its initial position at x, and $M^h(x,t) = (\partial X_i^h/\partial x_j)$ is the Jacobian matrix of the mapping $x \to X^h(x,t)$. Moreover,

$$\begin{pmatrix} \mathscr{d}_{1i}(t) \\ \mathscr{d}_{2i}(t) \end{pmatrix} = M^h(x_i,t)\begin{pmatrix} d_{1i} \\ d_{2i} \end{pmatrix}$$

and $\{c_i, d_{1i}, d_{2i}\}_{i\in I}$ are given by (2.4).

3. Error Estimates. In this section we will give a complete error estimate for the algorithm constructed in the last section. The analysis given here consists of two parts, one for estimating the consistency error and the other for stability error. The first part is based upon Sobolev space theory. The second part depends on analysis of the velocity kernel and the behavior of the cutoff function.

Notations and Definitions. The norms used for the analysis are discrete L^p-norms.

Definition 3.1. For $f(\cdot) \in [L^p(\Omega)]^2$ or $[L^p(\Omega)]^{2\times 2}$, define

$$\|f\|_{h,p} = \left[\sum_{i\in I} |f(x_i)|^p h^2\right]^{1/p}.$$

Let $x \to X(x,t)$ be the trajectory mapping and $x \to X^h(x,t)$ be the computed trajectory mapping. For $F(X(\cdot,t), X^h(\cdot,t),t) \in [L^p(\Omega)]^2$ or $[L^p(\Omega)]^{2\times 2}$, define

$$\|F(t)\|_{h,p} = \left[\sum_{i\in I} |F(X(x_i,t), X^h(x_i,t),t)|^p h^2\right]^{1/p}.$$

Let $e(X(\cdot,t), X^h(\cdot,t),t) = X(\cdot,t) - X^h(\cdot,t)$, and $E(X(\cdot,t), X^h(\cdot,t),t) = M(\cdot,t) - M^h(\cdot,t)$.

For the error estimate, we define $w^h(x)$ and $\omega^h(x)$ as

$$\begin{aligned} w^h(x) &= \sum_{j\in J}\sum_{k=1,4} [a_{jk}\delta(x - X(x_{jk},t)) + b_{1jk}(t)\delta_{x_1}(x - X(x_{jk},t)) \\ &\qquad\qquad + b_{2jk}(t)\delta_{x_2}(x - X(x_{jk},t))] \\ &= \sum_{i\in I}[c_i\delta(x - X(x_i,t)) + d_{1i}(t)\delta_{x_1}(x - X(x_i,t)) + d_{2i}(t)\delta_{x_2}(x - X(x_i,t))], \\ \omega^h(x) &= \sum_{j\in J}\sum_{k=1,4} [\mathscr{a}_{jk}\delta(x - X^h(x_{jk},t)) + \mathscr{b}_{1jk}(t)\delta_{x_1}(x - X^h(x_{jk},t)) \\ &\qquad\qquad + \mathscr{b}_{2jk}(t)\delta_{x_2}(x - X^h(x_{jk},t))] \\ &= \sum_{i\in I}[\mathscr{c}_i\delta(x - X^h(x_i,t)) + \mathscr{d}_{1i}(t)\delta_{x_1}(x - X^h(x_i,t)) \\ &\qquad\qquad + \mathscr{d}_{2i}(t)\delta_{x_2}(x - X^h(x_i,t))]. \end{aligned}$$

Then Table 1 contains all quantities we will use for the error estimate.

TABLE 1

	Trajectory	Vorticity	Velocity
Exact Solution	$x \to X(x,t)$	$w(x,t)$	$u = (K * w)$
Computed Solution	$x \to X^h(x,t)$	$\omega^h(x,t)$	$\mathscr{U}_\varepsilon^h = K_\varepsilon * \omega^h$
Intermediate Quantity		$w^h(x,t)$	$u_\varepsilon = K_\varepsilon * w$ $u_\varepsilon^h = K_\varepsilon * w^h$

For any $j \in I$, by using the system of ordinary differential equations for the particle trajectories, we have

$$\frac{dX(x_j,t)}{dt} - \frac{dX^h(x_j,t)}{dt} = u(X(x_j,t),t) - \mathscr{U}_\varepsilon^h(X^h(x_j,t),t)$$
$$= [u(X(x_j,t),t) - u_\varepsilon^h(X(x_j,t),t)] + [u_\varepsilon^h(X(x_j,t),t) - \mathscr{U}_\varepsilon^h(X^h(x_j,t),t)],$$

where the first bracketed expression is called the consistency error, and the second the stability error.

The Consistency Error. Let

$$\mathrm{I}(\cdot,t) = u(\cdot,t) - u_\varepsilon(\cdot,t) = K * w - K_\varepsilon * w$$

and

$$\mathrm{II}(\cdot,t) = u_\varepsilon(\cdot,t) - u_\varepsilon^h(\cdot,t) = K_\varepsilon * w - K_\varepsilon * w^h.$$

Then, the consistency error is $\mathrm{I} + \mathrm{II}$. For I, we have the following result.

THEOREM 3.1. *Assume that there exists an integer $k \geq 1$ such that*
(i) $\int_{\mathbf{R}^2} f(x)\,dx = 1$,
(ii) $\int_{\mathbf{R}^2} x^\alpha f(x)\,dx = 0$, $\forall \alpha \in N^2$, $1 \leq |\alpha| \leq k-1$,
(iii) $\int_{\mathbf{R}^2} |x|^k |f(x)|\,dx < \infty$.
Then there exists a constant $C = C(p,T,w_0) > 0$ such that $\|\mathrm{I}(\cdot,t)\|_{L^\infty(\mathbf{R}^2)} \leq C\varepsilon^k$ and $\|\mathrm{I}(t)\|_{h,p} \leq C\varepsilon^k$ for all $p \in [1,\infty], t \in [0,T]$.

Proof. See [18, Chapter II, Lemma 4.1]. □

In order to analyze $\mathrm{II}(\cdot,t)$, we need to discuss a few auxiliary results. First of all, consider some properties of the regularized kernel K_ε. These properties are also very useful for the stability error estimate. We begin by recalling a classical result.

LEMMA 3.1 (Calderon-Zygmund). *The convolution operator $f \to (\partial K/\partial x_i) * f$ is a bounded linear mapping from $L^p(\mathbf{R}^2)$ into $[L^p(\mathbf{R}^2)]^2$, for $i = 1,2$ and $1 < p < \infty$.* □

LEMMA 3.2. *Let $p \in (1,\infty)$ and $f \in W^{\ell-1,p}(\mathbf{R}^2)$ for some integer $\ell \geq 1$. Then there exists some constant C such that*

$$\|\partial^\alpha K_\varepsilon\|_{L^p(\mathbf{R}^2)} \leq \frac{C}{\varepsilon^{\ell-1+(2/q)}},$$

for $\alpha \in N^2$ with $|\alpha| = \ell$ and $(1/p) + (1/q) = 1$.

Proof. See [18, Chapter II, Lemma 3.2(ii)]. □

LEMMA 3.3. *Let ℓ be a nonnegative integer. The following properties hold for all $\alpha \in N^2$ with $|\alpha| = \ell$:*

(a) *If $f \in W^{\ell,1}(\mathbf{R}^2) \cap W^{\ell,\infty}(\mathbf{R}^2)$, we have $|\partial^\alpha K_\varepsilon(x)| \le C_1/\varepsilon^{\ell+1}$ for all $x \in \mathbf{R}^2$.*

(b) *If $f \in W^{\ell,1}(\mathbf{R}^2) \cap W^{\ell,\infty}(\mathbf{R}^2)$ satisfies in addition $|x|^{\ell+2}|\partial^\alpha f(x)| \le C_2$, then*

$$|\partial^\alpha K_\varepsilon(x)| \le \frac{C_3}{|x|^{\ell+1}} \quad \text{for all } |x| \ge \varepsilon.$$

Proof. See [18, Chapter II, Lemma 3.3]. □

LEMMA 3.4. *Let S be a compact set in $\mathbf{R}^2$. For any multi-index β, assume that there exists a constant $C_1 > 0$ such that $|x|^{|\beta|+2}|\partial^\beta f(x)| \le C_1$. Then there is a constant $C = C(S)$ such that for all $\varepsilon \le \frac{1}{2}$,*

$$\|\partial^\beta K_\varepsilon\|_{L^1(S)} \le \begin{cases} C(S), & |\beta| = 0, \\ C(S)|\log \varepsilon|, & |\beta| = 1, \\ C(S)\varepsilon^{1-|\beta|}, & |\beta| > 1. \end{cases}$$

Proof. Let $B_\varepsilon = \{x \in \mathbf{R}^n \mid |x| < \varepsilon\}$. Then,

$$\begin{aligned} \|\partial^\beta K_\varepsilon\|_{L^1(S)} &= \int_S |\partial^\beta K_\varepsilon(x)|\,dx = \int_{S\cap B_\varepsilon} |\partial^\beta K_\varepsilon(x)|\,dx + \int_{S\setminus B_\varepsilon} |\partial^\beta K_\varepsilon(x)|\,dx \\ &\le C_2\varepsilon^2/\varepsilon^{|\beta|+1} + \int_{S\setminus B_\varepsilon} \frac{C_3}{|x|^{|\beta|+1}}\,dx \quad \text{(Lemma 3.3)} \\ &\le C_2\varepsilon^{1-|\beta|} + \int_\varepsilon^{\operatorname{diam}(S)} \frac{C_4}{|r|^{|\beta|}}\,dr \\ &= C_2\varepsilon^{1-|\beta|} + C_4 \begin{cases} [\operatorname{diam}(S) - \varepsilon], & |\beta| = 0, \\ [\log(\operatorname{diam}(S)) - \log\varepsilon], & |\beta| = 1, \\ (1-|\beta|)[\operatorname{diam}(S)^{1-|\beta|} - \varepsilon^{1-|\beta|}], & |\beta| > 1. \; \square \end{cases} \end{aligned}$$

As we defined in the last section, the initial coefficients of δ functions in the approximated vorticity field are chosen according to a quadrature rule. Now we want to find a bound for the error which results from the numerical integration. We first state a classical result due to Bramble and Hilbert [9, Theorem 4.1.3]. Let k be a nonnegative integer, denote by P_k the space of all polynomials of degree $\le k$ in the n variables $x_1, \ldots, x_n$.

LEMMA 3.5 (Bramble-Hilbert). *Let Ω be an open bounded subset of $\mathbf{R}^n$ with a Lipschitz continuous boundary and let $L\colon \phi \to L(\phi)$ be a bounded linear functional on $W^{k,p}(\Omega)$, $k \ge 1$, $p \in [1,\infty]$, with norm $\|L\|$, which satisfies $L(\phi) = 0$ for all $\phi \in P_{k-1}$. Then there exists a constant $C > 0$ such that*

$$|L(\phi)| \le C\|L\||\phi|_{k,p,\Omega} \quad \forall \phi \in W^{k,p}(\Omega),$$

where

$$\|\phi\|_{k,p,\Omega} = \left(\sum_{|\alpha|\le k} \int_\Omega |\partial^\alpha \phi|^p\,dx\right)^{1/p}, \qquad |\phi|_{k,p,\Omega} = \left(\sum_{|\alpha|=k} \int_\Omega |\partial^\alpha \phi|\,dx\right)^{1/p}$$

and $\|L\| = \sup_{\|\phi\|_{k,p,\Omega}=1} |L(\phi)|$.

As a consequence of Lemma 3.5, we obtain

LEMMA 3.6. *Let $j \in J$ and B_j be a 2-D box as defined above. Assume that the center of B_j is $\mathscr{x} = (\mathscr{x}_1, \mathscr{x}_2) \in \mathbf{R}^2$. If $g(\cdot) \in W^{4,1}(B_j)$, then there exists a constant $C > 0$ independent of B_j such that*

$$\left| \int_{B_j} g(x)\,dx - Q_2(B_j, g) \right| \le Ch^4 |g|_{4,1,B_j}. \tag{3.1}$$

Proof. Let $\mathscr{B} = [-1, +1]^2$ and $B^1(\mathscr{B})$ be the set of functions whose derivatives through order one are bounded and continuous on $\mathscr{B}$. Then, for $\mathscr{g}(\cdot) \in B^1(\mathscr{B})$, define $\mathscr{E}(\mathscr{g}) = \int_{\mathscr{B}} \mathscr{g}(x)\,dx - Q_2(\mathscr{B}, \mathscr{g})$. By Theorem 2.2, $\mathscr{E}(\mathscr{g}) = 0$ for all $\mathscr{g} \in P_3(\mathscr{B})$.

It is very easy to check that $\mathscr{g} \to \mathscr{E}(\mathscr{g})$ is a bounded linear functional on $B^1(\mathscr{B})$. By Sobolev's embedding theorem, $W^{4,1}(\mathscr{B}) \subset B^1(\mathscr{B})$. So, $\mathscr{g} \to \mathscr{E}(\mathscr{g})$ is also a bounded linear functional on $W^{4,1}(\mathscr{B})$ which vanishes on $P_3(\mathscr{B})$. Thus, by the Bramble-Hilbert lemma, there is a constant $C_1 > 0$ such that

$$|\mathscr{E}(\mathscr{g})| \le C_1 |\mathscr{g}|_{4,1,\mathscr{B}}. \tag{3.2}$$

For a function $g(\cdot)$ defined on B_j, change variables by letting

$$x_i = \mathscr{x}_i + (h_{ji}/2)\xi_i, \qquad -1 \le \xi_i \le 1,\ i = 1, 2,$$

and define $\mathscr{g}(\xi) = g(x) = g(\mathscr{x}_1 + (h_{j1}/2)\xi_1, \mathscr{x}_2 + (h_{j2}/2)\xi_2)$. Then (3.1) follows from (3.2). □

Now consider the second part of the consistency error. Recall that

$$\mathrm{II}(\cdot, t) = u_\varepsilon(\cdot, t) - u_\varepsilon^h(\cdot, t) = K_\varepsilon * w - K_\varepsilon * w^h,$$
$$\mathrm{II}(X, t) = (K_\varepsilon * w)(X, t) - \sum_{j \in J} Q_2[B_j, K_\varepsilon(X - X(\cdot, t))w_0(\cdot)].$$

THEOREM 3.2. *Assume that $f \in W^{4,1}(\mathbf{R}^2) \cap W^{4,\infty}(\mathbf{R}^2)$.*

(a) *If there exists a constant $C_1 > 0$ such that*

$$|x|^{|\alpha|+2} |\partial^\alpha f(x)| \le C_1 \quad \textit{for all } |\alpha| \le 4 \textit{ and } x \in \mathbf{R}^2,$$

then there is a constant $C = C(\Omega, w_0, p, T) > 0$ such that

$$\|\mathrm{II}(\cdot, t)\|_{L^\infty(\Omega)} \le C \frac{h^4}{\varepsilon^3},$$
$$\|\mathrm{II}(t)\|_{h,p} \le C \frac{h^4}{\varepsilon^3} \quad \textit{for all } t \in [0, T] \textit{ and } 1 \le p \le \infty.$$

(b) *If we only assume that $|x|^2 |f(x)| \le C_1$, then there is a constant $C_s = C_s(\Omega, w_0, p, T) > 0$ such that*

$$\|\mathrm{II}(\cdot, t)\|_{L^\infty(\Omega)} \le C_s \frac{h^4}{\varepsilon^{3+s}},$$
$$\|\mathrm{II}(t)\|_{h,p} \le C_s \frac{h^4}{\varepsilon^{3+s}} \quad \textit{for all } t \in [0, T] \textit{ and } 1 \le p \le \infty,$$

where $s > 0$ is an arbitrary number.

Proof. (a) By the definition, we have $|\mathrm{II}(x,t)| = |K_\varepsilon * w(x,t) - K_\varepsilon * w^h(x,t)|$. So

$$\begin{aligned}|\mathrm{II}(x,t)| &= \left|\sum_{j\in J}\left\{\int_{B_j} K_\varepsilon(x - X(y,t))w_0(y)\,dy - Q_2[B_j, K_\varepsilon(x - X(\cdot,t))w_0(\cdot)]\right\}\right| \\ &\le \sum_{j\in J}\left|\left\{\int_{B_j} K_\varepsilon(x - X(y,t))w_0(y)\,dy - Q_2[B_j, K_\varepsilon(x - X(\cdot,t))w_0(\cdot)]\right\}\right| \\ &\le C_2h^4\sum_{j\in J}|K_\varepsilon(x - X(\cdot,t))w_0(\cdot)|_{4,1,Bj} \quad \text{(by Lemma 3.6)} \\ &\le C_2h^4 \sum_{|\alpha|+|\beta|=4}\int_\Omega |\partial_y^\alpha K_\varepsilon(x - X(y,t))\partial_y^\beta w_0(y)|\,dy.\end{aligned}$$

Using the smoothness of u, we have $|\partial_y^\alpha X_i(y,t)| \le C_3$ for $i = 1, 2$, $y \in \Omega$, $0 \le |\alpha| \le 4$. Now using the chain rule,

$$\begin{aligned}|\mathrm{II}(x,t)| &\le C_4h^4 \sum_{|\alpha|+|\beta|\le 4}\int_\Omega |\partial_X^\alpha K_\varepsilon(x - X(y,t))\partial_y^\beta w_0(y)|\,dy \\ &\le C_5h^4\sum_{|\alpha|\le 4}\int_\Omega |\partial_X^\alpha K_\varepsilon(x - X(y,t))|\,dy \\ &= C_5h^4\sum_{|\alpha|\le 4}\int_\Omega |\partial_X^\alpha K_\varepsilon(x - X)|\,dX \\ &= C_5h^4\sum_{|\alpha|\le 4}\|\partial^\alpha K_\varepsilon(x - \cdot)\|_{L^1(\Omega)} \quad (\det J = 1).\end{aligned}$$

If $x \in \Omega$ and $X \in \Omega$, since Ω is bounded there exists a compact set $\mathfrak{C} \subset \mathbf{R}^2$, such that $x - X \in \mathfrak{C}$. By Lemma 3.4, there is a constant $C_6 = C_6(\mathfrak{C})$ such that $\|\partial^\alpha K_\varepsilon(x - \cdot)\|_{L^1(\Omega)} \le \|\partial^\alpha K_\varepsilon(\cdot)\|_{L^1(\mathfrak{C})} \le C_6/\varepsilon^3$ for all $x \in \Omega$ and $|\alpha| \le 4$, and so,

$$|\mathrm{II}(x,t)| = |u_\varepsilon(x,t) - u_\varepsilon^h(x,t)| \le C_7\frac{h^4}{\varepsilon^3}, \quad \forall x \in \Omega.$$

This implies that

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon^h(\cdot,t)\|_{L^\infty(\Omega)} \le C_7\frac{h^4}{\varepsilon^3}.$$

For $1 \le p < \infty$, it follows that

$$\begin{aligned}\|\mathrm{II}(t)\|_{h,p} &= \left\{\sum_{i\in I} h^2|u_\varepsilon(X(x_i,t),t) - u_\varepsilon^h(X(x_i,t),t)|^p\right\}^{1/p} \\ &\le \|u_\varepsilon(\cdot,t) - u_\varepsilon^h(\cdot,t)\|_{L^\infty(\Omega)}\left\{\sum_{i\in I} h^2\right\}^{1/p} \\ &\le C_8(\text{measure}\,\Omega)^{1/p}h^4/\varepsilon^3 = C_9h^4/\varepsilon^3.\end{aligned}$$

(b) As in (a), we have

$$|\mathrm{II}(x,t)| \le C_4h^4 \sum_{|\alpha|+|\beta|\le 4}\int_\Omega |\partial_X^\alpha K_\varepsilon(x - X(y,t))\partial_y^\beta w_0(y)|\,dy.$$

This time, Lemma 3.4 is not directly available to estimate $\|\partial^\alpha K_\varepsilon(x-\cdot)\|_{L^1(\Omega)}$, for all $|\alpha|\le 4$. For $|\alpha|=0$ and $|\beta|\le 4$, by Lemma 3.4 as above,

$$\|\partial_X^\alpha K_\varepsilon(x-X(\cdot,t))\partial_y^\beta w_0(\cdot)\|_{L^1(\Omega)} \le C_{10}\|K_\varepsilon(x-X(\cdot,t))\|_{L^1(\Omega)} \le C_{11}(\mathfrak{C}).$$

For $|\alpha|\ge 1$ and $|\beta|\le 3$, using Hölder's inequality, we obtain

$$\begin{aligned}
&\|\partial_X^\alpha K_\varepsilon(x-X(\cdot,t))\partial_y^\beta w_0(\cdot)\|_{L^1(\Omega)}\\
&\quad\le \|\partial_X^\alpha K_\varepsilon(x-X(\cdot,t))\|_{L^{p'}(\Omega)}\|\partial_y^\beta w_0(\cdot)\|_{L^{q'}(\Omega)}\\
&\qquad\qquad\qquad\qquad (1/p'+1/q'=1 \text{ and } 1<p'<\infty)\\
&\quad\le C_{12}\|\partial_X^\alpha K_\varepsilon(x-X(\cdot,t))\|_{L^{p'}(\Omega)} \le C_{12}\|\partial_X^\alpha K_\varepsilon(\cdot)\|_{L^{p'}(\mathbf{R}^2)}.
\end{aligned}$$

Using Lemma 3.2,

$$\|\partial_X^\alpha K_\varepsilon(\cdot)\|_{L^{p'}(\mathbf{R}^2)} \le \frac{C_{12}}{\varepsilon^{|\alpha|-1+2/q'}} \le \frac{C_{12}}{\varepsilon^{3+2/q'}}.$$

Hence, $|\mathrm{II}(x,t)| = |u_\varepsilon(x,t)-u_\varepsilon^h(x,t)| \le C_{13}h^4/\varepsilon^{3+s}$ for all $x\in\Omega$, where $s=2/q'>0$. Then (b) follows. □

The Stability Error. By definition, this is

$$u_\varepsilon^h(X,t)-\mathscr{U}_\varepsilon^h(X^h,t) = [u_\varepsilon^h(X,t)-\mathscr{U}_\varepsilon^h(X,t)] + [\mathscr{U}_\varepsilon^h(X,t)-\mathscr{U}_\varepsilon^h(X^h,t)],$$

where $X=X(x,t)$ and $X^h=X^h(x,t)$ for $x\in\Omega$. We call the expression in the first bracket Part I, the other Part II.

Part I. By Table 1,

$$\begin{aligned}
u_\varepsilon^h(X,t)-\mathscr{U}_\varepsilon^h(X,t) &= K_\varepsilon * w^h(X,t) - K_\varepsilon * \omega^h(X,t)\\
&= \sum_{i\in I}[c_iK_\varepsilon(X-X(x_i,t)) - d_{1i}(t)K_{\varepsilon x_1}(X-X(x_i,t))\\
&\qquad\qquad\qquad - d_{2i}(t)K_{\varepsilon x_2}(X-X(x_i,t))]\\
&\quad -\sum_{i\in I}[c_iK_\varepsilon(X-X^h(x_i,t)) - \mathscr{d}_{1i}(t)K_{\varepsilon x_1}(X-X^h(x_i,t))\\
&\qquad\qquad\qquad - \mathscr{d}_{2i}(t)K_{\varepsilon x_2}(X-X^h(x_i,t))]\\
&= \left\{\sum_{i\in I}c_i[K_\varepsilon(X-X(x_i,t)) - K_\varepsilon(X-X^h(x_i,t))]\right\}\\
&\quad + \left\{-\sum_{i\in I}d_{1i}(t)[K_{\varepsilon x_1}(X-X(x_i,t)) - K_{\varepsilon x_1}(X-X^h(x_i,t))]\right\}\\
&\quad + \left\{-\sum_{i\in I}d_{2i}(t)[K_{\varepsilon x_2}(X-X(x_i,t)) - K_{\varepsilon x_2}(X-X^h(x_i,t))]\right\}\\
&\quad + \left\{-\sum_{i\in I}K_{\varepsilon x_1}(X-X^h(x_i,t))[d_{1i}(t)-\mathscr{d}_{1i}(t)]\right\}\\
&\quad + \left\{-\sum_{i\in I}K_{\varepsilon x_2}(X-X^h(x_i,t))[d_{2i}(t)-\mathscr{d}_{2i}(t)]\right\}\\
&= V_{11}(X,t)+V_{12}(X,t)+V_{13}(X,t)+V_{21}(X,t)+V_{22}(X,t).
\end{aligned}$$

Here, for $i = 1, 2, j = 1, 2$ (or 1, 2, and 3), we use $V_{ij}(X,t)$ to denote the terms in each pair of braces in the above equation, respectively. Since

$$K_{\varepsilon x_1} = \partial(K * f_\varepsilon)/\partial x_1 = K * (\partial f_\varepsilon/\partial x_1)$$

and

$$\partial f_\varepsilon/\partial x_1 = \frac{\partial}{\partial x_1}[f(x/\varepsilon)/\varepsilon^2] = f_{x_1}(x/\varepsilon)/\varepsilon^3,$$

where $f_{x_1} = \partial f(x)/\partial x_1$,

$$K_{\varepsilon x_1} = (K * f_{x_1 \varepsilon})/\varepsilon. \tag{3.3}$$

Similarly,

$$K_{\varepsilon x_2} = (K * f_{x_2 \varepsilon})/\varepsilon. \tag{3.4}$$

Concerning terms $V_{1j}(X,t)$, $j = 1, 2$ and 3, we have

$$\begin{aligned} V_{11}(X,t) &= \sum_{i \in I} c_i [K_\varepsilon(X - X(x_i,t)) - K_\varepsilon(X - X^h(x_i,t))] \quad \text{(by (2.4))} \\ &= \sum_{i \in I} |S_i| (w_0(x_i) + H_{i1} w_{0x_1}(x_i) + H_{i2} w_{0x_2}(x_i)) \\ &\qquad \cdot [K_\varepsilon(X - X(x_i,t)) - K_\varepsilon(X - X^h(x_i,t))], \end{aligned}$$

$$\begin{aligned} V_{12}(X,t) &= -\sum_{i \in I} d_{1i}(t) [K_{\varepsilon x_1}(X - X(x_i,t)) - K_{\varepsilon x_1}(X - X^h(x_i,t))] \\ &= \sum_{i \in I} |S_i| \{M_{11}(x_i,t) H_{i1}/\varepsilon + M_{12}(x_i,t) H_{i2}/\varepsilon\} w_0(x_i) \\ &\qquad \cdot [K * f_{x_1 \varepsilon}(X - X(x_i,t)) - K * f_{x_1 \varepsilon}(X - X^h(x_i,t))]. \end{aligned}$$

Similarly,

$$\begin{aligned} V_{13}(X,t) &= -\sum_{i \in I} d_{2i}(t) [K_{\varepsilon x_2}(X - X(x_i,t)) - K_{\varepsilon x_2}(X - X^h(x_i,t))] \\ &= \sum_{i \in I} |S_i| \{M_{21}(x_i,t) H_{i1}/\varepsilon + M_{22}(x_i,t) H_{i2}/\varepsilon\} w_0(x_i) \\ &\qquad \cdot [K * f_{x_2 \varepsilon}(X - X(x_i,t)) - K * f_{x_2 \varepsilon}(X - X^h(x_i,t))]. \end{aligned}$$

Define

$$V_1(X,t) = \sum_{i \in I} |S_i| \alpha_i \{K * g_\varepsilon(X - X(x_i,t)) - K * g_\varepsilon(X - X^h(x_i,t))\}, \tag{3.5}$$

where $\{\alpha_i\}_{i \in I}$ is a family of real numbers, $g(\cdot)$ is a cutoff function which can be $f(\cdot)$ or the partial derivatives of $f(\cdot)$, and $g_\varepsilon(x) = g(x/\varepsilon)/\varepsilon^2$.

LEMMA 3.7. *Assume the conditions*:

(1) $g(\cdot) \in W^{1,\infty}(\mathbf{R}^2)$ *and there are two constants* $C_1 > 0$ *and* $\gamma > 2$ *such that*

$$|\partial^\alpha g(x)| \le C_1 (1 + |x|)^{-\gamma} \quad \forall x \in \mathbf{R}^2, \ |\alpha| = 0, 1.$$

(2) *There is a constant* $C_2 > 0$ *such that* $h/\varepsilon \le C_2$.

(3) *There is a constant* $C_3 > 0$, $C_3 = C_3(w_0, T)$, *such that* $|\alpha_i| \le C_3$ *for all* $i \in I$.

Then, for $p \in (1, \infty)$, *there exists a constant* $C = C(p, T, w_0)$ *such that*

$$\|V_1(\cdot, t)\|_{L^p(\mathbf{R}^2)} + \varepsilon |V_1(\cdot, t)|_{1,p,\mathbf{R}^2} \le C(1 + \|e(t)\|_{h,\infty}/\varepsilon)^{2/q} \|e(t)\|_{h,p}, \tag{3.6}$$

where $1/p + 1/q = 1$ *and the discrete norms are defined in Definition* 3.1.

Proof. [18, Chapter II, Lemma 5.2 and Lemma 5.3: Substitute C_3 in (3) for $\|w_0\|_{L^\infty(\mathbf{R}^2)}$].

Remark. Although Lemma 5.2 and Lemma 5.3 in [18] are proved for uniform meshes, the generalization to the nonuniform case is straightforward. Several similar direct extensions are used below without comment. □

In order to find a bound for the discrete norm of V_1, we need the following standard result in finite element theory.

LEMMA 3.8. *For all* $p > 2$ *and all functions* $g \in W^{1,p}(\mathbf{R}^2)$,

$$\|g(t)\|_{h,p} = \left\{ h^2 \sum_{i \in Z^n} |g(X(x_i, t)|^p \right\}^{1/p} \le C\{\|g\|_{L^p(\mathbf{R}^2)} + h|g|_{1,p,\mathbf{R}^2}\}. \tag{3.7}$$

Proof. See [18, Chapter II, Lemma 5.4]. □

COROLLARY 3.1. *Assume conditions* (1), (2) *and* (3) *in Lemma* 3.7; *then, for* $2 < p < \infty$, *there exists a constant* $C = C(p, T, w_0)$ *such that*

$$\|V_1(t)\|_{h,p} \le C(1 + \|e(t)\|_{h,\infty}/\varepsilon)^{2/q} \|e(t)\|_{h,p}.$$

Proof. The proof follows directly from Lemma 3.7 and Lemma 3.8. □

THEOREM 3.3. *Assume the conditions*:

(1) $f(\cdot) \in W^{2,\infty}(\mathbf{R}^2)$ *and there are two constants* $C_1 > 0$ *and* $\gamma > 2$ *such that*

$$|\partial^\alpha f(x)| \le C_1(1 + |x|)^{-\gamma} \quad \forall x \in \mathbf{R}^2, \ |\alpha| = 0, 1, 2.$$

(2) *There is a constant* $C_2 > 0$ *such that* $h/\varepsilon \le C_2$.

Then, for $2 < p < \infty$, *there exists a constant* $C = C(p, T, w_0)$ *such that*

$$\|V_{11}(t)\|_{h,p} + \|V_{12}(t)\|_{h,p} + \|V_{13}(t)\|_{h,p} \le C(1 + \|e(t)\|_{h,\infty}/\varepsilon)^{2/q} \|e(t)\|_{h,p},$$

where $1/p + 1/q = 1$.

Proof. Since the solution of the Euler equation is assumed to be smooth for $t \in [0, T]$, $M(x, t)$, $w_0(x)$ and $\nabla w_0(x)$ are uniformly bounded for all $x \in \Omega$. So,

$$\begin{gathered} \{w_0(x_i) + H_{i1} w_{0x_1}(x_i) + H_{i2} w_{0x_2}(x_i)\}_{i \in I}, \\ \{[M_{11}(x_i, t) H_{i1}/\varepsilon + M_{12}(x_i, t) H_{i2}/\varepsilon] w_0(x_i)\}_{i \in I} \quad \text{and} \\ \{[M_{21}(x_i, t) H_{i1}/\varepsilon + M_{22}(x_i, t) H_{i2}/\varepsilon] w_0(x_i)\}_{i \in I} \end{gathered}$$

are all uniformly bounded by some constant which is independent of h and ε. Thus, Theorem 3.3 follows from Lemma 3.7, Lemma 3.8 and Corollary 3.1. □

For estimating V_{21} and V_{22}, the following lemmas are needed.

LEMMA 3.9. *Assume the following conditions:*

(1) $g(\cdot) \in W^{\ell,1}(\mathbf{R}^2) \cap W^{\ell,\infty}(\mathbf{R}^2)$ *and there is a constant* $C_1 > 0$ *such that for* $|\beta| = \ell$, $|x|^{\ell+2}|\partial^\beta g(x)| \le C_1$ *for all* $x \in \mathbf{R}^2$.

(2) *There is a constant* C_2 *such that* $h/\varepsilon \le C_2$.

Then, for any compact set $\mathscr{S} \in \mathbf{R}^2$, *there exists a constant* $C = C(\mathscr{S})$ *such that*

$$\sum_{i\in I, |y_i| \le \|e(t)\|_{h,\infty}} |\partial^\beta K * g_\varepsilon(x - X(x_i,t) + y_i)|h^2 \le B(\beta,\varepsilon)$$

for all $x \in \mathscr{S}$, $t \in [0, T_\varepsilon]$ *and* $|\beta| = \ell$, *where* $T_\varepsilon = \max\{t \in [0,T] \mid \|e(t)\|_{h,\infty} \le M\varepsilon;$ *M is an arbitrary constant*$\}$ *and*

$$B(\beta,\varepsilon) = \begin{cases} C, & |\beta| = 0, \\ C|\log\varepsilon|, & |\beta| = 1, \\ C\varepsilon^{1-|\beta|}, & |\beta| > 1. \end{cases}$$

Proof. For any $i \in I$, as in the proof of Lemma 5.2 in [18], the area of $S_i(t)$ is of order h^2. Let $a = \max_{i\in I}\max_{y\in S_i} |X(y,t) - X^h(x_i,t)|$; then

$$\|e(t)\|_{h,\infty} \le a \le C_3 h + \|e(t)\|_{h,\infty}. \tag{3.8}$$

For a fixed $x \in \mathscr{S}$, let $J_1 = \{i \in I \mid |x - X(x_i,t)| \le \varepsilon + a\}$. If $i \in J_1$, then $|x - X(x_i,t)| \le \varepsilon + a \le \varepsilon + C_3 h + \|e(t)\|_{h,\infty}$. So, $X(x_i,t) \in S(x, \varepsilon + C_3 h + \|e(t)\|_{h,\infty})$. This implies that $\operatorname{Card} J_1 \le C_4(\varepsilon/h + 1 + \|e(t)\|_{h,\infty}/h)^2$. So,

$$\begin{aligned}\sum_{i\in J_1, |y_i| \le \|e(t)\|_{h,\infty}} &|\partial^\beta K * g_\varepsilon(x - X(x_i,t) + y_i)|h^2 \le \operatorname{Card} J_1 h^2 \|\partial^\beta K * g_\varepsilon\|_{L^\infty(\mathbf{R}^2)} \\ &\le C_4 \frac{(\varepsilon/h + 1 + \|e(t)\|_{h,\infty}/h)^2 h^2}{\varepsilon^{\ell+1}} \le C_5(1 + \|e(t)\|_{h,\infty}/\varepsilon)^2 \cdot \varepsilon^{1-\ell} \\ &\le C_6 \varepsilon^{1-\ell} \quad \text{for all } t \in [0, T_\varepsilon] \text{ (Lemma 3.3(a))}.\end{aligned}$$

Let $J_2 = I \backslash J_1$. If $i \in J_2$, then $|x - X(x_i,t)| \ge \varepsilon + a$. So, $|x - X(x_i,t) + y_i| \ge \varepsilon + a - \|e(t)\|_{h,\infty} \ge \varepsilon$, because of (3.8). Using Lemma 3.3(b), we obtain

$$|\partial^\beta K * g_\varepsilon(x - X(x_i,t) + y_i)| \le \frac{C_7}{|x - X(x_i,t) + y_i|^{\ell+1}} \le \frac{C_7}{\{|x - X(x_i,t)| - a\}^{\ell+1}},$$

so that,

$$\sum_{i\in J_2, |y_i| \le \|e(t)\|_{h,\infty}} |\partial^\beta K * g_\varepsilon(x - X(x_i,t) + y_i)|h^2 \le C_7 \sum_{i\in J_2} \frac{h^2}{\{|x - X(x_i,t)| - a\}^{\ell+1}}.$$

When $h \to 0$,

$$\lim_{h\to 0} \sum_{i\in J_2} \frac{h^2}{\{|x - X(x_i,t)| - a\}^{\ell+1}} \le \int_{\substack{|x-y| \ge \varepsilon + a \\ y\in\Omega}} \frac{dy}{\{|x-y| - a\}^{\ell+1}}.$$

Since $x \in \mathscr{S}$, $y \in \Omega$, where $\mathscr{S}$ is compact and Ω is bounded, there exists a constant $R > 0$ such that $|x - y| \le R$, and therefore,

$$\begin{aligned}\int_{\substack{|x-y| \ge \varepsilon + a \\ y\in\Omega}} \frac{dy}{\{|x-y| - a\}^{\ell+1}} &\le \int_0^{2\pi}\int_{\varepsilon + a \le r \le R} \frac{r\,dr\,d\theta}{(r-a)^{\ell+1}} \\ &= \int_0^{2\pi}\int_\varepsilon^{R-a} \frac{(r+a)\,dr\,d\theta}{r^{\ell+1}}.\end{aligned}$$

Since $a \le C_3 h + \|e(t)\|_{h,\infty} \le C_3 h + M\varepsilon \le C_8\varepsilon \le C_8 r$ for all $t \in [0, T_\varepsilon]$, it follows that

$$\int_0^{2\pi}\int_\varepsilon^{R-a} \frac{(r+a)\,dr\,d\theta}{r^{\ell+1}} \le (1+C_8)2\pi \int_{\varepsilon\le r\le R-a} \frac{dr}{r^{\ell}}$$

$$= \begin{cases} (1+C_8)2\pi(R-a-\varepsilon), & |\beta| = 0, \\ (1+C_8)2\pi(\log(R-a) - \log\varepsilon), & |\beta| = 1, \\ (1+C_8)2\pi\dfrac{1}{\ell-1}\left[\dfrac{1}{\varepsilon^{\ell-1}} - \dfrac{1}{(R-a)^{\ell-1}}\right], & |\beta| > 1. \quad \square \end{cases}$$

Define

$$V_2(X,t) = \sum_{i\in I} |S_i|\alpha_i\{\partial^\beta K * g_\varepsilon(X - X^h(x_i,t))\}, \tag{3.9}$$

where $\{\alpha_i\}_{i\in I}$ and $g(\cdot)$ are the same as in (3.5). Then we have

LEMMA 3.10. *Assume the conditions* (1) *and* (2) *in Lemma* 3.9. *Then there exists a constant* $C = C(\Omega, p)$ *such that*

$$\|V_2(t)\|_{h,p} \le B(\beta,\varepsilon)\left\{\sum_{i\in I} |\alpha_i|^p h^2\right\}^{1/p} \qquad \forall p \in (1,\infty) \text{ and } t \in [0, T_\varepsilon],$$

where T_ε *and* $B(\beta,\varepsilon)$ *are the same as in Lemma* 3.9.

Proof. For $j \in I$ and $i \in I$,

$$X(x_j,t) - X^h(x_i,t) = [X(x_j,t) - X(x_i,t)] + [X(x_i,t) - X^h(x_i,t)].$$

Let $y_i = X(x_i,t) - X^h(x_i,t)$; then $|y_i| \le \|e(t)\|_{h,\infty}$. So, Lemma 3.9 implies that for $t \in [0, T_\varepsilon]$,

$$\sum_{i\in I} |\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))| h^2 \le B_1(\beta,\varepsilon).$$

Let $\mathscr{g}(x) = g(-x)$. Since $K(y) = -K(-y)$,

$$K * g_\varepsilon(x) = \int_{\mathbf{R}^2} K(y) g_\varepsilon(x-y)\,dy = \int_{\mathbf{R}^2} K(y)\mathscr{g}_\varepsilon(-x-y)\,dy = K * \mathscr{g}_\varepsilon(-x).$$

So we have

$$K * g_\varepsilon(X(x_j,t) - X^h(x_i,t)) = K * \mathscr{g}_\varepsilon(X^h(x_i,t) - X(x_j,t)).$$

Also,

$$X^h(x_i,t) - X(x_j,t) = [X(x_i,t) - X(x_j,t)] + [X^h(x_i,t) - X(x_i,t)],$$

and letting $y_i = X^h(x_i,t) - X(x_i,t)$, we have $|y_i| \le \|e(t)\|_{h,\infty}$. Substitute $\mathscr{g}$ for g in Lemma 3.9; then, for $t \in [0, T_\varepsilon]$, we have

$$\sum_{j\in I} |\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))| h^2$$
$$= \sum_{j\in I} |\partial^\beta K * \mathscr{g}_\varepsilon(X^h(x_i,t) - X(x_j,t))| h^2 \le B_2(\beta,\varepsilon),$$

$$|V_2(X(x_j,t),t)| = \left|\sum_{i\in I} |S_i|\alpha_i\{\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))\}\right|$$

$$\leq C_3 \sum_{i\in I} |\alpha_i||\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|^{1/p+1/q} h^{2/p+2/q}$$

$$\leq C_3 \left\{\sum_{i\in I} |\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right\}^{1/q}$$

$$\cdot \left\{\sum_{i\in I} |\alpha_i|^p|\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right\}^{1/p} \quad \text{(Hölder Inequality)}$$

$$\leq C_3(B_1(\beta,\varepsilon))^{1/q} \left\{\sum_{i\in I} |\alpha_i|^p|\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right\}^{1/p}.$$

From this, we obtain

$$|V_2(X(x_j,t),t)|^p$$

$$\leq (C_3)^p(B_1(\beta,\varepsilon))^{p/q} \left\{\sum_{i\in I} |\alpha_i|^p|\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right\}$$

and

$$\|V_2(t)\|^p_{h,p} = \sum_{j\in I} |V_2(X(x_j,t),t)|^p h^2$$

$$\leq (C_3)^p(B_1(\beta,\varepsilon))^{p/q} \left\{\sum_{j\in I} \left[\sum_{i\in I} |\alpha_i|^p|\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right] h^2\right\}$$

$$= (C_3)^p(B_1(\beta,\varepsilon))^{p/q} \left\{\sum_{i\in I} \left[\sum_{j\in I} |\partial^\beta K * g_\varepsilon(X(x_j,t) - X^h(x_i,t))|h^2\right] |\alpha_i|^p h^2\right\}$$

$$= (C_3)^p(B_1(\beta,\varepsilon))^{p/q} B_2(\beta,\varepsilon) \left\{\sum_{i\in I} |\alpha_i|^p h^2\right\}.$$

Thus,

$$\|V_2(t)\|_{h,p} \leq C_3(B_1(\beta,\varepsilon))^{1/q}(B_2(\beta,\varepsilon))^{1/p} \left\{\sum_{i\in I} |\alpha_i|^p h^2\right\}^{1/p} \quad \text{for all } t \in [0,T_\varepsilon]. \ \square$$

THEOREM 3.4. *Assume the following conditions:*

(1) $f(\cdot) \in W^{1,1}(\mathbf{R}^2) \cap W^{1,\infty}(\mathbf{R}^2)$. *There is a constant* $C_1 > 0$ *such that* $|x|^2|\partial^\alpha f(x)| \leq C_1$ *for all* $x \in \mathbf{R}^2$, $|\alpha| = 1$.

(2) *There is a constant* C_2 *such that* $h \leq C_2\varepsilon^2$.

Then there exists a constant $C = C(\Omega, p, w_0)$ *such that*

$$\|V_{21}(t)\|_{h,p} + \|V_{22}(t)\|_{h,p} \leq C\varepsilon\|E(t)\|_{h,p} \quad \forall t \in [0,T_\varepsilon].$$

Proof. By definition,

$$V_{21}(X,t) = -\sum_{i\in I} K_{\varepsilon x_1}(X - X^h(x_i,t))[d_{1i}(t) - \mathscr{d}_{1i}(t)]$$

$$= -(1/\varepsilon)\sum_{i\in I} K * f_{x_1\varepsilon}(X - X^h(x_i,t))[d_{1i}(t) - \mathscr{d}_{1i}(t)].$$

By Lemma 3.10 with $\beta = 0$ and $g = f_{x_1\varepsilon}$, there is a constant $C_3 = C_3(\Omega, p)$ such that

$$\|V_{21}(t)\|_{h,p} \leq (C_3/\varepsilon)\left\{\sum_{i\in I} |[d_{1i}(t) - \mathscr{d}_{1i}(t)]/|S_i||^p h^2\right\}^{1/p}.$$

According to the construction of the method in Section 2,

$$\begin{aligned}
&[d_{1i}(t) - \mathscr{d}_{1i}(t)]/|S_i| \\
&\quad = [M_{11}(x_i,t)d_{1i} + M_{12}(x_i,t)d_{2i} - M_{11}^h(x_i,t)d_{1i} - M_{12}^h(x_i,t)d_{2i}]/|S_i| \\
&\quad = [M_{11}(x_i,t) - M_{11}^h(x_i,t)](-H_{i1}w_0(x_i)) \\
&\qquad + [M_{12}(x_i,t) - M_{12}^h(x_i,t)](-H_{i2}w_0(x_i)),
\end{aligned}$$

so that

$$|d_{1i}(t) - \mathscr{d}_{1i}(t)|/|S_i| \leq 2h\|w_0\|_{L^\infty(\mathbf{R}^2)}|M(x_i,t) - M^h(x_i,t)|.$$

Hence, $\|V_{21}(t)\|_{h,p} \leq (C_4 h/\varepsilon)\|E(t)\|_{h,p} \leq (C_4C_2\varepsilon)\|E(t)\|_{h,p}$ (condition (2)), and in a similar way, $\|V_{22}(t)\|_{h,p} \leq (C_5C_2\varepsilon)\|E(t)\|_{h,p}$. □

Part II. Consider the difference

$$\begin{aligned}
&\mathscr{U}_\varepsilon^h(X,t) - \mathscr{U}_\varepsilon^h(X^h,t) = K_\varepsilon * \omega^h(X,t) - K_\varepsilon * \omega^h(X^h,t) \\
&\quad = \sum_{i\in I} [c_i K_\varepsilon(X - X^h(x_i,t)) - \mathscr{d}_{1i}(t)K_{\varepsilon x_1}(X - X^h(x_i,t)) \\
&\qquad\qquad - \mathscr{d}_{2i}(t)K_{\varepsilon x_2}(X - X^h(x_i,t))] \\
&\qquad - \sum_{i\in I} [c_i K_\varepsilon(X^h - X^h(x_i,t)) - \mathscr{d}_{1i}(t)K_{\varepsilon x_1}(X^h - X^h(x_i,t)) \\
&\qquad\qquad - \mathscr{d}_{2i}(t)K_{\varepsilon x_2}(X^h - X^h(x_i,t))] \\
&\quad = \left\{\sum_{i\in I} c_i[K_\varepsilon(X - X^h(x_i,t)) - K_\varepsilon(X^h - X^h(x_i,t))]\right\} \\
&\qquad + \left\{-\sum_{i\in I} \mathscr{d}_{1i}(t)[K_{\varepsilon x_1}(X - X^h(x_i,t)) - K_{\varepsilon x_1}(X^h - X^h(x_i,t))]\right\} \\
&\qquad + \left\{-\sum_{i\in I} \mathscr{d}_{2i}(t)[K_{\varepsilon x_2}(X - X^h(x_i,t)) - K_{\varepsilon x_2}(X^h - X^h(x_i,t))]\right\},
\end{aligned}$$

where we call the expression in the first pair of braces $W_{11}(X, X^h, t)$ and the other two $W_{21}(X, X^h, t)$ and $W_{22}(X, X^h, t)$. Now we define

$$W(X, X^h, t) = \sum_{i\in I} |S_i|\alpha_i[K * g_\varepsilon(X - X^h(x_i,t)) - K * g_\varepsilon(X^h - X^h(x_i,t))], \tag{3.10}$$

where $\{\alpha_i\}_{i\in I}$, g and g_ε are as before. For $\{\alpha_i\}_{i\in I}$, we have the following lemma, where the region Ω and $\mathscr{A}$ are as shown in Figure 5. $\mathscr{A}$ is a compact set which contains Ω and is bounded by mesh lines, and x_i, x_k are adjacent mesh points in $\mathscr{A}$.

LEMMA 3.12. *Consider the family of numbers* $\{\alpha_i \mid i \in I,\ \alpha_i = 0 \text{ if } x_i \notin \Omega\}$. *Assume the conditions*:

(1) *There exists a constant* C_1 *such that* $\max_{j\in J}(h_{j1}, h_{j2})/\min_{j\in J}(h_{j1}, h_{j2}) \leq C_1$ *and* $h \leq 1$.

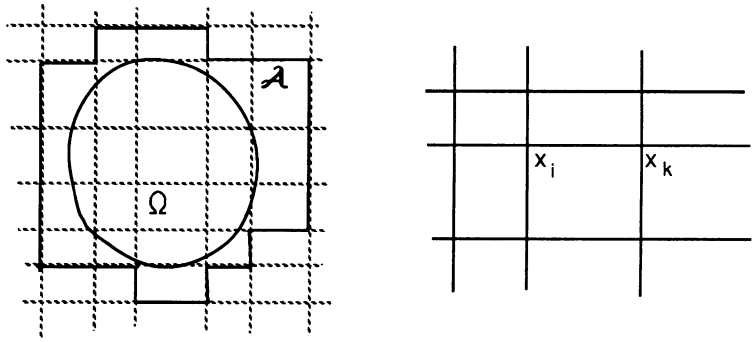

FIGURE 5

(2) *There exists a constant C_2 which is independent of h such that $|\alpha_i| \le C_2$ for all $i \in I$.*

(3) *For any adjacent x_i and x_k with $i, k \in I$, $|\alpha_i - \alpha_k| = \beta_{ik}|x_i - x_k|$, there exists a constant C_3 which is independent of h such that $|\beta_{ik}| \le C_3$.*

Then there exists a function $\omega_0(\cdot) \in B(\mathring{\mathscr{A}}) \cap W_0^{1,p}(\mathring{\mathscr{A}})$ ($\mathring{\mathscr{A}}$ is the interior of $\mathscr{A}$), $1 \le p \le \infty$, such that $\omega_0(x_i) = \alpha_i$ for all $i \in I$, and $\|\omega_0(\cdot)\|_{1,p,\mathring{\mathscr{A}}}$ is independent of h.

Proof. For any $j \in J$, take a local Cartesian coordinate system with the origin at x_{j1} (see Figure 6). Let $P_j(\varsigma_1, \varsigma_2) = A_j + B_j\varsigma_1 + C_j\varsigma_2 + D_j\varsigma_1\varsigma_2$, which satisfies

$$P_j(0,0) = \alpha_{j1}, \qquad P_j(h_{j1}, 0) = \alpha_{j2},$$
$$P_j(h_{j1}, h_{j2}) = \alpha_{j3}, \qquad P_j(0, h_{j2}) = \alpha_{j4}.$$

Then,

$$P_j(\varsigma_1, \varsigma_2) = \alpha_{j1} + \{(\alpha_{j2} - \alpha_{j1})/h_{j1}\}\varsigma_1 + \{(\alpha_{j4} - \alpha_{j1})/h_{j2}\}\varsigma_2 + \{[(\alpha_{j3} - \alpha_{j2}) + (\alpha_{j1} - \alpha_{j4})]/h_{j1}h_{j2}\}\varsigma_1\varsigma_2.$$

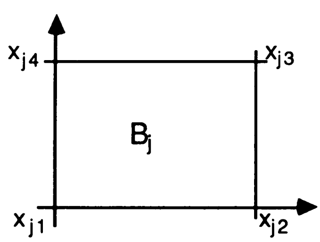

FIGURE 6

For the global coordinate system on B_j, $(x_1, x_2) = (x_{j11} + \varsigma_1, x_{j12} + \varsigma_2)$, where (x_{j11}, x_{j12}) is the coordinate for the point x_{j1}, $0 \le \varsigma_k \le h_{jk}$, $k = 1, 2$. On B_j, define $\rho_j(x_1, x_2) = \rho_j(x_{j11} + \varsigma_1, x_{j12} + \varsigma_2) = P_j(\varsigma_1, \varsigma_2)$. Let $\omega_0(\cdot)$ be a function defined on $\mathscr{A}$ in the following way:

$$\omega_0(x)|_{B_j} = \rho_j(x_1, x_2) = P_j(\varsigma_1, \varsigma_2) \quad \text{for all } j \in J.$$

By finite element theory, $\omega_0(\cdot) \in B(\mathscr{A})$ and $\omega_0(x)|_{\partial\mathscr{A}} = 0$, ω_0 is differentiable a.e., $\partial^\alpha \omega_0(x)|_{B_j} = \partial^\alpha \wp_j$ for $|\alpha| = 1$. So,

$$\begin{aligned}
&|P_j(\varsigma_1, \varsigma_2)| \\
&\quad \le |\alpha_{j1}| + |(\alpha_{j2} - \alpha_{j1})| + |(\alpha_{j4} - \alpha_{j1})| + |(\alpha_{j3} - \alpha_{j2})| + |(\alpha_{j1} - \alpha_{j4})| \\
&\quad \le |\alpha_{j1}| + |\beta_{j_2 j_1}| h_{j1} + |\beta_{j_4 j_1}| h_{j2} + |\beta_{j_3 j_2}| h_{j_2} + |\beta_{j_1 j_4}| h_{j1} \le C_2 + 4C_3 \quad \forall j \in J,
\end{aligned}$$

from which

$$\|\omega_0\|_{L^\infty(\mathscr{A})} \le C_2 + 4C_3. \tag{3.11}$$

Meanwhile,

$$\begin{aligned}
\partial \wp_j / \partial x_1 &= (\partial \wp_j / \partial \varsigma_1)(\partial \varsigma_1 / \partial x_1) + (\partial \wp_j / \partial \varsigma_2)(\partial \varsigma_2 / \partial x_1) \\
&= \{(\alpha_{j2} - \alpha_{j1})/h_{j1}\} + \{[(\alpha_{j3} - \alpha_{j2}) + (\alpha_{j1} - \alpha_{j4})]/h_{j1} h_{j2}\} \varsigma_2,
\end{aligned}$$

and so, $|\partial \wp_j / \partial x_1| \le |\beta_{j_2 j_1}| + |\beta_{j_3 j_2}| C_1 + |\beta_{j_1 j_4}| C_1 \le C_3(1 + 2C_1)$. Similarly, $|\partial \wp_j / \partial x_2| \le C_3(1 + 2C_1)$, and

$$\|\partial^\alpha \omega_0\|_{L^\infty(\mathscr{A})} \le C_3(1 + 2C_1), \qquad |\alpha| = 1. \tag{3.12}$$

(3.11) and (3.12) imply that $\omega_0 \in W_0^{1,\infty}(\mathscr{A})$. Since $\mathscr{A}$ is bounded, $\omega_0 \in W_0^{1,\infty}(\mathscr{A})$ implies that $\omega_0 \in W_0^{1,p}(\mathscr{A})$ for $1 \le p < \infty$ and $\|\omega_0\|_{\overset{\circ}{1,p,\mathscr{A}}}$ depends on C_1, C_2, C_3 and $\mathscr{A}$. □

LEMMA 3.12. *Assume the conditions*:
(1) $g(\cdot) \in W^{2,1}(\mathbf{R}^2) \cap W^{2,\infty}(\mathbf{R}^2)$ *and satisfies*

$$|\partial^\alpha g(x)| \le C_1(1 + |x|)^{-4} \quad \textit{for all } x \in \mathbf{R}^2 \textit{ and } |\alpha| = 2.$$

(2) $h/\varepsilon \le C_2$.
(3) $\{\alpha_i\}_{i \in I}$ *satisfies* (1), (2) *and* (3) *in Lemma* 3.11.
Then there exists a constant C *which is independent of* h *and* ε *such that*

$$\left| \sum_{i \in I} \alpha_i |S_i| \frac{\partial K * g_\varepsilon}{\partial x_k}(x - X^h(x_i, t)) \right| \le C(1 + \|e(t)\|_{h,\infty}/\varepsilon)^2$$
$$\forall x \in \mathbf{R}^2, \ t \in [0, T], \ k = 1, 2.$$

Proof. [18, Chapter II, Lemma 5.6: Substitute ω_0 in Lemma 3.11 for w_0]. □

LEMMA 3.13. *Assume the conditions* (1), (2) *and* (3) *in Lemma* 3.12. *Then there exists a constant* C *which is independent of* h *and* ε *such that for all* $t \in [0, T]$ *and* $p \in [0, \infty]$, $\|W(t)\|_{h,p} \le C \|e(t)\|_{h,p}(1 + \|e(t)\|_{h,\infty}/\varepsilon)^2$, *where* $W(X, X^h, t)$ *is defined by* (3.10) *and* $\|W(t)\|_{h,p}$ *is given by Definition* 3.1.

Proof. By (3.10) we have

$$
\begin{aligned}
&|W(X(x_j,t),X^h(x_j,t),t| \\
&= \left|\sum_{i\in I}|S_i|\alpha_i[K*g_\varepsilon(X(x_j,t)-X^h(x_i,t))-K*g_\varepsilon(X^h(x_j,t)-X^h(x_i,t))]\right| \\
&= \left|\sum_{i\in I}|S_i|\alpha_i\left\{\int_0^1 DK*g_\varepsilon[X(x_j,t)-X^h(x_i,t)+\theta(X^h(x_j,t)-X(x_j,t))]\,d\theta\right\}\right. \\
&\qquad\qquad \left.\cdot (X^h(x_j,t)-X(x_j,t))\right| \\
&\le \int_0^1 \left|\sum_{i\in I}|S_i|\alpha_i DK*g_\varepsilon[X(x_j,t)-X^h(x_i,t)+\theta(X^h(x_j,t)-X(x_j,t))]\right|\,d\theta \\
&\qquad\qquad \cdot |X^h(x_j,t)-X(x_j,t)| \\
&\le C_3(1+\|e(t)\|_{h,\infty}/\varepsilon)^2|X^h(x_j,t)-X(x_j,t)| \quad \text{(Lemma 3.12)},
\end{aligned}
$$

and hence, $\|W(t)\|_{h,p}\le C\|e(t)\|_{h,p}(1+\|e(t)\|_{h,\infty}/\varepsilon)^2$. □

THEOREM 3.5. *Assume the conditions:*

(1) $f(\cdot)\in W^{2,1}(\mathbf{R}^2)\cap W^{2,\infty}(\mathbf{R}^2)$ *and satisfies*

$$|\partial^\alpha f(x)|\le C_1(1+|x|)^{-4} \quad \textit{for all } x\in\mathbf{R}^2 \textit{ and } |\alpha|=2.$$

(2) $h/\varepsilon\le C_2$.

Then there exists a constant $C=C(w_0,p,T)$ *such that*

$$\|W_{11}(t)\|_{h,p}\le C(1+\|e(t)\|_{h,\infty}/\varepsilon)^2\|e(t)\|_{h,p} \quad \textit{for all } t\in[0,T],\ p\in[1,\infty].$$

Proof. By definition,

$$W_{11}(X,X^h,t)=\sum_{i\in I}c_i[K*f_\varepsilon(X-X^h(x_i,t))-K*f_\varepsilon(X^h-X^h(x_i,t))],$$

where $c_i=|S_i|(w_0(x_i)+H_{i1}w_{0x_1}(x_i)+H_{i2}w_{0x_2}(x_i))$. Let $\alpha_i=w_0(x_i)+H_{i1}w_{0x_1}(x_i)$ $+H_{i2}w_{0x_2}(x_i)$; then $|\alpha_i|\le C_3\|w_0\|_{W^{1,\infty}(\mathbf{R}^2)}$ for all $i\in I$. For two adjacent x_i and x_k with $i,k\in I$ we have

$$
\begin{aligned}
|\alpha_i-\alpha_k| &= |w_0(x_i)-w_0(x_k)+H_{i1}(w_{0x_1}(x_i)-w_{0x_1}(x_k)) \\
&\qquad + H_{i2}(w_{0x_2}(x_i)-w_{0x_2}(x_k))+w_{0x_1}(x_k)(H_{i1}-H_{k1}) \\
&\qquad\qquad + w_{0x_2}(x_k)(H_{i2}-H_{k2})| \\
&= |Dw_0(x_{ik})\cdot(x_i-x_k)+H_{i1}Dw_{0x_1}(y_{ik})\cdot(x_i-x_k) \\
&\qquad + H_{i2}Dw_{0x_2}(z_{ik})\cdot(x_i-x_k)+w_{0x_1}(x_k)(H_{i1}-H_{k1}) \\
&\qquad\qquad + w_{0x_2}(x_k)(H_{i2}-H_{k2})| \\
&\le C_4\|w_0\|_{W^{2,\infty}(\mathbf{R}^2)}|x_i-x_k|,
\end{aligned}
$$

where x_{ik}, y_{ik} and z_{ik} are intermediate points between x_i and x_k. Thus, by Lemma 3.13,

$$\|W_{11}(t)\|_{h,p}\le C_5\|e(t)\|_{h,p}(1+\|e(t)\|_{h,\infty}/\varepsilon)^2 \quad \forall t\in[0,T],\ p\in[0,\infty]. \ \square$$

For the estimation of $W_{21}(X,X^h,t)$ and $W_{22}(X,X^h,t)$, we have the following result.

THEOREM 3.6. *Assume the following conditions:*
(1) $f(\cdot) \in W^{2,1}(\mathbf{R}^2) \cap W^{2,\infty}(\mathbf{R}^2)$ *and satisfies*

$$|x|^4 |\partial^\alpha f(x)| \leq C_1 \quad \text{for all } x \in \mathbf{R}^2 \text{ and } |\alpha| = 2.$$

(2) $h \leq C_2 \varepsilon^2$.
Let $T_\varepsilon^* = \{t \in [0,T] \mid \|e(t)\|_{h,\infty} \leq M_1 \varepsilon, \|E(t)\|_{h,\infty} \leq M_2\}$, *where* M_1 *and* M_2 *are two arbitrary constants. Then there exists a constant* $C = C(w_0, T, p)$ *such that*

$$\|W_{21}(t)\|_{h,p} + \|W_{22}(t)\|_{h,p} \leq C \|e(t)\|_{h,p} \quad \text{for all } t \in [0, T_\varepsilon^*] \text{ and } p \in (1, \infty).$$

Proof. By definition,

$$\begin{aligned} W_{21}(X, X^h, t) &= -\sum_{i \in I} \mathscr{A}_{1i}(t) [K_{\varepsilon x_1}(X - X^h(x_i,t)) - K_{\varepsilon x_1}(X^h - X^h(x_i,t))] \\ &= -\sum_{i \in I} \frac{\mathscr{A}_{1i}(t)}{\varepsilon} [K * f_{x_1 \varepsilon}(X - X^h(x_i,t)) - K * f_{x_1 \varepsilon}(X^h - X^h(x_i,t))] \\ &= \sum_{i \in I} \frac{\mathscr{A}_{1i}(t)}{\varepsilon} \left[\int_0^1 DK * f_{x_1 \varepsilon}(X - X^h(x_i,t) + \theta(X - X^h))\, d\theta \right] \cdot (X^h - X), \end{aligned}$$

and

$$\begin{aligned} &W_{21}(X(x_j,t), X^h(x_j,t), t) \\ &\quad = \sum_{i \in I} \frac{\mathscr{A}_{1i}(t)}{\varepsilon} \left[\int_0^1 DK * f_{x_1 \varepsilon}(X(x_j,t) - X^h(x_i,t) + \theta(X(x_j,t) - X^h(x_j,t)))\, d\theta \right] \\ &\qquad \cdot (X^h(x_j,t) - X(x_j,t)). \end{aligned}$$

So,

$$\begin{aligned} &|W_{21}(X(x_j,t), X^h(x_j,t), t)| \\ &\quad \leq |X^h(x_j,t) - X(x_j,t)| \\ &\qquad \times \sum_{i \in I} \frac{|\mathscr{A}_{1i}(t)|}{\varepsilon} \max_{0 \leq \theta \leq 1} |DK * f_{x_1 \varepsilon}(X(x_j,t) - X^h(x_i,t) + \theta(X(x_j,t) - X^h(x_i,t)))| \\ &\quad = |X^h(x_j,t) - X(x_j,t)| \\ &\qquad \times \sum_{i \in I} \frac{|\mathscr{A}_{1i}(t)|}{\varepsilon} \max_{0 \leq \theta \leq 1} |DK * f_{x_1 \varepsilon}(X(x_j,t) + \theta(X(x_j,t) - X^h(x_j,t)) \\ &\qquad\qquad - X(x_i,t) + y_i)|, \end{aligned}$$

where $y_i = X(x_i,t) - X^h(x_i,t)$ and $|y_i| \leq \|e(t)\|_{h,\infty}$. By the definition of $\mathscr{A}_{1i}(t)$,

$$\mathscr{A}_{1i}(t)/\varepsilon = -[M_{11}^h(x_i,t) H_{i1} + M_{12}^h(x_i,t) H_{i2}] w_0(x_i) |S_i| / \varepsilon.$$

Since for $t \in [0, T_\varepsilon^*]$, $\|E(t)\|_{h,\infty} \leq M_2$, and $\|M(\cdot,t)\|_{L^\infty(\mathbf{R}^2)}$ is bounded for $t \in [0, T_\varepsilon^*]$, we have $|M^h(x_i,t)| \leq C_6$ for all $i \in I$ and $t \in [0, T_\varepsilon^*]$. Therefore,

$$|M_{11}^h(x_i,t) H_{i1} + M_{12}^h(x_i,t) H_{i2}| / \varepsilon \leq C_7 \varepsilon \quad \forall i \in I \text{ and } t \in [0, T_\varepsilon^*],$$

because $h \leq C_2 \varepsilon^2$, $H_{i1} = O(h)$ and $H_{i2} = O(h)$. Hence,

$$|\mathscr{A}_{1i}(t)/\varepsilon| \leq C_7 \varepsilon \|w_0\|_{L^\infty(\mathbf{R}^2)} |S_i| \quad \text{for all } i \in I.$$

Using Lemma 3.9 and Lemma 3.10 with $|\beta| = 1$,

$$\begin{aligned}
&|W_{21}(X(x_j,t), X^h(x_j,t),t)| \\
&\quad \le |X^h(x_j,t) - X(x_j,t)| \\
&\qquad \times \sum_{i\in I} \frac{|\mathscr{A}_{1i}(t)|}{\varepsilon} \max_{0\le\theta\le 1} |DK * f_{x_1\varepsilon}(X(x_j,t) + \theta(X(x_j,t) - X^h(x_j,t)) \\
&\qquad\qquad - X(x_i,t) + y_i)| \\
&\quad \le |X^h(x_j,t) - X(x_j,t)| C_7 \varepsilon \|w_0\|_{L^\infty(\mathbf{R}^2)} C_8 |\log \varepsilon| \le C_9 |X^h(x_j,t) - X(x_j,t)|.
\end{aligned}$$

This implies that $\|W_{21}(t)\|_{h,p} \le C_9 \|e(t)\|_{h,p}$ for all $t \in [0, T_\varepsilon^*]$, $p \in (1,\infty)$. Similarly, $\|W_{22}(t)\|_{h,p} \le C_{10} \|e(t)\|_{h,p}$ for all $t \in [0, T_\varepsilon^*]$, $p \in (1,\infty)$. □

The Error Bounds. Now we will give the error estimate for the 2-D vortex method constructed previously.

THEOREM 3.7 [Summary]. *Assume the following conditions:*

(1) (i), (ii) *and* (iii) *of Theorem* 3.1.

(2) $f(\cdot) \in W^{4,1}(\mathbf{R}^2) \cap W^{4,\infty}(\mathbf{R}^2)$ *and there exist constants* C_1, C_2 *and* $\gamma > 2$ *such that*

$$\begin{aligned}
|\partial^\alpha f(x)| &\le C_1 (1+|x|)^{-\gamma}, \qquad x \in \mathbf{R}^2, |\alpha| = 0, 1; \\
|\partial^\alpha f(x)| &\le C_2 (1+|x|)^{-4}, \qquad x \in \mathbf{R}^2, |\alpha| = 2.
\end{aligned}$$

(3) *There exists a constant* $C_3 > 0$ *such that* $h \le C_3 \varepsilon^2$.

Then, for $2 < p < \infty$, *there exists a constant* $C = C(p, T, s, w_0)$ *such that*

$$\left\| \frac{d}{dt} e(t) \right\|_{h,p} \le C \left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}} + \|e(t)\|_{h,p} + \varepsilon \|E(t)\|_{h,p} \right) \quad \forall t \in [0, T_\varepsilon^*],$$

where $s > 0$ *is an arbitrary number.*

Proof. This result is the content of Theorems 3.1–3.6. □

Since the error bound for $d\{e(X, X^h, t)\}/dt$ involves the term $\varepsilon \|E(t)\|_{h,p}$, we now need to analyze $d\{\varepsilon E(X, X^h, t)\}/dt$:

$$\frac{d}{dt} \varepsilon E(X, X^h, t) = \frac{d}{dt} \varepsilon \{M(x,t) - M^h(x,t)\}.$$

According to the systems of ordinary differential equations for M and M^h,

$$\begin{aligned}
\varepsilon \frac{d}{dt} \{M(x,t) - M^h(x,t)\} &= \varepsilon \{\nabla u(X,t) M(x,t) - \nabla \mathscr{U}_\varepsilon^h (X^h, t) M^h(x,t)\} \\
&= \varepsilon \{(\nabla u(X,t) - \nabla \mathscr{U}_\varepsilon^h(X^h,t)) M(x,t) + \nabla \mathscr{U}_\varepsilon^h(X^h,t)(M(x,t) - M^h(x,t))\} \\
&= \varepsilon \big\{ (\nabla u(X,t) - \nabla \mathscr{U}_\varepsilon^h(X^h,t)) M(x,t) \\
&\qquad + (\nabla \mathscr{U}_\varepsilon^h(X^h,t) - \nabla u(X,t))(M(x,t) - M^h(x,t)) \\
&\qquad\qquad + \nabla u(X,t)(M(x,t) - M^h(x,t)) \big\}.
\end{aligned}$$

Since we assume that the true solution is smooth,

$$\left\{\sum_{i\in I}|\nabla u(X(x_i,t),t)(M(x_i,t)-M^h(x_i,t))|^p h^2\right\}^{1/p} \le C_1\|E(t)\|_{h,p}, \tag{3.13}$$

$$\begin{aligned}&\left\{\sum_{i\in I}|(\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)-\nabla u(X(x_i,t),t))(M(x_i,t)-M^h(x_i,t))|^p h^2\right\}^{1/p}\\ &\quad\le \|E(t)\|_{h,\infty}\left\{\sum_{i\in I}|\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)-\nabla u(X(x_i,t),t)|^p h^2\right\}^{1/p}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\left\{\sum_{i\in I}|(\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)-\nabla u(X(x_i,t),t))M(x_i,t)|^p h^2\right\}^{1/p}\\ &\quad\le \|M(\cdot,t)\|_{L^\infty(\mathbf{R}^2)}\left\{\sum_{i\in I}|\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)-\nabla u(X(x_i,t),t)|^p h^2\right\}^{1/p}.\end{aligned} \tag{3.15}$$

So, for $t\in[0,T_\varepsilon^*]$, Eqs. (3.13), (3.14) and (3.15) imply that

$$\begin{aligned}&\left\|\frac{d}{dt}\varepsilon E(t)\right\|_{h,p}\\ &\quad\le C_2\left(\varepsilon\|E(t)\|_{h,p}+\varepsilon\left\{\sum_{i\in I}|\nabla u(X(x_i,t),t)-\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)|^p h^2\right\}^{1/p}\right).\end{aligned}$$

By norm equivalence,

$$\begin{aligned}&\left\{\sum_{i\in I}|\nabla u(X(x_i,t),t)-\nabla\mathscr{U}_\varepsilon^h(X^h(x_i,t),t)|^p h^2\right\}^{1/p}\\ &\quad\le C_3\left\{\left[\sum_{i\in I}\left|\frac{\partial u}{\partial x_1}(X(x_i,t),t)-\frac{\partial\mathscr{U}_\varepsilon^h}{\partial x_1}(X^h(x_i,t),t)\right|^p h^2\right]^{1/p}\right.\\ &\qquad\left.+\left[\sum_{i\in I}\left|\frac{\partial u}{\partial x_2}(X(x_i,t),t)-\frac{\partial\mathscr{U}_\varepsilon^h}{\partial x_2}(X^h(x_i,t),t)\right|^p h^2\right]^{1/p}\right\},\end{aligned}$$

$$\begin{aligned}\varepsilon\left(\frac{\partial u}{\partial x_k}(X,t)-\frac{\partial\mathscr{U}_\varepsilon^h}{\partial x_k}(X^h,t)\right)&=\varepsilon\left(K*w_{x_k}(X,t)-\frac{\partial\mathscr{U}_\varepsilon^h}{\partial x_k}(X^h,t)\right)\quad(k=1\text{ or }2)\\ &=\varepsilon\left\{[K*w_{x_k}(X,t)-K_\varepsilon*w_{x_k}(X,t)]+\left[K_\varepsilon*w_{x_k}(X,t)-\frac{\partial\mathscr{U}_\varepsilon^h}{\partial x_k}(X^h,t)\right]\right\}\\ &=\varepsilon\{\mathrm{I}_1+\mathrm{I}_2\}.\end{aligned}$$

I_1 can be bounded by using the method of Theorem 3.1, substituting $w_{x_1}(\cdot,t)$ or $w_{x_2}(\cdot,t)$ for $w(\cdot,t)$ in Theorem 3.1. Considering the second term I_2, for $k=1,2$ we have

$$
\begin{aligned}
\mathrm{I}_2 &= \varepsilon\left(K_\varepsilon * w_{x_k}(X,t) - \frac{\partial \mathscr{U}_\varepsilon^h}{\partial x_k}(X^h,t)\right) \\
&= \varepsilon\Bigg\{(\partial K_\varepsilon/\partial x_k) * w(X,t) \\
&\qquad - \sum_{i\in I}\left[c_i\frac{\partial K_\varepsilon}{\partial x_k}(X^h - X^h(x_i,t)) - \mathscr{A}_{1i}(t)\frac{\partial^2 K_\varepsilon}{\partial x_k \partial x_1}(X^h - X^h(x_i,t))\right. \\
&\qquad\qquad \left. -\mathscr{A}_{2i}(t)\frac{\partial^2 K_\varepsilon}{\partial x_k\partial x_2}(X^h - X^h(x_i,t))\right]\Bigg\} \\
&= (K * f_{x_k\varepsilon}) * w(X,t) \\
&\qquad - \sum_{i\in I}[c_i K * f_{x_k\varepsilon}(X^h - X^h(x_i,t)) \\
&\qquad\qquad - \mathscr{A}_{1i}(t)(K * f_{x_k\varepsilon})_{x_1}(X^h - X^h(x_i,t)) \\
&\qquad\qquad\qquad - \mathscr{A}_{2i}(t)(K * f_{x_k\varepsilon})_{x_2}(X^h - X^h(x_i,t))],
\end{aligned}
$$

where $f_{x_k\varepsilon}(x) = f_{x_k}(x/\varepsilon)/\varepsilon^2$. By substituting $f_{x_k\varepsilon}(\cdot)$ for $f_\varepsilon(\cdot)$ in Theorem 3.7, we obtain the following result.

THEOREM 3.8 [Corollary of Theorem 3.7]. *Assume the conditions*:

(1) (i), (ii) *and* (iii) *of Theorem* 3.1.

(2) $f(\cdot) \in W^{5,1}(\mathbf{R}^2) \cap W^{5,\infty}(\mathbf{R}^2)$. *There exist constants* C_1, C_2 *and* $\gamma > 2$ *such that*

$$
\begin{aligned}
|\partial^\alpha f(x)| &\le C_1(1+|x|)^{-\gamma}, \qquad x\in\mathbf{R}^2,\ |\alpha| = 0,1; \\
|\partial^\alpha f(x)| &\le C_2(1+|x|)^{-4}, \qquad x\in\mathbf{R}^2,\ |\alpha| = 2,3.
\end{aligned}
$$

(3) *There exists a constant* $C_3 > 0$ *such that* $h \le C_3\varepsilon^2$.

Then, for $2 < p < \infty$, *there exists a constant* $C = C(p,T,s,w_0)$ *such that*

$$
\begin{aligned}
&\left\|\frac{d}{dt}e(t)\right\|_{h,p} + \varepsilon\left\|\frac{d}{dt}E(t)\right\|_{h,p} \\
&\qquad \le C\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}} + \|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p}\right) \quad \forall t\in[0,T_\varepsilon^*],
\end{aligned}
$$

where $s > 0$ *is an arbitrary number.* □

For the main estimate, the Gronwall inequality is needed.

LEMMA 3.15 [Gronwall inequality]. *Let* $G\colon \mathbf{R}\to\mathbf{R}$ *be a smooth function. Let* $\|\cdot\|$ *be a norm on* $\mathbf{R}^n$ *and let* $\mathscr{e}$ *be a continuously differentiable* n*-vector function on* $[0,T^*]$ *such that* $\mathscr{e}(0) = 0$ *and* $\|d\mathscr{e}(t)/dt\| \le G(\|\mathscr{e}(t)\|)$. *Let* y *be the real-valued function defined by* $dy(t)/dt = G(y(t))$ *and* $y(0) = 0$. *Then for* $t\in[0,T^*]$, $\|\mathscr{e}(t)\| \le y(t)$.

Proof. See [11, Section I.6]. □

THEOREM 3.9 [The main estimate]. *Assume the following conditions*:

(1) *Conditions* (1) *and* (2) *of Theorem* 3.8 *with* $k \ge 2$.

(2) *There exist three constants* $C_3 > 0$, α *and* β *such that*

$$\alpha \geq \beta \geq 2 \quad \textit{and} \quad C_3^{-1}\varepsilon^\alpha \leq h \leq C_3\varepsilon^\beta.$$

Then we have the following results:

(a) *For* $2 < p < \infty$, *there exists a constant* $C_s = C_s(p, w_0, T)$ *such that*

$$\|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p} \leq C_s\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right), \tag{3.16}$$

$$\|u(t) - \mathscr{U}_\varepsilon^h(t)\|_{h,p} + \varepsilon\|\nabla u(t) - \nabla\mathscr{U}_\varepsilon^h(t)\|_{h,p} \leq C_s\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right). \tag{3.17}$$

(b) *There exists a constant* $C_s = C_s(p, w_0, T)$ *such that*

$$\|e(t)\|_{h,\infty} + \varepsilon\|E(t)\|_{h,\infty} \leq \frac{C_s}{\varepsilon^s}\left(\varepsilon^k + \frac{h^4}{\varepsilon^3}\right), \tag{3.18}$$

$$\begin{aligned} &\|u(t) - \mathscr{U}_\varepsilon^h(t)\|_{h,\infty} + \varepsilon\|\nabla u(t) - \nabla\mathscr{U}_\varepsilon^h(t)\|_{h,\infty} \\ &\leq \frac{C_s}{\varepsilon^s}\left(\varepsilon^k + \frac{h^4}{\varepsilon^3}\right) \quad \forall t \in [0, T], \end{aligned} \tag{3.19}$$

where $s > 0$ *is an arbitrary number.*

Proof. By Theorem 3.8, for $p \in (0, \infty)$ there exists a constant $C_{1s} = C_{1s}(p, w_0, T)$ such that for all $t \in [0, T_\varepsilon^*]$,

$$\left\|\frac{d}{dt}e(t)\right\|_{h,p} + \varepsilon\left\|\frac{d}{dt}E(t)\right\|_{h,p} \leq C_{1s}\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}} + \|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p}\right), \tag{3.20}$$

where $s > 0$ is an arbitrary number. For (3.16), let $\mathscr{e}(X, X^h, t) = (e(X, X^h, t), E(X, X^h, t))$ and define $\|\mathscr{e}(t)\| = \|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p}$. Then,

$$\|d\mathscr{e}(t)/dt\| \leq C_{1s}\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}} + \|\mathscr{e}(t)\|\right) \quad \forall t \in [0, T_\varepsilon^*].$$

Define $G\colon \mathbf{R} \to \mathbf{R}$ by setting

$$G(a) = C_{1s}\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}} + a\right) = C_{1s}(a + \varepsilon_1).$$

Then, $\|d\mathscr{e}(t)/dt\| \leq G(\|\mathscr{e}(t)\|)$ by (3.20). Solve the initial value problem

$$dy(t)/dt = G(y(t)) = C_{1s}(y(t) + \varepsilon_1), \qquad y(0) = 0$$

to get

$$y(t) = (\exp(C_{1s}t) - 1)\varepsilon_1.$$

By Lemma 3.15,

$$\|\mathscr{e}(t)\| \leq y(t) \leq (\exp(C_{1s}T) - 1)\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right) \quad \text{for all } t \in [0, T_\varepsilon^*].$$

So,

$$\|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p} \leq C_s\left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right) \quad \forall t \in [0, T_\varepsilon^*],\ 2 < p < \infty.$$

The above result implies (3.16) for $t \in [0, T_\varepsilon^*]$, and (3.17) follows from (3.16) and (3.20). For (3.18) and (3.19), consider any $p \in (2, \infty)$ and note that

$$\|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p} \geq h^{2/p}(\|e(t)\|_{h,\infty} + \varepsilon\|E(t)\|_{h,\infty}).$$

Then,

$$\begin{aligned}
&\|e(t)\|_{h,\infty} + \varepsilon\|E(t)\|_{h,\infty} \le h^{-2/p}(\|e(t)\|_{h,p} + \varepsilon\|E(t)\|_{h,p}) \\
(3.21)\qquad &\le \frac{1}{C_3^{-1}\varepsilon^{2\alpha/p}} C_s \left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right) \\
&\le \frac{1}{C_3^{-1}\varepsilon^{2\alpha/p+s}} C_s \left(\varepsilon^k + \frac{h^4}{\varepsilon^{3}}\right) = \frac{Cs'}{\varepsilon^{s'}}\left(\varepsilon^k + \frac{h^4}{\varepsilon^3}\right) \quad \forall t \in [0, T_\varepsilon^*].
\end{aligned}$$

By the same procedure we can show that

$$\|u(t) - \mathscr{U}_\varepsilon^h(t)\|_{h,\infty} + \varepsilon\|\nabla u(t) - \nabla\mathscr{U}_\varepsilon^h(t)\|_{h,\infty} \le \frac{Cs'}{\varepsilon^{s'}}\left(\varepsilon^k + \frac{h^4}{\varepsilon^3}\right) \quad \forall t \in [0, T_\varepsilon^*],$$

so that (3.18) and (3.19) are satisfied for all $t \in [0, T_\varepsilon^*]$.

The remaining problem is to prove that $T_\varepsilon^* = T$. Recall that

$$T_\varepsilon^* = \max\{t \in [0,T] \mid \|e(t)\|_{h,\infty} \le M_1\varepsilon \text{ and } \|e(t)\|_{h,\infty} \le M_2\}.$$

Since $u, \mathscr{U}_\varepsilon^h, \nabla u$ and $\nabla\mathscr{U}_\varepsilon^h$ are continuous and bounded uniformly for $(x,t) \in \Omega \times [0,T]$ and $e(X, X^h, 0) = 0$, if $T_\varepsilon^* < T$, then $\|e(T_\varepsilon^*)\|_{h,\infty} = M_1\varepsilon$ or $\|E(T_\varepsilon^*)\|_{h,\infty} = M_2$. But, by (3.21),

$$\|e(t)\|_{h,\infty} + \varepsilon\|E(t)\|_{h,\infty} \le \frac{1}{C_3^{-1}\varepsilon^{2\alpha/p}} C_s \left(\varepsilon^k + \frac{h^4}{\varepsilon^{3+s}}\right) \quad \text{with } k \ge 2.$$

Choose p so large that $k - (2\alpha/p) > 1$. Then,

$$\|e(t)\|_{h,\infty} + \varepsilon\|E(t)\|_{h,\infty} \le \max(M_1, M_2)\varepsilon^{k-(2\alpha/p)}$$

for suitable ε and h, for all $t \in [0, T_\varepsilon^*]$. This is a contradiction, and it follows that $T_\varepsilon^* = T$. □

Remark. The result of Theorem 3.9 may be compared with the analogous result, using simple δ functions. For the latter case, the error bound would be $C_s(\varepsilon^k + h^2/\varepsilon^{s+1})$. Hence, substantial improvements can be obtained by suitable choices for ε. Some numerical results for this algorithm are given in [8].

4. Conclusion. A higher-order vortex algorithm is defined for two-dimensional incompressible inviscid flows. This algorithm uses gradients of δ functions (vortex dipoles) in addition to the usual δ functions (point vortices) with appropriate smoothing. Error estimates are proved, demonstrating the higher orders of convergence on arbitrary (graded) meshes for assignment of the initial vorticity distribution.

Department of Mathematics
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213
E-mail: 1044rn0m@vb.cc.cmu.edu

1. C. ANDERSON & C. GREENGARD, "On vortex methods," *SIAM J. Numer. Anal.*, v. 22, 1985, pp. 413–440.

2. J. T. BEALE & A. MAJDA, "Vortex methods I: Convergence in three dimensions," *Math. Comp.*, v. 39, 1982, pp. 1–27.

3. J. T. BEALE & A. MAJDA, "Vortex methods II: Higher order accuracy in two and three dimensions," *Math. Comp.*, v. 39, 1982, pp. 29–52.

4. J. T. BEALE & A. MAJDA, "Rates of convergence for viscous splitting of the Navier-Stokes equations," *Math. Comp.*, v. 37, 1981, pp. 243–259.

5. G. H. COTTET, *Méthodes Particulaires pour l'équation d'Euler dans le Plan*, Thèse de 3ème cycle, Université Paris VI, 1982.

6. A. J. CHORIN, "Numerical study of slightly viscous flow," *J. Fluid Mech.*, v. 57, 1973, pp. 785–796.

7. A. J. CHORIN & J. MARSDEN, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, New York, 1979.

8. S. CHOUDHURY & R. A. NICOLAIDES, "Vortex multipole methods for viscous incompressible flows," *10th International Conf. on Numerical Methods in Fluid Dynamics* (F. G. Zhuang and Y. L. Zhu, eds.), Lecture Notes in Phys., vol. 264, Springer-Verlag, Berlin, 1986.

9. P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

10. O. H. HALD, "Convergence of vortex methods for Euler's equations. II," *SIAM J. Numer. Anal.*, v. 16, 1979, pp. 726–755.

11. J. K. HALE, *Ordinary Differential Equations*, Wiley Interscience, New York, 1969.

12. F. H. HARLOW, "The particle in cell computing method for fluid dynamics," *Methods in Computational Physics* (B. Alder, S. Fernbach & M. Rotenberg, eds.), Vol. 3, Academic Press, New York, 1964.

13. R. W. HOCKNEY & J. W. EASTWOOD, *Computer Simulation Using Particles*, McGraw-Hill, New York, 1981.

14. T. KATO, "Nonstationary flows of viscous and ideal fluids in R^3," *J. Funct. Anal.*, v. 9, 1972, pp. 296–305.

15. A. LEONARD, "Vortex methods for flow simulations," *J. Comput. Phys.*, v. 37, 1980, pp. 289–335.

16. F. J. MCGRATH, "Nonstationary plane flow of viscous and ideal fluids," *Arch. Rational Mech. Anal.*, v. 27, 1968, pp. 328–348.

17. R. A. NICOLAIDES, "Construction of higher order accurate vortex and particle methods," *Appl. Numer. Math.*, v. 2, 1986, pp. 313–320.

18. P. A. RAVIART, "An analysis of particle methods," *Numerical Methods in Fluid Dynamics*, Como, July, 1983.

19. R.TEMAM, "Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids," *Turbulence and Navier-Stokes Equations* (R. Temam, ed.), Lecture Notes in Math., vol. 565, Springer-Verlag, Berlin, 1976.