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MONOTONICITY OF SOME FUNCTIONS INVOLVING THE GAMMA AND PSI FUNCTIONS

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ABSTRACT. Let $L(x) := x - \frac{\Gamma(x+t)}{\Gamma(x+s)} x^{s-t+1}$, where $\Gamma(x)$ is Euler's gamma function. We determine conditions for the numbers s, t so that the function $\Phi(x) := -\frac{\Gamma(x+s)}{\Gamma(x+t)} x^{t-s-1} L''(x)$ is strongly completely monotonic on $(0, \infty)$. Through this result we obtain some inequalities involving the ratio of gamma functions and provide some applications in the context of trigonometric sum estimation. We also give several other examples of strongly completely monotonic functions defined in terms of Γ and $\psi := \Gamma'/\Gamma$ functions. Some limiting and particular cases are also considered.

1. Introduction and results

Let $\Gamma(x)$ be Euler's gamma function defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ and let $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ be its logarithmic derivative, which is also known as the psi or digamma function. The derivatives $\psi^{(n)}(x)$ are called polygamma functions. Over the years, these functions have been the subject of intensive study by many researchers in view of their importance in applications in various fields. In particular, there is an extensive bibliography on inequalities involving these functions. See for example [2], [4], [11], [12], [14], [15], [16], [19] [25] and the references given therein. Most of the inequalities for the gamma and polygamma functions are obtained through monotonicity or convexity properties of functions which are expressed in terms of them. Several functions of this type have been shown to be completely monotonic. We recall that a function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, \quad \text{for all } x > 0 \text{ and } n = 0, 1, 2, \ldots \tag{1.1}$$

J. Dubourdieu [13] proved that if a nonconstant function f is completely monotonic, then strict inequality holds in (1.1). See also [18] for a simpler proof of this result.

A characterization of completely monotonic functions is given by Bernstein's theorem (see [27, p. 161]) which states that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. Completely monotonic functions attracted the attention of many authors

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and a plethora of results on completely monotonic functions have been obtained, most of them concerning functions defined in terms of the gamma and polygamma functions. We refer the reader to [5], [3], [17] [18], [20], [23] for some interesting results, applications, bibliography and helpful information regarding such functions.

Here we are interested in the class of strongly completely monotonic functions, introduced in [26]. A function $g : (0, \infty) \to \mathbb{R}$ is called strongly completely monotonic if it satisfies the more restrictive condition $(-1)^n x^{n+1} g^{(n)}(x)$ is nonnegative and decreasing on $(0, \infty)$ for all $n = 0, 1, 2, \ldots$. A characterization of strongly completely monotonic functions is the following:

The function $g(x)$ is strongly completely monotonic on $(0, \infty)$ if and only if

$$g(x) = \int_0^\infty e^{-xt}\, p(t)\, dt, \tag{1.2}$$

where $p(t)$ is nonnegative and increasing and the integral converges for all x in $(0, \infty)$; see [26, Theorem 1]. Note that if a function $g(x)$ is strongly completely monotonic, then the function $x\, g(x)$ is completely monotonic. The strongly completely monotonic functions considered in this paper satisfy (1.2) with the additional property that the function $p(t)$ is convex and $p(0) = 0$. This implies that $x^2\, g(x)$ is completely monotonic on $(0, \infty)$ (see Lemma 2 in Section 2) a fact that plays a key role in the present work.

The purpose of this paper is to contribute some more inequalities for ratios of gamma functions, differences of digamma and polygamma functions, obtained via monotonicity of certain special functions, and indicate some applications of the derived inequalities in the estimation of certain trigonometric sums. It turns out that several of the functions involved are additionally strongly completely monotonic in the sense described above.

Let s, t be positive real numbers such that $s - t \neq 1$. The main result we prove here is the following.

Theorem 1. *Let*

$$L(x) := x - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1},$$

and suppose that either

(i) $s \geq 1 > t$ *and* $0 < s - t \leq \frac{1}{2}$, *or*
(ii) $t \geq 1$ *and* $0 < s - t < 1$, *or*
(iii) $\frac{\sqrt{3}}{6} + \frac{1}{2} \leq t < s < 1$.

Then

(1) *the function*

$$\Phi(x) := -\frac{\Gamma(x+s)}{\Gamma(x+t)}\, x^{t-s-1}\, L''(x)$$

is strongly completely monotonic on $(0, \infty)$.

(2) *The function* $-L''(x)$ *is completely monotonic on* $(0, \infty)$. *In particular, the function* $L(x)$ *is strictly increasing and concave on* $(0, \infty)$, *and the inequality*

$$0 < x - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1} < \frac{(s-t)(s+t-1)}{2}, \tag{1.3}$$

holds for all $x > 0$.

As an application of inequality (1.3) we give the following proposition which was the starting point of our investigations considered here. It provides a substantial generalization of [21, Lemma 3] and [22, Proposition 4] which are indispensable in the estimation of certain trigonometric sums arising in the context of starlike functions.

Proposition 1. *For $m, n \in \mathbb{N}$ with $m > n$, let*

$$U_{n,m}(x) := \sum_{k=n}^{m} \frac{(t)_k}{(s)_k}\, e^{ikx}, \quad V_{n,m}(x) = \frac{\Gamma(s)}{\Gamma(t)} \sum_{k=n}^{m} \frac{1}{k^{s-t}}\, e^{ikx},$$

where $(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol.

If the numbers s, t satisfy (i) or (ii) of Theorem 1, then for $\pi/n \le x < \pi$, $n > 1$ we have the estimate

$$|U_{n,m}(x) - V_{n,m}(x)| < \frac{1}{n^{s-t}} \frac{\Gamma(s)}{\Gamma(t)} \frac{(s-t)(s+t-1)}{2}\,. \tag{1.4}$$

The observation

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim e^{(t-s)\psi(x)}, \text{ as } x \to \infty$$

led us to the following limiting case of Theorem 1.

Theorem 2. *Consider the function*

$$M(x) := x - e^{a\psi(x)}\, x^{1-a}, \quad a \neq 0.$$

Then (1) *the function*

$$\xi(x) := -x^{a-1}\, e^{-a\,\psi(x)}\, M''(x)$$

is strongly completely monotonic on $(0, \infty)$ if and only if $a \in (-\infty, 0) \cup \big[2/3, \infty\big)$. When $0 < a < 2/3$ the function $\xi(x)$ changes sign on $(0, \infty)$.

(2) *The function $M(x)$ is strictly increasing and concave on $(0, \infty)$ precisely when $a \in (-\infty, 0) \cup \big[2/3, \infty\big)$, so that*

$$x - e^{a\psi(x)}\, x^{1-a} < \frac{a}{2}, \quad \textit{for all} \quad x > 0. \tag{1.5}$$

Note that $\lim_{x \to 0^+} M(x) = 0$, *for $a > 0$ and* $\lim_{x \to 0^+} M(x) = -\infty$, *for $a < 0$.*

(3) *If $a < 0$, then the function $-M''(x)$ is completely monotonic on $(0, \infty)$.*

Several existing results on digamma and polygamma functions are immediate consequences or special cases of this theorem. These are described in the final section of the paper. In the next section we prove two necessary lemmas. In Section 3, we give proofs of Theorem 1, Theorem 2 and Proposition 1. In Section 3 we also prove some closely related results, give some corollaries and make some additional comments on relevant considerations.

2. Some lemmas

In order to prove our main results we need the following:

Lemma 1. (1) *Let*

$$f(x) := \frac{e^{\alpha\, x} - 1}{e^x - 1}.$$

This function is strictly decreasing and convex on $(0, \infty)$ for $0 < \alpha \le 1/2$, and strictly increasing and concave on the same interval for $-1 < \alpha < 0$.

(2) *The function*

$$g(x) := \frac{e^{\alpha\, x} - e^{\beta\, x}}{e^x - 1}$$

is strictly decreasing and convex on $(0, \infty)$ *for* (i) $\beta \le 0 < \alpha,\ 0 < \alpha - \beta \le \frac{1}{2}$, (ii) $\alpha \le 0,\ 0 < \alpha - \beta < 1$ *and* (iii) $0 < \beta < \alpha \le \frac{1}{2} - \frac{\sqrt{3}}{6}$.

(3) *The function* $g(x)$ *is log-concave on* $(-\infty, \infty)$ *when* $0 < \alpha - \beta < 1$ *and log-convex on* $(-\infty, \infty)$ *when* $\alpha - \beta > 1$.

Proof. (1) Differentiating twice we get

$$f''(x) = \frac{1}{(e^x-1)^3}\left[(\alpha-1)^2\, e^{(\alpha+2)x} + (-2\alpha^2+2\alpha+1)e^{(\alpha+1)x} + \alpha^2 e^{\alpha x} - e^{2x} - e^x\right].$$

It follows from this that

$$f''(x) = \left(\frac{x}{e^x-1}\right)^3 \sum_{n=3}^{\infty} P_n(\alpha)\,\frac{x^{n-3}}{n!}\,, \tag{2.1}$$

where

$$P_n(\alpha) := (\alpha-1)^2\,(\alpha+2)^n + (-2\alpha^2+2\alpha+1)\,(\alpha+1)^n + \alpha^{n+2} - 2^n - 1,\quad n = 3, 4, \ldots.$$

It is easy to see that

$$P_3(\alpha) = \alpha\,(\alpha-1)\,(2\alpha-1),$$

hence

$$P_3(\alpha) \ge 0 \ \text{ for } \ 0 < \alpha \le \frac{1}{2},\ \ P_3(\alpha) < 0, \text{ for } \ -1 < \alpha < 0. \tag{2.2}$$

Next we shall prove that

$$P_n(\alpha) > 0, \ \text{ for all } \ n \ge 4, \quad 0 < \alpha \le \frac{1}{2}, \tag{2.3}$$

and

$$P_n(\alpha) < 0, \ \text{ for all } \ n \ge 4, \quad -1 < \alpha < 0. \tag{2.4}$$

First we assume that $0 < \alpha \le 1/2$. Clearly,

$$P_n(\alpha) > Q_n(\alpha), \tag{2.5}$$

where

$$Q_n(\alpha) := (\alpha-1)^2\,(\alpha+2)^n - 2^n\,.$$

We shall show that

$$Q_n(\alpha) > 0, \ \text{ for all } \ n \ge 7, \quad 0 < \alpha \le \frac{1}{2}. \tag{2.6}$$

We have

$$Q_n'(\alpha) = (\alpha+2)^{n-1}\,(\alpha-1)\,\Big((n+2)\alpha - n + 4\Big). \tag{2.7}$$

It follows from this that $Q_n'(\alpha) > 0$ for $n \ge 10$. Since $Q_n(0) = 0$ we infer that (2.6) is valid for all $n \ge 10$. For $n = 7, 8, 9$ we see from (2.7) that $Q_n'(\alpha)$ has a unique root on $(0, 1/2)$. Then by an elementary argument we deduce that $Q_n(\alpha) > 0$ for $0 < \alpha < 1/2$.

The positivity of the polynomials $P_n(\alpha)$ on $0 < \alpha < 1/2$, for $n = 4, 5, 6$ can be checked by a direct computation. Note that

$$P_4(\alpha) = 6\alpha\,(\alpha-1)\,(\alpha^2+\alpha-1),$$

$$P_5(\alpha) = \alpha\,(\alpha-1)\,(12\alpha^3 + 27\alpha^2 + 7\alpha - 23),$$

$$P_6(\alpha) = \alpha\,(\alpha-1)\,(20\alpha^4 + 68\alpha^3 + 73\alpha^2 - 17\alpha - 72).$$

The proof of (2.3) is complete. In a similar way we prove (2.4). We check directly that $P_4(\alpha) < 0$, for $-1 < \alpha < 0$. Suppose that $n \geq 5$, $Q_n(\alpha)$ is as above and $-1 < \alpha < 0$. It is clear that

$$P_n(\alpha) < Q_n(\alpha) + (\alpha+1)^n - 1, \quad \text{when } n \text{ is odd} \tag{2.8}$$

and

$$P_n(\alpha) < Q_n(\alpha) + (\alpha+1)^n - 1 + \alpha^2, \quad \text{when } n \text{ is even.} \tag{2.9}$$

It follows from (2.7) that $Q_n(\alpha)$ is strictly increasing on $(-1,\, 0)$. Hence the function $T_n(\alpha) := Q_n(\alpha) + (\alpha+1)^n - 1$ is strictly increasing on the same interval so that $T_n(\alpha) < T_n(0) = 0$. This in combination with (2.8) establishes (2.4) for all odd $n \geq 5$. Suppose that n is even and $n \geq 6$. Using again (2.7) we see that

$$Q_n'(\alpha) + 2\alpha > (\alpha+2)^{n-1}(n-4) + 2\alpha > n - 6 \geq 0\,.$$

Therefore the function $U_n(\alpha) := Q_n(\alpha) + (\alpha+1)^n - 1 + \alpha^2$ is strictly increasing on $(-1, 0)$ and so $U_n(\alpha) < U_n(0) = 0$. We then use (2.9) to complete the proof of (2.4).

Taking into consideration (2.2), (2.3) and (2.4), we conclude from (2.1) that the function $f(x)$ is convex on $(0,\, \infty)$ when $0 < \alpha \leq 1/2$, and concave on the same interval when $-1 < \alpha < 0$. We also obtain from

$$f'(x) = \frac{\alpha e^{\alpha x}}{e^x - 1} - \frac{e^x(e^{\alpha x} - 1)}{(e^x - 1)^2}$$

that

$$\lim_{x\to\infty} f'(x) = 0$$

when $-1 < \alpha < 0$ or $0 < \alpha \leq 1/2$.

(2) In the case (i) write

$$g(x) = e^{\beta x}\,\frac{e^{(\alpha-\beta)x} - 1}{e^x - 1}$$

and use part (1) of the lemma to see that $g(x)$ is strictly decreasing and convex on $(0,\, \infty)$. Similarly, in the case (ii) we write

$$g(x) = e^{\alpha x}\,\frac{1 - e^{(\beta-\alpha)x}}{e^x - 1}$$

and use again part (1) of the lemma to deduce the desired result. In the case (iii), we write

$$g(x) = \frac{e^{\alpha\, x} - 1}{e^x - 1} - \frac{e^{\beta\, x} - 1}{e^x - 1},$$

then use (2.1) to see that

$$g''(x) = \left(\frac{x}{e^x - 1}\right)^3 \sum_{n=3}^{\infty} [P_n(\alpha) - P_n(\beta)]\,\frac{x^{n-3}}{n!}\,.$$

Now $g''(x) > 0$ for $x > 0$, follows from the fact that the polynomials $P_n(\alpha)$ are strictly increasing functions of α on $(0,\, \frac{1}{2} - \frac{\sqrt{3}}{6})$, for $n = 3, 4, \ldots$. Indeed, for $n = 3, 4, 5$ inequality $P_n'(\alpha) > 0$ can be checked by a direct calculation, the case $n = 3$ being sharp. For $n \geq 6$ we set $r_n(\alpha) := (\alpha-1)^2(\alpha+2)^n$ and observe that $P_n(\alpha) - r_n(\alpha)$ obviously increases for this range of α. Then we use (2.7) to conclude that $Q_n'(\alpha) = r_n'(\alpha) > 0$, for $n \geq 6$ and $\alpha \in (0,\, \frac{1}{2} - \frac{\sqrt{3}}{6})$. Since in this case we have $0 < \beta < \alpha < 1$ an elementary calculation shows that $\lim_{x\to\infty} g'(x) = 0$ and completes the proof of part (2) of the lemma.

(3) This follows easily by observing that

$$g''(x)\, g(x) - g'(x)^2 = -(\alpha-\beta)^2 \sinh^2 \frac{x}{2}\, \frac{4\, e^{(\alpha+\beta+1)x}}{(e^x-1)^4} \left[1 - \Big(\frac{\sinh(\alpha-\beta)\,\frac{x}{2}}{(\alpha-\beta)\,\sinh\frac{x}{2}}\Big)^2\right].$$

A different proof of part (3) of this lemma can be found in [23].

The proof of Lemma 1 is complete. □

We also need the following:

Lemma 2. *Let*

$$h(x) := \int_0^\infty e^{-xt}\, p(t)\, dt.$$

If the function $p(t)$ satisfies the conditions

$$p(0) = 0,\;\; p(t) > 0,\;\; p'(t) > 0,\;\; p''(t) > 0,\;\; \textit{for all}\;\; t > 0, \tag{2.10}$$

then the function $x^2\, h(x)$ is completely monotonic on $(0,\infty)$.

Proof. We write

$$x^2\, h(x) = \int_0^\infty e^{-t}\, x\, p\Big(\frac{t}{x}\Big)\, dt. \tag{2.11}$$

In order to prove that the left hand side of (2.11) is a decreasing function of x in $(0,\infty)$ it is sufficient to show that the function $q(u) := \frac{p(u)}{u}$ is strictly increasing on $(0,\infty)$. Set $r(u) := u\, p'(u) - p(u)$. Since $r'(u) = u\, p''(u) > 0$, we deduce that $r(u)$ is strictly increasing from $r(0) = 0$. This implies $q'(u) > 0$ for all $u > 0$ as desired. Therefore,

$$\frac{d}{dx}\big(x^2\, h(x)\big) < 0, \quad \text{for all} \quad x \in (0,\infty).$$

For $n \geq 2$ applying Leibniz' rule we obtain

$$\big(x^2\, h(x)\big)^{(n)} = \frac{1}{x^{n-2}}\, \big(x^n\, h^{(n-2)}(x)\big)''. \tag{2.12}$$

We also have

$$(-1)^n\, x^n\, h^{(n-2)}(x) = \int_0^\infty t^{n-2}\, e^{-t}\, x\, p\Big(\frac{t}{x}\Big)\, dt. \tag{2.13}$$

Combining (2.12) with (2.13) we see that the inequality

$$(-1)^n\, \big(x^2\, h(x)\big)^{(n)} > 0, \quad \text{for all} \quad x \in (0,\infty), \tag{2.14}$$

follows from

$$\frac{d^2}{dx^2}\Big(x\, p\Big(\frac{t}{x}\Big)\Big) = \frac{t^2}{x^3} p''\Big(\frac{t}{x}\Big) > 0, \quad \text{for all} \quad x \in (0,\infty),$$

which is clearly true. The proof of Lemma 2 is complete. □

Remark. The condition $p(0) = 0$ is necessary for the validity of Lemma 2 as the simple example $p(t) = (t+1)^2$ shows. The inequality (2.14), however, holds true for all $n \geq 2$ without the assumption $p(0) = 0$.

3. Proofs of the main results

We first give a proof of Theorem 1.

Proof. Differentiating we get

$$L'(x) = 1 - \frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t} \Big(x \big(\psi(x+t) - \psi(x+s) \big) + s - t + 1 \Big)$$

and

$$\begin{aligned} (3.1) \quad L''(x) \quad &= \quad -\frac{\Gamma(x+t)}{\Gamma(x+s)}\, x^{s-t+1} \\ &\times \quad \Big\{ \big(\psi(x+t) - \psi(x+s) \big)^2 + \psi'(x+t) - \psi'(x+s) \\ &+ \quad 2\, \frac{s-t+1}{x} \Big(\psi(x+t) - \psi(x+s) \Big) + \frac{(s-t)\,(s-t+1)}{x^2} \Big\}. \end{aligned}$$

It follows from this that

$$(3.2) \quad \Phi(x) = \Big(\psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} \Big)^2 + \Big(\psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} \Big)'.$$

Using the well-known formulae

(3.3)

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-u} - e^{-xu}}{1 - e^{-u}}\, du, \quad \text{and} \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xu} \frac{u^n}{1 - e^{-u}}\, du,$$

where $x > 0$ and γ is Euler's constant and $n = 1, 2, \ldots$, (cf. [1, pp. 259-260] or [8, p. 26]), we get

$$(3.4) \qquad \psi(x+t) - \psi(x+s) = -\int_0^\infty e^{-xu}\ \phi(u)\, du$$

and

$$\psi'(x+t) - \psi'(x+s) = \int_0^\infty e^{-xu}\ u\, \phi(u)\, du,$$

where

$$\phi(u) := \frac{e^{(1-t)u} - e^{(1-s)u}}{e^u - 1}.$$

Defining

$$\phi(0) = s - t, \quad \sigma(u) := \phi(0) - \phi(u) + 1,$$

and using the above we see that

$$\psi(x+t) - \psi(x+s) + \frac{s-t+1}{x} = \int_0^\infty e^{-xu}\, \sigma(u)\, du,$$

so that the application of convolution theorem for the Laplace transform in (3.2) yields

$$(3.5) \qquad \Phi(x) = \int_0^\infty e^{-xu}\, F(u)\, du,$$

where

$$F(u) := \int_0^u \sigma(u-v)\, \sigma(v)\, dv - u\, \sigma(u)\,.$$

It follows easily that

$$F'(u) = \int_0^u \sigma'(u-v)\,\sigma(v)\,dv - u\,\sigma'(u)\,,$$

and

$$\begin{aligned} F''(u) &= \int_0^u \sigma''(u-v)\,\sigma(v)\,dv + \sigma'(0)\sigma(u) - \sigma'(u) - u\,\sigma''(u) \\ &= u\,\phi''(u) + \int_0^u \phi''(u-v)\,\phi(v)\,dv + \phi(0)\phi(u)\Big(\frac{\phi'(0)}{\phi(0)} - \frac{\phi'(u)}{\phi(u)}\Big) \\ &= u\,\phi''(u) + \int_0^u \phi'(u-v)\,\phi'(v)\,dv\,. \end{aligned} \tag{3.6}$$

It follows from Lemma 1 that when the numbers s, t satisfy one of the conditions (i)–(iii), the function $\phi(u)$ defined above is positive, strictly decreasing, convex and log-concave on $(0, \infty)$. Hence we infer from the second (or third) relation of (3.6) that $F''(u) > 0$ for all $u > 0$. Since $F'(0) = F(0) = 0$ we deduce that $F'(u) > 0$ and $F(u) > 0$ for all $u > 0$. In view of Theorem 1 of [26] and relation (3.5) we conclude that the function $\Phi(x)$ is strongly completely monotonic on $(0, \infty)$. In addition, it follows from Lemma 2 that the function $x^2\,\Phi(x)$ is completely monotonic on $(0, \infty)$. From (3.1) we have that

$$-L''(x) = x^{s-t-1}\,\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^2\Phi(x)\,.$$

It is also well known that for the ratio of two gamma functions we have the representation (see for example [8, p. 615])

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} = \frac{1}{\Gamma(s-t)}\int_0^\infty e^{-xu}\,e^{-tu}\,(1-e^{-t})^{s-t-1}\,du\,.$$

Since in our considerations here we have $0 < s-t < 1$ we conclude that the function $-L''(x)$ is completely monotonic on $(0, \infty)$ as a product of completely monotonic functions.

Finally, using the asymptotic formulae

$$\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^{s-t} = 1 - \frac{(s-t)(s+t-1)}{2x} + O\Big(\frac{1}{x^2}\Big), \quad x \to \infty \tag{3.7}$$

and

$$\psi(x+t) - \psi(x+s) = -\frac{s-t}{x} + \frac{(s-t)\,(s+t-1)}{2\,x^2} + O\Big(\frac{1}{x^3}\Big), \quad x \to \infty, \tag{3.8}$$

(see [1] or [8]), we obtain

$$\lim_{x\to\infty} L'(x) = 0\,.$$

Therefore, the function $L(x)$ is strictly increasing and concave on $(0, \infty)$. Formula (3.7) also gives

$$\lim_{x\to\infty} L(x) = \frac{(s-t)(s+t-1)}{2},$$

which in combination with the above entails inequality (1.3) and completes the proof of Theorem 1. $\square$

From Theorem 1 we derive the following:

Corollary 1. *If* s, t *satisfy one of the conditions* (i)–(iii) *of Theorem* 1*, then the inequality*

$$\psi(x+t)-\psi(x+s)+\frac{s-t+1}{x}<\frac{\Gamma(x+s)}{\Gamma(x+t)}\,x^{t-s-1}, \tag{3.9}$$

holds for all $x>0$ *and the function*

$$\psi(x+s)-\psi(x+t)-\frac{s-t}{x}+\frac{(s-t)(s+t-1)}{2x^2}$$

is completely monotonic on $(0,\infty)$.

Proof. Inequality (3.9) is an immediate consequence of the fact that for the function $L(x)$ defined above we have $L'(x)>0$ for all $x>0$. For the second assertion we observe that $\phi(0)=s-t$ and $\phi'(0)=-\frac{1}{2}(s-t)(s+t-1)$. Since the function $\phi(u)$ is convex on $(0,\infty)$ we have $\phi(u)-\phi(0)-\phi'(0)\,u>0$. Taking Laplace transformation and using (3.4) we complete the proof. □

Using the convexity of the function ϕ defined above we are able to prove the following:

Proposition 2. *Let*

$$K(x):=\psi'(x+t)-\psi'(x+s)+\frac{2}{x}\left[\psi(x+t)-\psi(x+s)\right]+\frac{s-t}{x^2}$$

and

$$\Lambda(x):=x\,\log\Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^{s-t}\Big).$$

If the numbers s, t *satisfy one of the conditions* (i)–(iii) *of Theorem 1, then:*

(1) *the function* $K(x)$ *is strongly completely monotonic on* $(0,\infty)$*. In particular, the functions* $x^2\,K(x)$ *and* $\Lambda''(x)$ *are completely monotonic on* $(0,\infty)$.

(2) *The function* $\Lambda(x)$ *is strictly decreasing and convex on* $(0,\infty)$*, so that*

$$-\frac{(s-t)(s+t-1)}{2}<x\,\log\Big(\frac{\Gamma(x+t)}{\Gamma(x+s)}\,x^{s-t}\Big)<0,\quad \textit{for all}\quad x>0. \tag{3.10}$$

Proof. It is easy to see that

$$K(x)=\int_0^\infty e^{-xu}\,\rho(u)\,du,$$

where

$$\rho(u):=u\,\phi(u)-2\int_0^u\phi(v)\,dv+(s-t)\,u$$

where $\phi(u)$ as in the proof of Theorem 1. Then we observe that

$$\rho(0)=0,\ \ \rho'(0)=0,\ \ \text{and}\ \rho''(u)=u\,\phi''(u).$$

When the numbers s,t satisfy one of the conditions (i)–(iii) of Theorem 1, by Lemma 1 we deduce the function $\phi(u)$ is convex on $(0,\infty)$. Applying Lemma 2 we conclude that the function $x^2\,K(x)$ is completely monotonic on $(0,\infty)$. Hence the function $x\,K(x)=\Lambda''(x)$ is also completely monotonic on $(0,\infty)$. Using the asymptotic formula

$$\log\Gamma(x)=\Big(x-\frac{1}{2}\Big)\log x-x+\frac{1}{2}\log(2\pi)+\frac{1}{12\,x}+O\big(\frac{1}{x^3}\big),\ \ x\to\infty$$

(cf. [1, p. 257] or [8, p. 20]), we obtain $\lim_{x\to\infty}\Lambda'(x)=0$ and also inequality (3.10). □

We now give a proof of Proposition 1.

Proof. We observe the assumptions $s \geq 1$ and $0 < s - t < 1$ imply that

$$\Big(\frac{n}{n+1}\Big)^{s-t} < 1 + \frac{t-s}{n+1} \leq \frac{n+t}{n+s},$$

which is to say that the sequence $x_n := n^{s-t}\,\dfrac{(t)_n}{(s)_n}$ is strictly increasing. Also, we clearly have $\lim_{n\to\infty} x_n = \Gamma(s)/\Gamma(t)$. Hence the sequence

$$\Delta_n = \frac{1}{n^{s-t}}\Big[\frac{\Gamma(s)}{\Gamma(t)} - n^{s-t}\,\frac{(t)_n}{(s)_n}\Big] = \frac{1}{n^{s-t}}\,\frac{\Gamma(s)}{\Gamma(t)}\Big[1 - \frac{\Gamma(n+t)}{\Gamma(n+s)}\,n^{s-t}\Big],$$

is positive and strictly decreasing. Obviously $\lim_{n\to\infty} \Delta_n = 0$. Then a summation by parts yields

$$|U_{n,m}(x) - V_{n,m}(x)| = \Big|\sum_{k=n}^{m} \Delta_k\, e^{ikx}\Big| \leq \frac{\Delta_n}{\sin\frac{x}{2}}\,. \tag{3.11}$$

It is easy to see that when $\pi/n \leq x < \pi$ we have $\sin\frac{x}{2} \geq 1/n$ for $n > 1$. Now, from (3.11) and (1.3) we conclude that

$$|U_{n,m}(x) - V_{n,m}(x)| \leq n\,\Delta_n < \frac{1}{n^{s-t}}\,\frac{\Gamma(s)}{\Gamma(t)}\,\frac{(s-t)(s+t-1)}{2}\,,$$

which establishes (1.4). $\square$

Next we give a proof of Theorem 2.

Proof. We have

$$M'(x) = 1 - x^{-a}\,e^{a\,\psi(x)}\,(a\,x\,\psi'(x) + 1 - a)$$

and

$$\begin{aligned} M''(x) &= -\,e^{a\,\psi(x)}\,x^{1-a} \\ &\times \Big\{a^2\,[\psi'(x)]^2 + a\,\psi''(x) + 2\,\frac{a\,(1-a)}{x}\,\psi'(x) - \frac{a\,(1-a)}{x^2}\Big\}, \end{aligned}$$

so that

$$\xi(x) = a^2\,[\psi'(x)]^2 + a\,\psi''(x) + 2\,\frac{a\,(1-a)}{x}\,\psi'(x) - \frac{a\,(1-a)}{x^2}\,.$$

Using the second relation of (3.3) and the convolution theorem for the Laplace transform, we easily find that

$$\xi(x) = \int_0^\infty e^{-xu}\,G(u)\,du, \tag{3.12}$$

where

$$G(u) := a^2\int_0^u \delta(u-v)\,\delta(v)\,dv + 2a(1-a)\int_0^u \delta(v)\,dv - a\,u\,\delta(u) - a(1-a)u$$

and

$$\delta(u) := \frac{u\,e^u}{e^u - 1}, \quad \delta(0) := 1.$$

It follows from the above that

$$G'(u) = a^2\int_0^u \delta'(u-v)\,\delta(v)\,dv - a\,u\,\delta'(u) + a(1-a)\big(\delta(u) - 1\big)$$

and

$$G''(u) = a^2 \Big(\int_0^u \delta''(u-v)\,\delta(v)\,dv + \delta'(0)\,\delta(u) - \delta'(u)\,\delta(0) \Big) - a\,u\,\delta''(u) \tag{3.13}$$
$$= a^2 \Big(\int_0^u \delta'(u-v)\,\delta'(v)\,dv \Big) - a\,u\,\delta''(u).$$

In an elementary way we can verify that

$$\delta'(u) = \frac{e^u\,(e^u - 1 - u)}{(e^u - 1)^2} > 0, \quad \delta''(u) = \frac{e^u\,(-2e^u + 2 + ue^u + u)}{(e^u - 1)^3} > 0,$$
$$\delta'''(u) = -\frac{e^u\,(-3e^{2u} + 3 + ue^{2u} + 4ue^u + u)}{(e^u - 1)^4} < 0, \quad \text{for all } u \geq 0.$$

We also have

$$\delta'(0) = \frac{1}{2}, \quad \delta''(0) = \frac{1}{6}, \quad \lim_{u\to\infty} \delta''(u) = 0$$

and

$$\big(\log \delta(u)\big)'' = \frac{e^u\,(u^2+2) - e^{2u} - 1}{u^2\,(e^u - 1)^2} < 0, \quad \text{for all } u \geq 0.$$

Hence the function $\delta(u)$ is strictly increasing, convex and log-concave on $[0, \infty)$. Thus, when $a < 0$, (3.13) immediately implies that $G''(u) > 0$ for $u > 0$. In the case where $a \geq 2/3$, in order to prove that $G''(u) > 0$ for $u > 0$, we observe that this is a consequence of the inequality

$$\frac{2}{3} \Big(\int_0^u \delta'(u-v)\,\delta'(v)\,dv \Big) - u\,\delta''(u) > 0, \quad \text{for all } u > 0 \tag{3.14}$$

which, in turn, follows easily by $\delta'(u) \geq 1/2$ and $\delta''(u) \leq 1/6$.

When $0 < a < 2/3$, the function $G''(u)$ assumes negative values sufficiently close to zero, because

$$\lim_{u\to 0^+} \frac{G''(u)}{u} = \frac{a(3a-2)}{12} < 0.$$

Therefore, we have shown that $G''(u) > 0$ for all $u > 0$ precisely when $a \in (-\infty, 0) \cup \big[2/3, \infty\big)$. Since $G'(0) = 0$ and $G(0) = 0$ we infer that $G'(u) > 0$ and $G(u) > 0$ for all $u > 0$. From (3.12) we deduce that the function $\xi(x)$ is strongly completely monotonic on $(0, \infty)$ and by Lemma 2 we conclude that the function $x^2\,\xi(x)$ is completely monotonic on $(0, \infty)$ for this range of a.

It follows from the above that the function $M(x)$ is strictly concave on $(0, \infty)$ when $a \in (-\infty, 0) \cup [2/3, \infty)$. Using the well-known asymptotic formulae

$$\psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + O\Big(\frac{1}{x^4}\Big), \quad x \to \infty \tag{3.15}$$

and

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + O\Big(\frac{1}{x^5}\Big), \quad x \to \infty, \tag{3.16}$$

(cf. [1, pp. 259–260]), we find that

$$\lim_{x\to\infty} M'(x) = 0\,,$$

hence $M(x)$ is strictly increasing for the same range of a. Again, by (3.15) we get

$$\lim_{x\to\infty} M(x) = \frac{a}{2}\,,$$

so that (1.5) is valid for all $x > 0$ when $a \in (-\infty, 0) \cup [2/3, \infty)$.

By the asymptotic formulae (3.15) and (3.16) together with

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + O\Big(\frac{1}{x^6}\Big), \quad x \to \infty$$

we also obtain

$$\lim_{x\to\infty} x^2\, M'(x) = \frac{a}{8}\left(a - \frac{2}{3}\right)$$

and

$$\lim_{x\to\infty} x^4\, \xi(x) = \frac{a}{4}\left(a - \frac{2}{3}\right).$$

On the other hand, using the functional equations

$$\psi^{(n)}(x) = \psi^{(n)}(x+1) + (-1)^{n+1}\,\frac{n!}{x^{n+1}}, \quad x > 0, \;\; n = 0, 1, 2, \ldots \tag{3.17}$$

we find that for $a > 0$,

$$\lim_{x\to 0^+} M'(x) = 1\,,$$

while

$$\lim_{x\to 0^+} x^4\, \xi(x) = a^2\,.$$

It follows from the above that the functions $M'(x)$, $M''(x)$ and $\xi(x)$ change sign on $(0, \infty)$ when $0 < a < \frac{2}{3}$ and therefore for such a, (1.5) fails to hold for appropriate $x > 0$.

Finally, when $a < 0$ we write

$$-M''(x) = \lambda(x)\, x^2 \xi(x), \tag{3.18}$$

where

$$\lambda(x) := \frac{e^{a\,\psi(x)}}{x^{a+1}}.$$

Then we observe that

$$(-\log \lambda(x))' = \frac{a+1}{x} - a\,\psi'(x) = \int_0^\infty e^{-xu}\,\big[1 + a\,(1 - \delta(u))\big]\, du.$$

Since $1 + a\,(1 - \delta(u)) > 0$ for $a < 0$, it follows that the function $(-\log \lambda(x))'$ is completely monotonic on $(0, \infty)$ and this implies that $\lambda(x)$ is also completely monotonic on $(0, \infty)$ (cf. [5, Lemma 2.4]). Therefore, when $a < 0$ by (3.18) we conclude that the function $-M''(x)$ is completely monotonic on $(0, \infty)$ as a product of completely monotonic functions.

The proof of Theorem 2 is complete. $\square$

Remark. The fact that the function $e^{\psi(x+t)} - x$ is decreasing on $(0, \infty)$ for all $t > 0$ has been established in [14]. By this the authors derived the estimate

$$\psi'(x) < \exp\{-\psi(x)\}, \quad x > 0. \tag{3.19}$$

See [4] and [11] for some extensions of (3.19) for higher order derivatives of $\psi(x)$ and also [10] for a different proof of (3.19).

Here we observe that Theorem 2 enables us to obtain the following extension of (3.19).

Corollary 2. *The inequality*

$$x^{1-a}\left\{a\,\psi'(x)+\frac{1-a}{x}\right\}<\exp\{-a\,\psi(x)\},$$

holds for all $x>0$ *if and only if* $a\in(-\infty,\,0)\cup[2/3,\,\infty)$.

Proof. It follows immediately from the fact that $M'(x)>0$ for all $x>0$ holds if and only if $a\in(-\infty,\,0)\cup[2/3,\,\infty)$. □

By Theorem 2 we can also derive the following:

Corollary 3. *The functions*

$$\begin{aligned} f_1(x)&:=[\psi'(x)]^2+\psi''(x),\\ f_2(x)&:=\frac{2}{3}\left(\psi'(x)+\frac{1}{2\,x}\right)^2+\psi''(x)-\frac{1}{2\,x^2},\\ f_3(x)&:=-\psi''(x)-\frac{2}{x}\,\psi'(x)+\frac{1}{x^2}, \end{aligned}$$

are strongly completely monotonic on $(0,\,\infty)$. *In particular, the functions* $x^2\,f_i(x)$, $i=1,2,3$ *are completely monotonic on* $(0,\,\infty)$.

Proof. In order to obtain the conclusions for the function $f_1(x)$ we repeat the proof of Theorem 2 taking $a=1$ while for the function $f_2(x)$ we take $a=\frac{2}{3}$ in the same proof.

We also have

$$f_3(x)=\int_0^\infty e^{-xu}\,\rho_3(u)\,du,$$

where

$$\rho_3(u):=u\,\delta(u)-2\int_0^u\delta(v)\,dv+u$$

and $\delta(u)$ as in the proof of Theorem 2. Since

$$\rho_3(0)=0,\;\;\rho_3'(0)=0,\;\;\text{and}\;\;\rho_3''(u)=u\,\delta''(u).$$

Recalling that the function $\delta(u)$ is strictly convex on $(0,\,\infty)$, an application of Lemma 2 completes the proof. □

Remarks. (1) Inequality

$$f_1(x)=[\psi'(x)]^2+\psi''(x)>0,\quad\text{for all }\;x>0,\tag{3.20}$$

has been established in [4] and also in [10]. It corresponds to the convexity of the function $e^{\psi(x)}-x$ on $(0,\,\infty)$. In [24] this result has been extended by proving that the function $f_1(x)$ is completely monotonic on $(0,\,\infty)$. Thus, the conclusions of Corollary 3 concerning the function $f_1(x)$ may be considered as a further extension of the above cited results. A generalization of (3.20) for higher order derivatives of the function $\psi(x)$ has been obtained in [11]. The q-analogue of (3.20) has been established in [6]. Several interesting inequalities for gamma and polygamma functions have been obtained in [4], [6], [9], [10] and [11] using inequality (3.20).

(2) Inequality

$$f_3(x)=-\psi''(x)-\frac{2}{x}\,\psi'(x)+\frac{1}{x^2}>0,\quad\text{for all }\;x>0,$$

expresses the convexity of the function $\nu(x):=x\,(\log x-\psi(x))$ on $(0,\,\infty)$, which has been proved in [7, Theorem 3.1]. In addition, this result has been strengthened in

[2, Theorem 1] by showing that the function $\nu(x)$ is strictly completely monotonic on $(0, \infty)$, which implies that the function $\nu''(x) = x\, f_3(x)$ is also strictly completely monotonic on $(0, \infty)$. The conclusion of Corollary 3 regarding the function $f_3(x)$ provides a further extension of the theorems cited above.

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