The Legacy of Vladimir Andreevich Steklov

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Vladimir Andreevich Steklov, an outstanding Russian mathematician whose 150th anniversary is celebrated this year, played an important role in the history of mathematics. Largely due to Steklov's efforts, the Russian mathematical school that gave the world such giants as N. Lobachevsky, P. Chebyshev, and A. Lyapunov, survived the revolution and continued to flourish despite political hardships. Steklov was the driving force behind the creation of the Physical–Mathematical Institute in starving Petrograd in 1921, while the civil war was still raging in the newly Soviet Russia. This institute was the predecessor of the now famous mathematical institutes in Moscow and St. Petersburg bearing Steklov's name.

Steklov's own mathematical achievements, albeit less widely known, are no less remarkable than his contributions to the development of science. The Steklov eigenvalue problem, the Poincaré–Steklov operator, the Steklov function—there exist probably a dozen mathematical notions associated with Steklov. The present article highlights some of the milestones of his career, both as a researcher and as a leader of the Russian scientific community.

V. A. Steklov in the 1920s.

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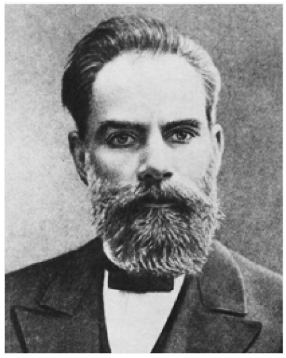


Aleksandr Mikhaylovich Lyapunov in the 1900s.

The article is organized as follows. It starts with a brief biography of V. A. Steklov written by N. Kuznetsov. The next section, written by N. Kuznetsov, A. Nazarov, and S. Poborchi, focuses on Steklov's work related to several celebrated inequalities in mathematical physics. The remaining two sections are concerned with some recent developments in the study of the Steklov eigenvalue problem, which is an exciting and rapidly developing area on the interface of spectral theory, geometry and mathematical physics. The "high spots" problem for sloshing eigenfunctions is discussed in the section written by T. Kulczycki, M. Kwaśnicki, and B. Siudeja. In particular, the authors explain why it is easier to spill coffee from a mug than to spill wine from a snifter. An overview of some classical and recent results on isoperimetric inequalities for Steklov eigenvalues is presented in the last section, written by I. Polterovich.

Many topics to which Steklov contributed in a major way are beyond the scope of the present article. For further references, see [14], [24], [31] and [64].

A Biographical Sketch of V. A. Steklov

Vladimir Andreevich Steklov was born in Nizhni Novgorod on January 9, 1864 (= December 28, 1863, old style). His grandfather and great-grandfather on the father's side were country clergymen. His father, Andrei Ivanovich Steklov, graduated from the Kazan Theological Academy and taught history and Hebrew at the Theological Seminary in Nizhni Novgorod. Steklov's mother, Ekaterina Aleksandrovna (née Dobrolyubova), was a daughter of a country clergyman as well. Her brother, Nikolay Aleksandrovich, was a prominent literary critic and one of the leaders of the democratic movement that aimed to abolish serfdom in Russia.

At ten years of age, Steklov enrolled into the Alexander Institute (a gymnasium that had many notable alumni, including the famous Russian composer M. Balakirev) in Nizhni Novgorod. Steklov's critical thinking manifested itself at a very early age. In his diaries, Steklov describes how he was chastised by the school principal for a composition deemed "disrespectful" towards the Russian empress Catherine II.

> I said to myself: "Aha! It occurs to me that I have my own point of view on historical events which is different from that of my schoolmates and teachers. […] It was the principal himself who proved that I am, in some sense, a self-maintained thinker and critic." This was the initial impact that led to my mental awakening; I realized that I am a human being able to reason and, what is important, to reason freely. […] Soon, my free thinking encompassed the religion as well. […] Thus, the cornerstone was laid for my future complete lack of faith.

After graduating from school in 1882, Steklov entered the Faculty of Physics and Mathematics of Moscow University. Failing to pass an examination in 1883, he left Moscow and the same year entered a similar faculty in Kharkov. There he met A. M. Lyapunov, and this encounter became a turning point in his life. Steklov graduated in 1887, but remained at the university working under Lyapunov's supervision towards obtaining his Master's Degree. In the beginning of 1890, Steklov married Olga Nikolaevna Drakina, who was a music teacher; their marriage lasted for 31 years. In the fall of the same year, he was appointed Lecturer in Elasticity Theory. In 1891, the Steklovs' daughter Olga was born and, presumably, this event delayed the defence of his Master's thesis, *On the motion of a solid body in a fluid*, until 1893. The same year, Steklov began lecturing at Kharkov Institute of Technology, combining it with his work at the university; the goal was to improve his family's financial situation, given that his wife had to leave her job after giving birth to their child. The sudden death of their daughter in 1901 was a heavy blow to Steklov and his wife, and caused a six-month break in his research activities.

He was appointed to an extraordinary professorship in mechanics in 1896. The first in a series of full-length papers, which formed the core of his dissertation for the Doctor of Science degree, appeared in print the same year (not to mention numerous brief notes in *Comptes rendus*). The dissertation entitled *General methods of solving fundamental problems in mathematical physics* was published as a book in 1901 by the Kharkov Mathematical Society [53].

At the time of completing his DSc dissertation, Steklov began to publish his results in French. Since then, most of his papers were written in French — the language widely used by Russian mathematicians to make their results accessible in Europe. Unfortunately, this did not prevent some of his results from remaining unnoticed. In particular, this concerns the so-called Wirtinger's inequality which was published by Steklov in 1901 in *Annales fac. sci. Toulouse* (see details in the next section). Even before that, Steklov became very active in corresponding with colleagues abroad (J. Hadamard, A. Kneser, A. Korn, T. Levi-Civita, E. Picard, S. Zaremba, and many others were among his correspondents); these contacts were of great importance for him, residing in a provincial city. In 1902, Steklov was appointed to an ordinary professorship in applied mathematics and was elected a corresponding member of the Academy of Sciences in St. Petersburg the next year.

In 1903, the Steklovs went on a summer vacation to Europe. Some details of this trip are described in one of Steklov's letters to Lyapunov (see [60], letter 29). In particular, the meeting with J. Hadamard in Paris:

> Somehow, Hadamard found me himself; presumably, he had learned my address from A. Hermann [the well-known publisher]. Once he missed me, but the next day he came at half past eight in the morning when we had just awakened. He arrived to Paris to stay for two days examining for "baccalauréat" [at some lycée]; on the day of his returning to the countryside, where he spends summer, he called on me before examination. His visit lasted only half-an-hour, but he told as much as another person would tell in a whole day. He is a model Parisian, very agile and swift to react; he behaved so as we are old friends who had not seen each other for some time.

In 1908, the Lyapunovs and the Steklovs travelled to Italy together, where A. M. and V. A. participated in the Rome ICM. At the Cambridge ICM (1912), Steklov was elected a vice president of the congress (Hadamard and Volterra were the other two vice presidents). The Toronto ICM (1924) was the third and the last one for Steklov.

Let us turn to the Petersburg–Petrograd–Leningrad period of Steklov's life. In 1906, he succeeded (after several attempts) in moving to St. Petersburg. It is a remarkable coincidence that a group of very talented students entered the university the same year. In the file of M. F. Petelin, who was one of them, this fact was commented on by Steklov as follows:

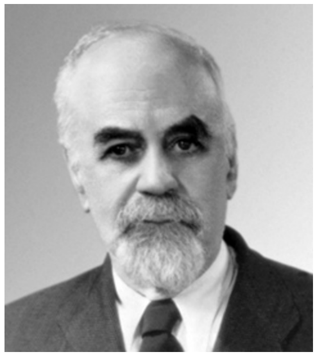
> I should note that the class of 1910 is exceptional. In the class of 1911 and among the fourth-year students who are about to graduate there is no one equal in knowledge and abilities to Messrs. Tamarkin, Friedmann, Bulygin, Petelin, Smirnov, Shohat, and others. There was no such case during the fifteen years of teaching at Kharkov University either. This favorable situation should be used for the benefit of the University.

Aleksandr Aleksandrovich Friedmann in the 1920s.

Steklov had done his best to nurture his students (see [66]). His dedication as an advisor was rewarded by the outstanding achievements of the members of his group, the most famous of which is Friedmann's solution of Einstein's equations in the general theory of relativity. The future fate of Steklov's students varied greatly; two of them (Bulygin and Petelin) died young. Tamarkin and Shohat emigrated to the USA and became prominent mathematicians there. It is worth noting that J.D. Tamarkin has more than 1500 mathematical descendants, and through him the St. Petersburg mathematical tradition had a profound impact on the American mathematics. Tamarkin's escape from the Soviet Union was quite an adventure. While secretly crossing the frozen Chudskoe Lake in order to reach Latvia, he was fired on by the Soviet border guards. As E. Hille wrote:

> One of J. D.'s best stories told how he tried to convince the American consul in Riga of his identity: the consul attempted to examine him in analytic geometry, but ran out of questions and gave up.

Friedmann and Smirnov became prominent scientists staying in Leningrad. Together with their colleagues and students (N. M. Günther, A. N. Krylov, V. A. Fock, N. E. Kochin, S. G. Mikhlin, and S. L. Sobolev to name a few) they organized the school of mathematical physics in Leningrad–St. Petersburg, the foundation of which was laid by Steklov.



Vladimir Ivanovich Smirnov in the 1950s.

Steklov's scientific career was advancing. In 1910, he was elected an adjunct member of the Academy of Sciences; two years later he was elected extraordinary and then ordinary academician within a few months. After his election to the executive committee of the Academy in 1916, Steklov reduced his work at the university and abandoned it completely in 1919, being elected vice president of the Academy. It would take too much space to describe everything he had accomplished at this post by the time of his unexpected and untimely death on May 30, 1926. (His wife died in 1920 from illness caused by undernourishment.) Therefore, only his role in establishing the Physical–Mathematical Institute—the predecessor of institutes named after him—will be outlined.

In January 1919, a memorandum was submitted to the Academy in which Steklov, A.A. Markov Sr., and A.N. Krylov proposed to establish a Mathematical cabinet as an initial stage in further development of the Academy's Department of Physics and Mathematics (the Physical cabinet existed in the Academy since its foundation in 1724, and it was reorganized into a laboratory in 1912). Later the same year, the initiative was supported, and Steklov became the head of the new institution named after P.L. Chebyshev and A.M. Lyapunov. In January 1921, Steklov submitted another memorandum, pointing out the necessity to merge the Physical laboratory and the recently established Mathematical cabinet. As he writes, "Mathematics and physics have now merged to such an extent that it is sometimes difficult to find the line that divides them." Nowadays, this viewpoint is shared by many mathematicians; however, at the time it was quite unusual.

The same January 1921, Steklov, S.F. Oldenburg (the Permanent Secretary of the Academy), and V.N. Tonkov (Head of the Military Medical Academy) visited V.I. Lenin in Moscow. In his recollections about Lenin, Maxim Gorky (the famous Russian writer close to the Bolsheviks), who had been present at this meeting, wrote (see also [67]):

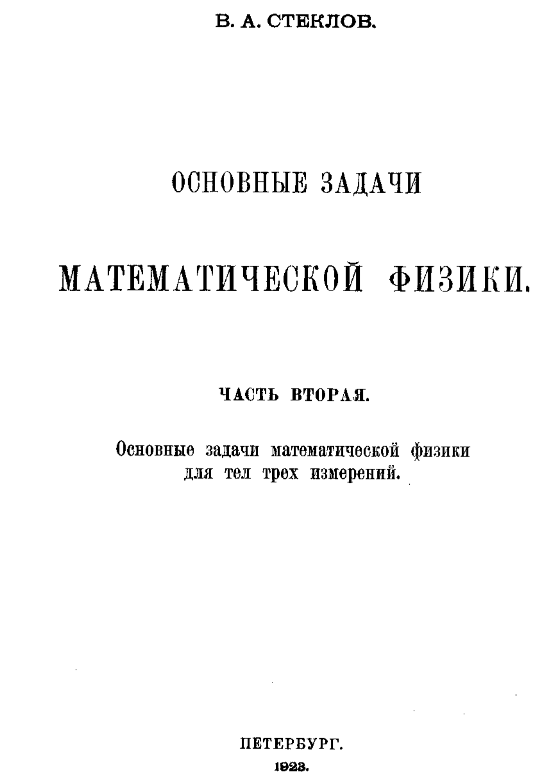
> They talked about the necessity to reorganize the leading scientific institution in Petersburg [the Academy]. After seeing off his visitors, Lenin said with satisfaction.
> 'What clever men! Everything is simple for them, everything is formulated rigorously; it is clear immediately that they know well what they want. It is a pleasure to work with such people. The best impression I've got from …'
> He named one of the most prominent Russian scientists; two days later, he told me by phone.
> 'Ask S[teklov] whether he is going to work with us.'
> When S[teklov] accepted the offer, this was a real joy for Lenin; rubbing his hands, he joked.
> 'Just wait! One Archimedes after the other, we'll gain support of all of them in Russia and in Europe, and then the World, willingly or unwillingly, will turn over!'

Indeed, after the Bolshevik government declared the so-called "New Economic Policy", the deadlock over Academy funding was broken thanks to the improving economic situation in the country. This resulted in the creation of the Physical–Mathematical Institute later in 1921, and Steklov was appointed its first director. In his recollections [59] completed in 1923, he writes:

> Another achievement of mine for the benefit of the Academy and the development of science in general is establishing the Physical–Mathematical Institute with the following divisions: mathematics, physics, magnitology and seismology. Its work is still in the process of being organized, the funding is scarce and difficult to obtain. The number of researchers is still negligible […], but a little is better than nothing.

After Steklov's death the institute was named after him. In 1934, simultaneously with relocation of the Academy from Leningrad to Moscow the Physical–Mathematical Institute was divided into the following two: the P.N. Lebedev Physical Institute and the V.A. Steklov Mathematical Institute (even before that the division of seismology became a separate institute). The Leningrad (now St. Petersburg) Department of the latter was founded in 1940, and it is an independent institute since 1995. To summarize, it must be said that the role of V.A. Steklov in Petrograd was similar to that of R. Courant who organized mathematical institutes, first in Göttingen and then in New York.

In 1922 and 1923, Steklov's monograph [58] was published; it summarizes many of his results in mathematical physics. Based on the lectures


В. А. СТЕКЛОВ.

ОСНОВНЫЕ ЗАДАЧИ

МАТЕМАТИЧЕСКОЙ ФИЗИКИ.

ЧАСТЬ ВТОРАЯ.

Основные задачи математической физики
для тел трех измерений.

ПЕТЕРБУРГ.
1923.

The title page of Steklov's monograph [58].

given in 1918–1920, this book is written in Russian, despite the fact that the corresponding papers originally appeared in French. More material was presented in the lecture course than was included in [58], and Steklov planned to publish the 3rd volume about his results concerning "fundamental" functions (that is, eigenfunctions of various spectral problems for the Laplacian) and some applications of these functions. Unfortunately, the administrative duties prevented him from realizing this project. However, one gets an idea about the probable contents of the unpublished 3rd volume from the lengthy article [55], in which Steklov developed his approach to "fundamental" functions (it is briefly outlined by A. Kneser [28], section 5). It is based on two different kinds of Green's function, and this allowed Steklov to apply the theory of integral equations worked out by E. I. Fredholm and D. Hilbert shortly before that. It is worth mentioning that [58] was listed among the most important mathematical books published during the period from 1900 to 1950 (see "Guidelines 1900–1950" in [45]; another item concerning mathematical physics published the same year is Hadamard's *Lectures on Cauchy's problem in linear partial differential equations*).

A great part of the material presented in the second volume of [58] is taken from the article [54], which is concerned with boundary value problems for the Laplace equation. It is Steklov's most cited work, but, confusingly, his initials are given incorrectly in many citations of this paper. Indeed, R. Weinstock found the following spectral problem

$$(1) \qquad \Delta u = 0 \ \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda \varphi u \text{ on } \partial D,$$

in [54], and for this reason he called it the *Stekloff problem*; here n is the exterior unit normal on ∂D and φ is a non-negative bounded weight function. In fact, Steklov introduced this problem in his talk at a session of the Kharkov Mathematical Society in December 1895; it was also studied in his DSc dissertation. Nowadays, it is mainly referred to as the Steklov problem, but, sometimes, is still called the Stekloff problem. In [69], Weinstock initiated the study of this problem, but, unfortunately, citing [54], he supplied Steklov's surname with wrong initials, which afterwards were reproduced elsewhere. Weinstock's result and its later developments are discussed in detail in the last section.

It must be emphasized that the legacy of Steklov is multifaceted (see [67]). He wrote biographies of Lomonosov and Galileo, an essay about the role of mathematics, the travelogue of his trip to Canada, where he participated in the 1924 ICM, his correspondence—published (see [60] and [61]) and unpublished; the recollections [59] and still unpublished diaries. Fortunately, many excerpts from Steklov's diaries are quoted in [66] and some of them appeared in [43]. The most expressive is dated September 2, 1914, one month after the Russian government declared war:

> St. Petersburg has been renamed Petrograd by Imperial Order. Such trifles are all our tyrants can do—religious processions and extermination of the Russian people by all possible means. Bastards! Well, just you wait. They will get it hot one day!

What happened in Russia during several years after that confirms clearly how right was Steklov in his assessment of the Tsarist regime. In his recollections [59] written in 1923, he describes vividly and, at the same time, critically "the complete bacchanalia of power" preceding the collapse of "autocracy and [Romanov's] dynasty" in February 1917 (old style), "the shameful Provisional government headed by Kerensky, the fast end of which can be predicted by every sane person", and how "the Bolshevik government [...] decided to accomplish the most Utopian socialistic ideas in the multi-million Russia." The list can be easily continued.

The pinpoint characterization of Steklov's personality was given by A. Kneser (see [28]):

> Everybody who maintained contact with Steklov was impressed by his personality. He was highly educated in the traditions of the European culture, but at the same time maintained distinctive features characteristic to his nation. He was not only a deep mathematician, but also a connoisseur of music and art. [...] Besides, he was a skillful mediator between scientists and the new government in Moscow. Thus, his role was crucial for the survival of the Russian science and its restoration (predominantly, in the Academy and its institutes) after the revolutions and the Civil War.

To conclude this section, we list some of Steklov's awards and distinctions. He was a member of the Russian Academy of Sciences, a corresponding member of the Göttingen Academy of Science, and a Doctor honoris causa of the University of Toronto. The mathematical Societies in Kharkov, Moscow, and St. Petersburg (in Petrograd, it was reorganized into the Physical–Mathematical Society) counted him among their members, as well as the Circolo Matematico di Palermo.

V. A. Steklov and the Sharp Constants in Inequalities of Mathematical Physics

In 1896, Lyapunov established that the trigonometric Fourier coefficients of a bounded function that is Riemann integrable on $(-\pi, \pi)$ satisfy the closedness equation. He presented this result at a session of the Kharkov Mathematical Society, but left it unpublished. The same year, Steklov had taken up studies of the closedness equation initiated by his teacher; Steklov's extensive work on this topic lasted for 30 years until his death. For this reason A. Kneser [28] referred to this equation as "Steklov's favorite formula". It should be mentioned that the term *closedness equation* was introduced by Steklov for general orthonormal systems, but only in 1910 (see brief announcements [56] and the full-length paper [57]).

The same year (1896), Steklov [50] proved that the following inequality (nowadays often referred to as Wirtinger's inequality)

$$\int_0^l u^2(x)\,\mathrm{d}x \leqslant \left(\frac{l}{\pi}\right)^2 \int_0^l [u'(x)]^2\,\mathrm{d}x \tag{2}$$

holds for all functions which are continuously differentiable on $[0, l]$ and have zero mean. For this purpose, he used the closedness equation for the Fourier coefficients of u (the corresponding system is $\{\cos(k\pi x/l)\}_{k=0}^{\infty}$ normalised on $[0, l]$). Inequality (2) was among the earliest inequalities with a sharp constant that appeared in mathematical physics. It was then applied to justify the Fourier method for initial-boundary value problems for the heat equation in two dimensions with variable coefficients independent of time. Later, Steklov justified the Fourier method for the wave equation as well. The fact that the constant in (2) is sharp was emphasized by Steklov in [52], where he gave another proof of this inequality (see pp. 294–296). There is another result proved in [52] (see pp. 292–294); it says that (2) is true for continuously differentiable functions vanishing at the interval's end-points, and again the constant is sharp. In the first volume of his monograph [58], Steklov presented inequality (2) along with its generalization.

The problem of finding and estimating sharp constants in inequalities attracted much attention from those who work in theory of functions and mathematical physics (see, for example, the classical monographs [20] and [49]). It is worth mentioning that in the famous book by Hardy, Littlewood, and Polya [20, section 7.7], inequality (2) is proved under either type of conditions proposed by Steklov; however, the authors call it Wirtinger's inequality and refer to the book of Blaschke (who was a student of Wirtinger) published in 1916 [5, p. 105], twenty years after the publication of Steklov's paper [50]. This terminology became standard. However, the controversy does not end here; we refer to [41] for other historical aspects of this inequality.

S. G. Mikhlin (he graduated from Leningrad University a few years after Steklov's death; see his recollections of student years [40]) emphasized the role of sharp constants in his book [39]. Let us quote the review [44] of the German version of [39]:

> [This book] is devoted to appraising the (best) constants—exact results or explicit (numerical) estimates—in various inequalities arising in "analysis" (=PDE). [...] This is the most original work, a bold attack in a direction where still very little is known.

In 1897, Steklov published the article [51], in which the following analogue of inequality (2) was proved:

$$\int_D u^2\,\mathrm{d}x \leqslant C \int_D |\nabla u|^2\,\mathrm{d}x. \tag{3}$$

Here ∇ stands for the gradient operator and the integral on the right-hand side is called the Dirichlet integral. The assumptions made by Steklov are as follows: D is a bounded three-dimensional domain whose boundary is piecewise smooth and u is a real C^1-function on $\bar{D}$ vanishing on ∂D. Again, inequality (3) was obtained by Steklov with the sharp constant equal to $1/\lambda_1^D$, where λ_1^D is the smallest eigenvalue of the Dirichlet Laplacian in D. In the early 1890s, H. Poincaré [46] and [47] obtained (3) using different assumptions, namely, u

Figure 1. High spot in a coffee cup.

Figure 2. High spot in a snifter.

has zero mean over D which is a union of a finite number of smooth convex two- and three-dimensional domains, respectively. In the latter case, the sharp constant in (3) is $1/\lambda_1^N$, where λ_1^N is the smallest positive eigenvalue of the Neumann Laplacian in D.

If u vanishes on ∂D (this is understood as follows: u can be approximated in the norm $\|\nabla u\|_{L^2(D)}$ by smooth functions having compact support in a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, of finite volume), then (3) is often referred to as the *Friedrichs inequality*. In fact K. O. Friedrichs [13] obtained a slightly different inequality under the assumption that $D \subset \mathbb{R}^2$. Namely, he proved:

$$\text{(4)} \qquad \int_D u^2\,\mathrm{d}x \leqslant C\left[\int_D |\nabla u|^2\,\mathrm{d}x + \int_{\partial D} u^2\,\mathrm{d}S\right],$$

where $\mathrm{d}S$ denotes the element of length of ∂D. Generally speaking, (4) holds for all bounded domains in $\mathbb{R}^n$ for which the divergence theorem is true (see [38], p. 24).

Inequality (3) for functions u with a zero mean value over D is equivalent to the following one (it is called the *Poincaré inequality*):

$$\text{(5)} \qquad \|u - \langle u\rangle\|_{L^2(D)} \leqslant C\|\nabla u\|_{L^2(D)}, \quad \text{where } \langle u\rangle = \frac{\int_D u(x)\,\mathrm{d}x}{\operatorname{meas}_n D},$$

here $\operatorname{meas}_n D$ is the n-dimensional measure of D. Note that the sharp constant here is $1/\sqrt{\lambda_1^N}$. Some requirements must be imposed on D for the validity of (5). Indeed, as early as 1933 O. Nikodým [42] (see also [38], p. 7) constructed a bounded two-dimensional domain D and a function with finite Dirichlet integral over D such that inequality (5) is not true. Another example of a domain with this property is given in the classical book [10] by Courant and Hilbert (see ch. 7, sect. 8.2).

In conclusion, we consider the following "boundary analogue" of inequality (5):

$$\|u - \langle u\rangle_G\|_{L^2(G)} \leqslant C\|\nabla u\|_{L^2(D)}, \quad \text{where } \langle u\rangle_G = \frac{\int_G u(x)\,\mathrm{d}S}{\operatorname{meas}_{n-1} G}.$$

Here D is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$, whereas G is a part of ∂D possibly coinciding with ∂D. The sharp constant in this inequality is equal to $1/\sqrt{\lambda_1^S}$, where λ_1^S is the smallest positive eigenvalue of the following mixed (unless $G = \partial D$) Steklov problem:

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = \lambda u \text{ on } G, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \setminus G.$$

If D is a special two- or three-dimensional domain with a particular choice of G, then the eigenvalues of this problem give rise to the *sloshing frequencies*, that is, the frequencies of free oscillations of a liquid in channels or containers; see, for example, [35, Chapter IX]. The sloshing problem is discussed in detail in the next section.

Spilling from a Wineglass and a Mixed Steklov Problem

The 2012 Ig Nobel Prize for Fluid Dynamics was awarded to R. Krechetnikov and H. Mayer for their work [37] on the dynamics of liquid sloshing. They investigated why coffee so often spills while people walk with a filled mug. In their study, oscillations of coffee are modeled by an appropriate mixed Steklov problem which is usually referred to as the *sloshing problem*. They realized that within this model one of the main reasons for spilling coffee can be described as follows. In a typical mug, the sloshing mode corresponding to the lowest eigenfrequency of the problem tends to get excited during walking.

However, there is another reason for spilling coffee from a mug of typical shape. Namely, a *high spot* is present on the boundary of the free surface, that is, the maximal elevation of the surface is always located on the mug's wall (see Figure 1), provided oscillations are free and their frequency is the lowest one. The latter effect (combined with that described in [37]) makes it even easier to spill coffee from a mug. On the other hand, in a

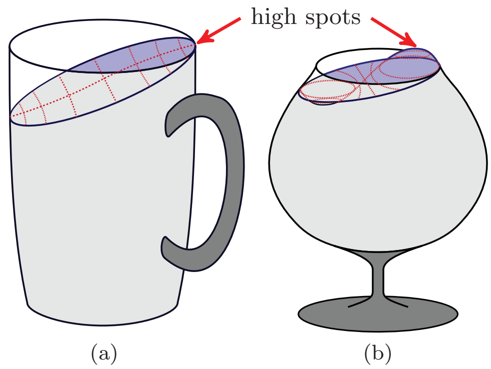


Figure 3. A schematic sketch of location of high spots in a coffee cup (a) and in a snifter (b).

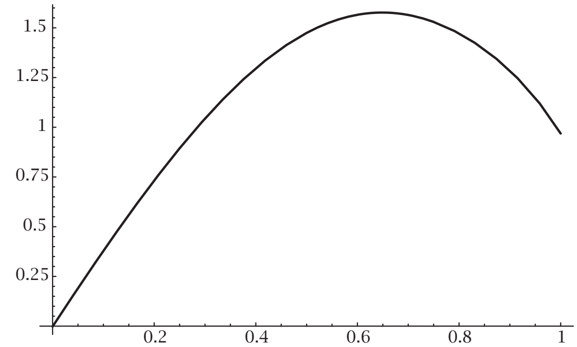


Figure 4. The highest radial free surface profile in an ice-fishing hole.

bulbous wineglass both antisymmetric sloshing modes corresponding to the lowest eigenfrequency are such that their maximal elevations (high spots) are attained inside the free surface, but not on the wall (see Figure 2). This reduces the risk of spilling from a snifter. Thus, the position of a high spot depends on the container's shape as is schematically shown in Figure 3 for a coffee cup (a) and a snifter (b); theorems guaranteeing these kinds of behavior are proved in [34].

The natural limiting case of bulbous containers is an infinite ocean covered by ice with a single circular hole (the corresponding sloshing problem is usually referred to as the *ice-fishing* problem). The question about the shape of the free surface when water oscillates at the lowest eigenfrequency in an ice-fishing hole was answered in [30]. This shape is similar to that in a snifter (see Figures 2 and 3 (b)). The highest free surface profile existing in radial directions of an ice-fishing hole was computed numerically and is plotted in Figure 4. One finds that the maximal amplitude is attained at some point located approximately $\frac{2}{3}r$ away from the hole's center (r is the hole's radius). This amplitude is over 50% larger than at the boundary.

Let us turn to the exact statement of the sloshing problem which is the mathematical model describing small oscillations of an inviscid, incompressible and heavy liquid in a bounded container. The liquid domain W is bounded by a free surface (its mean position F is horizontal) and by the wetted rigid part of ∂W, say B (bottom). Choosing Cartesian coordinates (x, y, z) so that the z-axis is vertical and points upwards, we place the two-dimensional domain F into the plane $z = 0$.

The water motion is assumed to be irrotational and the surface tension is neglected on F. In the framework of linear water wave theory, one seeks sloshing modes and frequencies as eigenfunctions and eigenvalues, respectively, of the following mixed Steklov problem:

$$\Delta\varphi = 0 \text{ in } W, \quad \frac{\partial\varphi}{\partial z} = \nu\varphi \text{ on } F, \tag{6}$$

$$\frac{\partial\varphi}{\partial \boldsymbol{n}} = 0 \text{ on } B, \quad \int_F \varphi \, \mathrm{d}x\mathrm{d}y = 0. \tag{7}$$

The last condition is imposed to exclude the eigenfunction identically equal to a non-zero constant and corresponding to the zero eigenvalue that exists for the problem including only the Laplace equation and the above boundary conditions.

In terms of (ν, φ) found from problem (6), (7), the velocity field of oscillations is given by

$$\cos(\omega t + \alpha)\nabla\varphi(x, y, z).$$

Here α is a certain constant, t stands for the time variable and $\omega = \sqrt{\nu g}$ is the radian frequency of oscillations (as usual, g denotes the acceleration due to gravity). Furthermore, the elevation of the free surface is proportional to $\sin(\omega t + \alpha)\,\varphi(x, y, 0)$, and so high spots are located at the points, where the restriction of $|\varphi|$ to F attains its maximum values.

It is known that if W and F are Lipschitz domains, then problem (6), (7) has a sequence of eigenvalues:

$$0 < \nu_1 \leqslant \nu_2 \leqslant \ldots \nu_n \leqslant \ldots, \quad \nu_n \to \infty.$$

For all n, $\varphi_n \in H^1(W)$, whereas their restrictions to F form (together with a non-zero constant) a complete orthogonal system in $L^2(F)$. In hydrodynamics, eigenfunctions corresponding to ν_1 play an important role because the rate of their decay (which is caused by non-ideal effects for real-life liquids) is least.

Modelling a mug by the following vertical-walled container $W = \{(x, y, z) : x^2 + y^2 < 1,\ z \in (-h, 0)\}$, in which case $F = \{(x, y, 0) : x^2 + y^2 < 1\}$ (cf. [37]), one finds all solutions of problem (6), (7) explicitly. In particular, there are two linearly independent

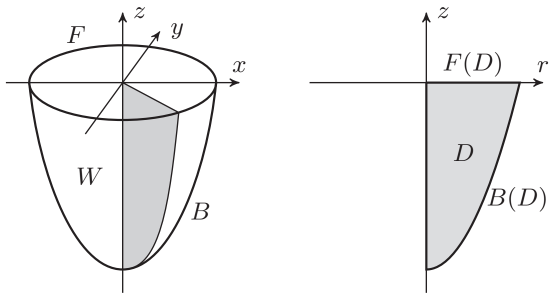


Figure 5. A body of revolution (left) and its radial cross-section (right).

eigenfunctions

$$\varphi_1 = J_1(j'_{1,1}r)\sin\theta\cosh j'_{1,1}(z+h),$$
$$\varphi_2 = J_1(j'_{1,1}r)\cos\theta\cosh j'_{1,1}(z+h),$$

corresponding to $\nu_1 = \nu_2 = j'_{1,1}\tanh j'_{1,1}h$. Here (r,θ,z) are the cylindrical coordinates such that θ is counted from the x-axis (see the left-hand side of Figure 5), J_1 is the Bessel function of the first kind and $j'_{1,1} \approx 1.8412$ is the first positive zero of J'_1. It is clear that $\varphi_1(x,y,0)$ is an odd, increasing function of y, and so it attains extreme values (high spots) at the boundary points $(0,1)$ and $(0,-1)$; similarly, φ_2 has its high spots at $(1,0)$ and $(-1,0)$. Moreover, all linear combinations of φ_1 and φ_2 have high spots on the boundary of F.

Using the finite element method, one can obtain approximate positions of high spots for more complicated domains. We used FEniCS [36] to implement a trough (see [33] for the corresponding rigorous result), which is short and has a hexagonal cross-section. Such a trough is shown in Figure 6, where several level surfaces of the lowest-frequency mode φ_1 are also plotted. It is clear

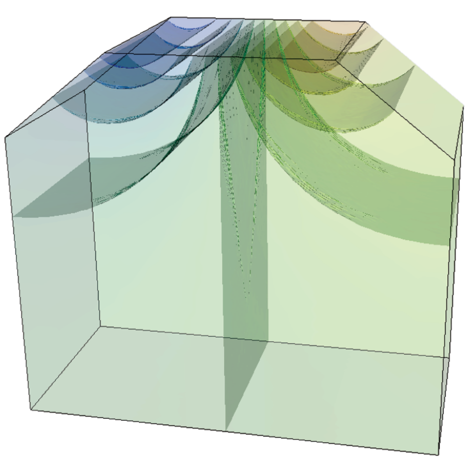


Figure 6. A short trough and level surfaces of its fundamental eigenfunction.

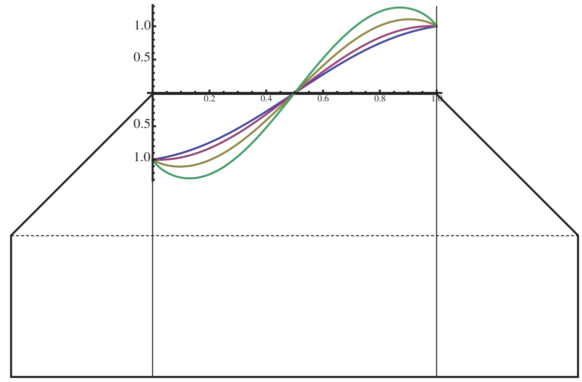


Figure 7. The cross-section of a channel is given by solid segments. The length of the free surface (respectively, bottom) is equal to 2 $(2\,(x+1)$**, respectively); the depth of the channel (respectively, its rectangular part) is equal to** $x+1$ **(**x**, respectively). The free surface profile plotted in blue corresponds to an isosceles trapezoid which is 50% wider at the top than at the bottom; other profiles correspond to the shown hexagonal cross-section with** $x = 0.1, 1, 10$**.**

that the maximum of $\varphi_1(x,y,0)$ is not on the boundary.

Since the previous example is essentially two-dimensional (see [33]), we exploited this reduction to obtain numerically more accurate free surface profiles plotted in Figure 7. The blue curve corresponds to an isosceles trapezoid which is 50% wider at the top than at the bottom; its maximum is on the boundary (as for a coffee mug). Other profiles correspond to hexagons with different slopes of side walls. Notice that the point of maximum moves towards the center for more horizontal slopes, whereas the high spot becomes more pronounced.

Photos of water oscillations in bulbous and other containers were made (examples are given in Figures 1 and 2). It proved difficult to illustrate the high spot effect by photographing in a conventional way because of the nonlinearity caused by relatively large amplitude of oscillations and non-ideal nature of liquid. Therefore, along with photos shown in Figures 1 and 2, we also photographed a reflection of a dotted piece of paper on a slightly disturbed surface of the liquid (see Figure 8, bottom). Images produced using sufficiently long exposure time mostly consist of blurred segments with just a few clearly visible dots (see Figure 8, top). The reason for this is the fact that planes tangent to the water surface oscillate almost everywhere creating a segment path for each dot. The exceptional points are those where the sloshing surface has its local extrema, and so the corresponding tangent planes

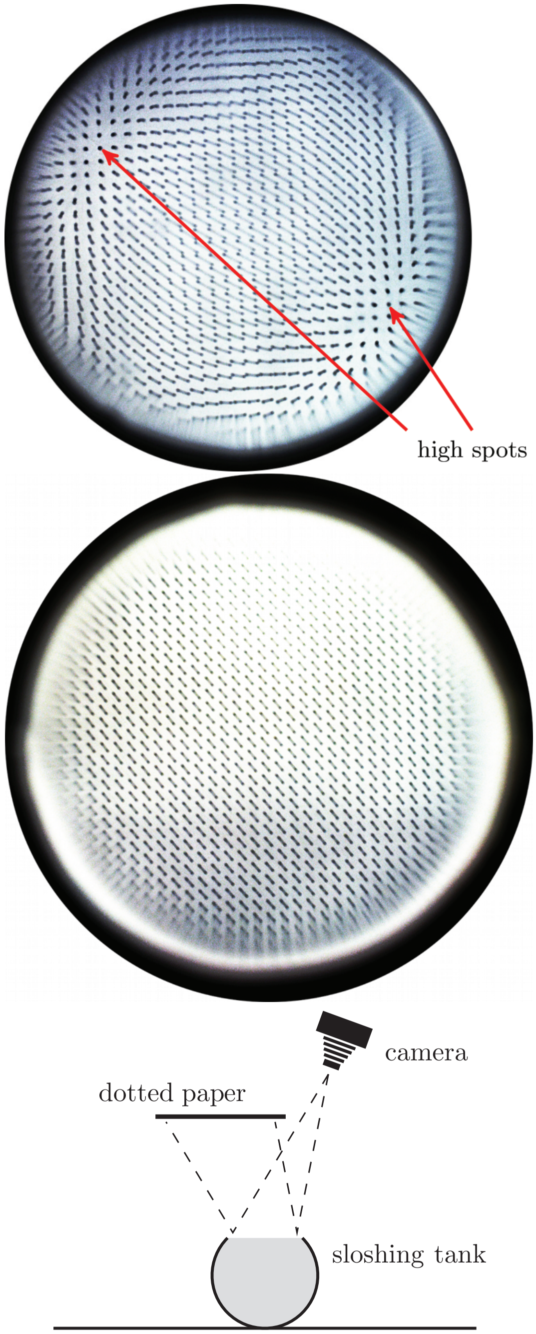


Figure 8. (top) Image of reflected light from the sloshing surface of water in a fish bowl. (middle) Similar image for a cocktail glass, showing no points with vanishing gradient. (bottom) Setup for photos.

are always horizontal, which makes these dots sharp. The image obtained for a bulbous container (a fish bowl) has two clearly visible points of extrema located away from the boundary (see Figure 8, top), which is in agreement with Figures 2 and 3 (b). The similar image for a conical tank (a cocktail glass) consists exclusively of almost the same blurred segment paths for all dots (see Figure 8, middle), and this agrees with Figures 1 and 3 (a).

Let us turn to discussing results proved rigorously for bodies of revolution. If W is obtained by rotating a two-dimensional domain D that has a horizontal segment on the top and is attached to the z-axis around this axis (see Figure 5), then the free surface F is a disk in the (x, y)-plane. We assume that the fundamental modes are antisymmetric; many domains have this property (see below). Nevertheless, examples of rotationally symmetric fundamental eigenfunctions also exist. For example, this takes place for the following domain: halves of a ball and a spherical shell joined by a small vertical pipe so that all of them are coaxial.

Under antisymmetry assumption about the fundamental modes, there are two of them

$$\varphi_1 = \psi(r, z) \cos\theta, \qquad \varphi_2 = \psi(r, z) \sin\theta,$$

that correspond to $\nu_1 = \nu_2$ and are linearly independent; here $\psi(r, z)$ is defined on D.

In [34] (see Theorems 1.1 and 1.2), the following is proved. If W is a convex body of revolution confined to the cylinder $\{(x, y, z) : (x, y, 0) \in F,\ \ z \in \mathbb{R}\}$ (this condition was introduced by F. John in 1950), then three assertions hold: (i) $\nu_1 = \nu_2$; (ii) the corresponding eigenfunctions φ_1 and φ_2 are antisymmetric; (iii) the high spots of these modes are attained on ∂F.

On the other hand (see [34], Proposition 1.3), if the angle between B and F is bigger than $\frac{\pi}{2}$ and smaller than π then $\varphi_1(x, y, 0)$, $\varphi_2(x, y, 0)$ attain their extrema inside F as is shown in Figure 8, top.

The proof of (ii) and (iii) is based on the technique of domain deformation used by D. Jerison and N. Nadirashvili [26], who studied the *hot spots* conjecture. The latter was posed by J. Rauch in 1974 (see a description by I. Stewart in his *Nature* article [62], and Terence Tao's Polymath project [65] for current developments). Roughly speaking, the hot spots conjecture states that in a thermally insulated domain, for "typical" initial conditions, the hottest point will move towards the boundary of the domain as time passes. The mathematical formulation of the hot spots conjecture is as follows: *every fundamental eigenfunction of the Neumann Laplacian in an n-dimensional domain D attains its extrema on ∂D*. It was proved for sufficiently regular planar domains (see, for example, [3] and [26]), disproved for some domains with holes (see, for example, [7]) and is still open for arbitrary convex planar domains.

There is a remarkable relationship between high and hot spots (see, for example, [32], Proposition 3.1). If $\varphi_1, \ldots, \varphi_k$ are the sloshing eigenfunctions corresponding to an eigenvalue ν in a vertical cylinder $W = \{(x, y, z) : (x, y) \in F, z \in (-h, 0)\}$, then μ is an eigenvalue of the Neumann Laplacian in $D = F$, if and only if $\nu = \sqrt{\mu} \tanh \sqrt{\mu} h$. Moreover, every eigenfunction $\psi(x, y)$ corresponding to μ is defined by some φ_j, $j = 1, \ldots, k$, in the following way: $\varphi_j(x, y, z) = \psi(x, y) \cosh \sqrt{\mu}(z + h)$. This relationship between the two problems implies, in particular, that the results obtained in [3] and [26] for planar convex domains with two orthogonal axes of symmetry (e.g., ellipses) can be reformulated as follows. In a vertical-walled, cylindrical tank with such free surface, high spots of fundamental sloshing modes are located on the free surface's boundary.

Isoperimetric Inequalities for Steklov Eigenvalues

It was mentioned above that Steklov's major contribution to mathematical physics appeared in 1902, namely, the article [54], in which he introduced an eigenvalue problem (1). This problem with the spectral parameter in the boundary condition turned out to have numerous applications. Moreover, the Steklov problem—as it is referred to nowadays—provides a new playground for exciting interactions between geometry and spectral theory, exhibiting phenomena that could not be observed in other eigenvalue problems. For the sake of simplicity, it is assumed throughout this section that $\varphi \equiv 1$ in formula (1).

In two dimensions, problem (1) can be viewed as a "cousin" of the Neumann problem in a bounded domain D. Indeed, the latter problem describes the vibration of a homogeneous free membrane, whereas the Steklov problem models the vibration of a free membrane with all its mass concentrated along the boundary (see [2], p. 95). Steklov eigenvalues

$$0 = \lambda_0 < \lambda_1(D) \leqslant \lambda_2(D) \leqslant \lambda_3(D) \leqslant \cdots \nearrow \infty$$

correspond to the frequencies of oscillations. As in the Neumann case, the Steklov spectrum starts with zero, and in order to ensure discreteness of the spectrum it is sufficient to assume that the boundary is Lipschitz.

Isoperimetric inequalities for eigenvalues is a classical topic in geometric spectral theory that goes back to the ground-breaking results of Rayleigh–Faber–Krahn and Szegő–Weinberger on the first Dirichlet and the first non-zero Neumann eigenvalues. The problem is to find a shape that extremizes (minimizes for Dirichlet and maximizes for Neumann) the first eigenvalue among all shapes of fixed volume. In both cases, the unique extremal domain is a ball, similarly to the classical isoperimetric inequality in Euclidean geometry.

Szegő's proof of the isoperimetric inequality for the first Neumann eigenvalue on a simply connected planar domain D is based on the Riemann mapping theorem and a delicate construction of trial functions using eigenfunctions on a disk [63]. In 1954 (the same year Szegő's paper was published), R. Weinstock [69] realized that this approach could be adapted to prove a sharp isoperimetric inequality for the first Steklov eigenvalue. Weinstock showed that the first nonzero Steklov eigenvalue is maximized by a disk among all simply connected planar domains of fixed *perimeter*. Note that for the Steklov problem, the perimeter is proportional to the mass of the membrane, like the area in the Neumann problem. In fact, Weinstock's proof is easier than Szegő's, because the first Steklov eigenfunctions on a disk are just coordinate functions, not Bessel functions as in the Neumann case. In a way, Weinstock's argument is a first application of the "barycentric method" that is being widely used in geometric eigenvalue estimates.

The analogy between isoperimetric inequalities for Neumann and Steklov eigenvalues is far from being complete, which makes the study of Steklov eigenvalues particularly interesting. For instance, as was shown by Weinberger [68], Szegő's inequality for the first Neumann eigenvalue can be generalized to arbitrary Euclidean domains of any dimension. At the same time, Weinstock's result fails for non-simply connected planar domains: if one digs a small hole in the center of a disk, the first Steklov eigenvalue of the corresponding annulus, normalized by the perimeter, is bigger than the normalized first Steklov eigenvalue of a disk [19].

Another major distinction from the Neumann case is that for simply connected planar domains, sharp isoperimetric inequalities are known for *all* Steklov eigenvalues. It was shown in [16] that the inequality $\lambda_n(D)\, L(\partial D) \leqslant 2\pi n$, $n = 1, 2, 3 \ldots$, proved in [23] is sharp, with the equality attained in the limit by a sequence of domains degenerating to a disjoint union of n identical disks; here $L(\partial D)$ denotes the perimeter of D. For Neumann eigenvalues, a similar result holds for $n = 2$ [15], but the situation is quite different for $n \geqslant 3$ (see [1] and [48]).

In 1970, J. Hersch [22] developed the approach of Szegő in a more geometric direction. He proved that among all Riemannian metrics on a sphere of given area, the first eigenvalue of the corresponding Laplace–Beltrami operator is maximal for the standard round metric. Note that the first eigenspace on a round sphere is generated by coordinate functions, which allows one to prove Hersch's theorem in a similar way as Weinstock's inequality [17]. The result of Hersch stimulated a whole direction of research on extremal metrics for Laplace–Beltrami eigenvalues on surfaces.

Recently, Fraser and Schoen (see [11] and [12]) extended the theory of extremal metrics to Steklov eigenvalues on surfaces with boundary. They have studied extremal metrics for the first Steklov eigenvalue on a surface of genus zero with l boundary components, and proved the existence of maximizers for all $\ell \geqslant 1$. Weinstock's inequality covers the case $\ell = 1$, but already for $\ell = 2$ the result is quite unexpected. The maximizer is given by a "critical catenoid"; it is a certain metric of revolution on an annulus such that the first Steklov eigenvalue has multiplicity three. Interestingly enough, this is the maximal possible multiplicity for the first eigenvalue on an annulus (see [12], [27] and [25]). The critical catenoid admits the following characterization: it is a unique free boundary minimal annulus embedded into a Euclidean ball by the first Steklov eigenfunctions [12].

Maximizers for higher Steklov eigenvalues on surfaces, as well as sharp isoperimetric inequalities for eigenvalues on surfaces of higher genus, are still to be found. Some bounds were obtained in [11], [29], [18], [21] in terms of the genus and the number of boundary components. At the same time, it was shown in [9] that there exists a sequence of surfaces of fixed perimeter, such that the corresponding first eigenvalues of the Steklov problem tend to infinity.

Apart from the two-dimensional vibrating membrane model discussed above, there is another physical interpretation of the Steklov problem, which is valid in arbitrary dimension. It describes the stationary heat distribution in a body D, under the condition that the heat flux through the boundary is proportional to the temperature. In this context, it is meaningful to use the volume of D as a normalizing factor. Weinstock's inequality combined with the classical isoperimetric inequality implies that the disk maximizes the first Steklov eigenvalue among all simply connected planar domains of given area. Generalizations of this result were obtained in [6] and [4]. In particular, it was shown that in any dimension, the ball maximizes the first Steklov eigenvalue among all Euclidean domains of given volume.

Yet another interpretation of the Steklov spectrum involves the concept of the Dirichlet-to-Neumann map (sometimes called the Poincaré–Steklov operator), which is important in many applications, such as electric impedance tomography, cloaking, etc. The Dirichlet-to-Neumann map acts on functions on the boundary of a domain D (or, more generally, of a Riemannian manifold), and assigns to each function the normal derivative of its harmonic extension into D. The spectrum of this operator is given precisely by the Steklov eigenvalues. Since the Dirichlet-to-Neumann map acts on ∂D, it is natural to normalize the eigenvalues by the volume of the boundary. If the volume of ∂D is fixed, the corresponding Steklov eigenvalues λ_n of a Euclidean domain D can be bounded in terms of n and the dimension [8]. However, no sharp isoperimetric inequalities of this type are known at the moment in dimensions higher than two.

Mathematical notions often lead a life of their own, independent of the will of their creators. When Steklov introduced the eigenvalue problem that now bears his name, he was motivated mainly by applications. It is hard to tell whether he could foresee the interest in the problem coming from geometric spectral theory. There is no doubt, however, that the past and future work of many mathematicians on isoperimetric inequalities for Steklov eigenvalues owes a lot to Steklov's insight.

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