To the Memory of Lars Hörmander (1931–2012)

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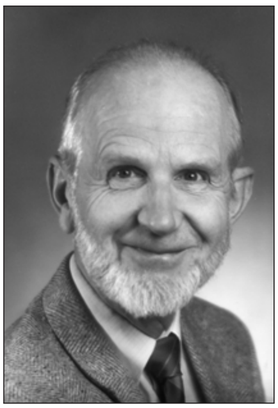

Lars Hörmander 1996.

The eminent mathematician Lars Valter Hörmander passed away on November 25, 2012. He was the leading figure in the dramatic development of the theory of linear partial differential equations during the second half of the twentieth century, and his L^2 estimates for the $\bar{\partial}$ equation became a revolutionary tool in complex analysis of several variables. He was awarded the Fields Medal in 1962, the Wolf Prize in 1988, and the Leroy P. Steele Prize for Mathematical Exposition in 2006. He published 121 research articles and 9 books, which have had a profound impact on generations of analysts.

Student in Lund

Lars Hörmander was born on January 24, 1931, in Mjällby in southern Sweden. His talents became apparent very early. He skipped two years of elementary school and moved to Lund in 1946 to enter secondary school (gymnasium). He was one of the selected students who were offered the chance to cover three years' material in two years by staying only three hours per day in school and devoting the rest of the day to individual studies, a scheme he found ideal. His mathematics teacher, Nils Erik Fremberg (1908–52), was also docent at Lund University and in charge of the undergraduate program there. When graduating from gymnasium in the spring of 1948 Lars had already completed the first-semester university courses in mathematics.

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Unless otherwise noted, all photographs are from Lars Hörmander's personal collection, courtesy of Sofia Broström (Hörmander).

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Lars entered Lund University in the fall of 1948 to study mathematics and physics. Marcel Riesz (1886–1969) became his mentor. Lars studied most of the courses on his own, but followed the lectures on analysis given by Riesz during 1948–50. In the early fall of 1949 he completed a bachelor's degree and a master's degree in the spring of 1950, which could have allowed him to earn his living as a secondary school teacher at the age of only nineteen.

In October 1951 Lars completed the licentiate degree with a thesis entitled "Applications of Helly's theorem to estimates of Tchebycheff type", written under the supervision of Riesz. Shortly afterwards Riesz retired from his professorship and moved to the US to remain there during most of the coming ten years. We have often been asked who was the PhD thesis advisor of Lars Hörmander. The correct answer is no one and it needs a little bit of explanation. As a young man Marcel Riesz worked mainly on complex and harmonic analysis, but later in life he became interested in partial differential equations and mathematical physics, so partial differential equations were certainly studied in Lund in the early 1950s. Moreover, Lars Gårding (1919–2014) and Åke Pleijel (1913–89) were appointed professors in mathematics in Lund during this period. They had good international contacts, and the young Lars concluded that partial differential equations would give him the best opportunities in Lund.

Lars defended his doctorate thesis, "On the theory of general partial differential operators," on October 22, 1955, with Jacques-Louis Lions as opponent, and it was published in *Acta Mathematica* the same year. His work was highly independent. Lars chose his problems himself and solved them. Gårding was often mentioned as the thesis advisor of Hörmander, but the fact is that he was no more than a formal advisor, in the sense that he was an important source of inspiration and that he served as a chairman at Lars's thesis defense. It is no exaggeration to say that the thesis opened a new era of the subject of partial differential equations.

Some of the most important results were quite original and had not even been envisioned before.

Professor in Stockholm, Stanford, and Princeton

Lars spent the year 1956 in the United States. In January 1957 he became professor at Stockholm University. Since there was no activity in partial differential equations at Stockholm University at this time, he had to start from the beginning and lecture on distribution theory, Fourier analysis, and functional analysis. In 1961–62 he lectured on the material that was to become his 1963 book, *Linear Partial Differential Operators*. He quickly gathered a large group of students around him. Among them were Christer Kiselman and Vidar Thomée. One of us (JB) had the privilege to be a member of that group. Lars's lectures were wonderful, and there was excitement and enthusiasm around him. His vast knowledge, fast thinking, and overwhelming capacity for work inspired all of us, but also sometimes scared his students. The clattering from his typewriter, which was constantly heard through his door, is famous.

Lars spent the summer of 1960 in Stanford and the following academic year at the Institute for Advanced Study (IAS) in Princeton. The summer of 1960 was both pleasant and productive for Lars, and both he and the members of the department in Stanford were interested in a continuation on a more permanent basis, but Lars was not ready to leave Sweden at that time. An arrangement was made so that he would be on leave from Stockholm for April and May and combine the position in Stockholm with a permanent professorship at Stanford, where he would work for the spring and summer quarters. In Stockholm 1962–63 he gave a lecture series on complex analysis which he developed further in Stanford, 1964 and published as *An Introduction to Complex Analysis in Several Variables* in 1966. Revised editions were published in 1973 and 1990. It is remarkable that this book has kept its position as one of the main references on the subject for almost fifty years.

At Stanford in the summer of 1963 Lars received a letter which would change his life. Robert Oppenheimer, the director of the IAS, made him an offer to become a professor and a permanent member of the institute. It took Lars some time to make up his mind. Attempts were made by Lennart Carleson (b. 1928), Otto Frostman (1907–77), and Lars Gårding to arrange for a research position in Lund, but when this was declined by the Swedish government, Lars decided to accept the offer. He spent the summer of 1964 at Stanford and started his work at the institute in September. He soon found that ongoing conflicts had created a bad climate at the institute. He also felt strong pressure to produce a steady stream of high-quality research, and this he found paralyzing. This appears especially paradoxical, since his accomplishments during his period in Princeton were truly remarkable. Already in early 1966 he had made up his mind to return to an ordinary professorship in Sweden as soon as an opportunity came up.

Back in Lund

Lars left Princeton in the spring of 1968 and became a professor in Lund, a position he held until his retirement in 1996. He always kept good contact with IAS and other universities in the US. In 1977–78 Lars was in charge of a special program on microlocal analysis at IAS. Among Lars's prominent students in Lund were Johannes Sjöstrand, Anders Melin, Nils Dencker, and Hans Lindblad.

Photograph by Nils Åslund

Student in Lund around 1950.

During the years 1979–84 Lars worked on the four volumes of *The Analysis of Partial Differential Operators* published in 1983 and 1985. This was the time of study of the younger of us (RS) in Lund. Lars gave a wonderful lecture series on various parts of the manuscript. The students corrected (rare) errors and in return got superb private lessons on the parts they did not understand. The four volumes, written in the well-known compact Hörmander style, contain enough material to fill eight volumes rather than four. The amount of work needed to complete the project was clearly formidable, and Lars later looked back at this period as six years of slave labor.

In the fall of 1984 Lars succeeded Lennart Carleson as director of the Institut Mittag-Leffler and became the managing editor for *Acta Mathematica*. He conducted a two-year program on nonlinear partial differential equations. Back in Lund in the fall of 1986 Lars started a series of lectures on nonlinear problems. His notes from 1988 were widely circulated and finally published in revised form as the book *Lectures on Nonlinear Hyperbolic Differential Equations* in 1997. In 1991–92 Lars gave lectures which he later developed into the book *Notions of Convexity*, published in 1994. In this book he shows again the depth and breadth of his knowledge of analysis.

Lars became emeritus on January 1, 1996. From the beginning of the 1990s his research was not as focused on partial differential equations as it had been before. He looked back on his career and took up the study of various problems that he had dealt with and continued publishing interesting papers. Lars was always interested in the Nordic cooperation of mathematicians. He published his first paper in the proceedings of the Scandinavian Congress of Mathematicians held

With John Forbes Nash, Jean Leray, and Lars Gårding in Paris, 1958.

Photograph by Vidar Thomée.

With Bogdan Bojarski and Vidar Thomée in Norway, 1965.

in Lund in 1953, the second one in *Mathematica Scandinavica* in 1954, and it is symbolic that his last paper was published there in 2008. In 1997 Mikael Passare (1959–2011) initiated the annual Nordan conferences, which is a platform for Nordic researchers in complex analysis and related topics. Lars participated the first few years. He gave his last conference talk in Reykjavik in 2002. The subject was unusual for him: he talked about his L^2 method from a historical perspective. It was published in *Journal of Geometric Analysis* in 2003.

Lars Hörmander had a huge influence on our development as mathematicians, first as a teacher and advisor and later as a colleague and friend. We always admired him for his great knowledge, his sharp mind, and his masterful way of communicating mathematics in speech and writing. Until his death, he kept his great spirit and memory. His interests in mathematics, science, nature, and history were always the same. We kept regular contact with him and it was always a pleasure to talk to him. We are very grateful for having had the opportunity to work with him.

In this series of memorial articles we have assembled nine contributions from Lars's colleagues, friends, students, and his daughter, Sofia. Nicolas Lerner writes on Lars's contributions to partial differential equations, and Jean-Pierre Demailly on his work in complex analysis.

Nicolas Lerner

On Lars Hörmander's Work on Partial Differential Equations

The Beginning

Lars Hörmander wrote a PhD thesis under the guidance of L. Gårding, and the publication of that thesis, "On the theory of general partial differential operators" [29], in *Acta Mathematica* in 1955 can be considered as the starting point of a new era for partial differential equations. Among other things, very general theorems of local existence were established without using an analyticity hypothesis of the coefficients. Hörmander's arguments relied on a priori inequalities combined with abstract functional analytic arguments. Let us cite L. Gårding in [26]: *It was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions even for an operator with analytic coefficients. This reduces the interest of the Cauchy-Kowalevski theorem which …does not distinguish between classes of differential operators which have, in fact, very different properties such as the Laplace operator and the Wave operator.*

L. Ehrenpreis in [23] and B. Malgrange in [58] had proven a general theorem on the existence of a fundamental solution for any constant coefficients PDE, and the work [30] by Hörmander provided another proof along with some improvement on the regularity properties, whereas [29] gave a characterization of hypoelliptic constant coefficients PDE via properties of the algebraic variety

$$\operatorname{char} P = \{\zeta \in \mathbb{C}^n, P(\zeta) = 0\}.$$

The operator $P(D)$ is hypoelliptic if and only if $\{|\zeta| \to \infty \text{ on } \operatorname{char} P \implies |\operatorname{Im} \zeta| \to \infty.\}$ Here hypoellipticity means

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu \tag{1}$$

for the C^∞ singular support. The characterization of hypoellipticity of the constant coefficient operator $P(D)$ by a simple algebraic property of the characteristic set is a tour de force, technically and conceptually: in the first place, nobody had conjectured such a result or even remotely suggested a link between the two properties, and next, the proof provided by Hörmander relies on a very subtle study of the characteristic set, requiring an extensive knowledge of real algebraic geometry.

In 1957, Hans Lewy made a stunning discovery [57]: the equation $\mathcal{L}u = f$ with

$$\mathcal{L} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + i(x_1 + ix_2)\frac{\partial}{\partial x_3} \tag{2}$$

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Photograph by Ragnar Sigurdsson.

With Lars Gårding and Tomas Claesson, 2008.

does not have local solutions for most right-hand sides f. The surprise came in particular from the fact that the operator $\mathcal{L}$ is a nonvanishing vector field with a very simple expression and also, as the Cauchy-Riemann operator on the boundary of a pseudo-convex domain, it is not a cooked-up example. Hörmander started working on the Lewy operator (2) with the goal to get a general geometric understanding of a class of operators displaying the same defect of local solvability. The two papers [34], [33], published in 1960, achieved that goal. Taking a complex-valued homogeneous symbol $p(x,\xi)$, the existence of a point (x,ξ) in the cotangent bundle such that

$$p(x,\xi) = 0, \quad \{\bar{p}, p\}(x,\xi) \neq 0 \tag{3}$$

ruins local solvability at x (here $\{\cdot,\cdot\}$ stands for the Poisson bracket). With this result, Hörmander nonetheless gave a generalization of the Lewy operator, but above all provided a geometric explanation for that nonsolvability phenomenon.

A.-P. Calderón's 1958 paper [17] on the uniqueness in the Cauchy problem was somehow the starting point for the renewal of singular integrals methods in local analysis. Calderón proved in [17] that an operator with real principal symbol with simple characteristics has the Cauchy uniqueness property; his method relied on a pseudodifferential factorization of the operator which can be handled thanks to the simple characteristic assumption. It appears somewhat paradoxical that Hörmander, who later became one of the architects of pseudodifferential analysis, found a generalization of Calderón's paper using only a local method, inventing a new notion to prove a Carleman estimate. He introduced in [32], [31] the notion of pseudoconvexity of a hypersurface with respect to an operator and was able to handle the case of tangent characteristics of order two. A large array of counterexamples, due to P. Cohen [18], A. Pliś [73], and later to S. Alinhac [1] and S. Alinhac and M. S. Baouendi [2], showed the relevance of the pseudoconvexity hypothesis for Cauchy uniqueness.

In 1962, at the age of thirty-one, Hörmander was awarded the Fields Medal. His impressive work on PDE, in particular his characterization of

Receiving the Fields Medal from King Gustav VI Adolf.

Opening ceremony of ICM in Stockholm, 1962. From left: Lars Gårding, Lars Hörmander, John Milnor, Hassler Whitney, Åke Pleijel, Harald Cramér, Otto Frostman. Standing at the rostrum: Rolf Nevanlinna. Gårding and Whitney presented the work of the Fields Medalists Hörmander and Milnor, resp. The organizing committee consisted of Frostman (chair), Cramér, Gårding, and Pleijel. Nevanlinna was president of IMU and chairman of the Fields Medal Committee.

hypoellipticity for constant coefficients and his geometrical explanation of the Lewy nonsolvability phenomenon, were certainly very strong arguments for awarding him the medal. Also his new point of view on PDE, which combined functional analysis with a priori inequalities, had led to very general results on large classes of equations which had been out of reach in the early 1950s.

The Microlocal Revolution, Act I

Pseudodifferential Equations

The paper [17] of Calderón led to renewed interest in singular integrals, and the notion of pseudodifferential operators along with a symbolic calculus was introduced in the 1960s by several authors: J. J. Kohn and L. Nirenberg in [53], and A. Unterberger and J. Bokobza in [81]. Hörmander wrote in 1965 a synthetic account of the nascent pseudodifferential methods with the article [36]. A pseudodifferential operator $A = a(x,D)$ with symbol a is defined by the formula

$$(4) \qquad (Au)(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi(2\pi)^{-n},$$

say for $u \in C_c^\infty(\mathbb{R}^n)$. The symbol a is a smooth function on the phase space which should satisfy some estimates, e.g., $\exists m, \forall \alpha, \beta$,

$$\sup_{x,\xi}(1+|\xi|)^{-m+|\beta|}|(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi)| < \infty.$$

This type of operator, initially used to construct parametrices of elliptic operators, soon became a key tool in the analysis of PDE.

Hypoellipticity

In 1934, A. Kolmogorov introduced the operator in $\mathbb{R}^3_{t,x,v}$,

$$(5) \qquad \mathcal{K} = \partial_t + v\partial_x - \partial_v^2,$$

to provide a model for Brownian motion in one dimension. Hörmander took up the study of general operators

$$(6) \qquad \mathcal{H} = X_0 - \sum_{1\le j\le r} X_j^2,$$

where the $(X_j)_{0\le j\le r}$ are smooth real vector fields whose Lie algebra generates the tangent space at each point: the rank of the X_j and their iterated Poisson brackets is equal to the dimension of the ambient space (for $\mathcal{K}$, we have $X_0 = \partial_t + v\partial_x, X_1 = \partial_v, [X_1, X_0] = \partial_x$). These operators were proven to be hypoelliptic in the 1967 article [37]: (1) holds with $P = \mathcal{H}$ for the C^∞ singular support. This paper was the starting point of many studies, including numerous articles in probability theory, and the operators $\mathcal{H}$ soon became known as *Hörmander's sum of squares.* Their importance in probability came from the fact that these operators appeared as a generalization of the heat equation where the diffusion term $\sum_{1\le j\le r} X_j^2$ was no longer elliptic but had instead some hypoelliptic behavior. Chapter XXII in Hörmander's book [45] is concerned with hypoelliptic pseudodifferential operators: on the one hand, operators with a pseudodifferential parametrix, such as the hypoelliptic constant coefficient operators, and on the other hand generalizations of the Kolmogorov operators (6). Results on lower bounds for pseudodifferential operators due to A. Melin [60] are a key tool in this analysis. Results of L. Boutet de Monvel [13], J. Sjöstrand [76], L. Boutet de Monvel, and A. Grigis and B. Helffer [14] are also given in that chapter. Chapter XIX in [45] deals with elliptic operators on a manifold without boundary and the index theorem. In the Notes of Chapter XVIII, Hörmander writes: *It seems likely that it was the solution by Atiyah and Singer* [5] *of the index problem for elliptic operators which led to the revitalization of the theory of singular integral operators.*

Spectral Asymptotics

The article [38], published in 1968, provides the best possible estimates for the remainder term in the asymptotic formula for the spectral function of an arbitrary elliptic differential operator. This is achieved by means of a complete description of the singularities of the Fourier transform of the spectral function for low frequencies. The method of proof for a positive elliptic operator P of order m on a compact manifold is using the construction of a parametrix for the hyperbolic equation $i\partial_t + P^{1/m}$, which is formally $\exp itP^{1/m}$, an operator that can be realized as a *Fourier integral operator.*

The Microlocal Revolution, Act II

Propagation of Singularities

The fact that singularities should be classified according to their spectrum was recognized first in the early 1970s by three Japanese mathematicians: the Lecture Notes [75] by M. Sato, T. Kawai, and M. Kashiwara set the basis for the analysis in the phase space and microlocalization. The analytic wave-front set was defined in algebraic terms and elliptic regularity, and propagation theorems were proven in the analytic category. The paper [15] by J. Bros and D. Iagolnitzer gave a formulation of the analytic wave-front set that was more friendly to analysts. The definition of the C^∞ wave-front set was given by Hörmander in [40]. For Ω an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, $(x_0, \xi_0) \in \Omega \times \mathbb{S}^{n-1}$ belongs to the *complement of* WFu means that there exists a neighborhood $U \times V$ of (x_0, ξ_0) such that $\forall \chi \in C_c^\infty(U), \forall N \in \mathbb{N}$,

$$(7) \qquad \sup_{\lambda\ge 1, \xi \in V} |\widehat{\chi u}(\lambda\xi)|\lambda^N < \infty.$$

The propagation of singularities theorem for real principal-type operators (see [75] for the analytic wave-front set and Hörmander's [41] for the C^∞ wave-front set) represents certainly the apex of microlocal analysis. Since the seventeenth century, with the works of Huygens and Newton, the mathematical formulation for propagation of linear waves lacked correct definitions. The wave-front set provided the ideal framework: for P a real principal-type operator with smooth coefficients (e.g., the wave equation) and u a function such that $Pu \in C^\infty$, WFu is invariant by the flow of the Hamiltonian vector field of the principal symbol of P.

Fourier Integral Operators

The propagation results found new proofs via Hörmander's articles on Fourier integral operators [39] and [20] (joint work with J. Duistermaat). It is interesting to quote at this point the introduction of [39] (the reference numbers are those of our reference list): *The work of Egorov is actually an application of ideas from Maslov* [59] *who stated at* *the International Congress in Nice that his book actually contains the ideas attributed here to Egorov* [22] *and Arnold* [4] *as well as a more general and precise operator calculus than ours. Since the book is highly inaccessible and does not appear to be quite rigorous we can only pass this information on to the reader, adding a reference to the explanations of Maslov's work given by Buslaev* [16]. *In this context we should also mention that the "Maslov index" which plays an essential role in Chapters III and IV was already considered quite explicitly by J. Keller* [51]. *It expresses the classical observation in geometrical optics that a phase shift of $\pi/2$ takes place at a caustic. The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential operators.*

The simplest example of a Fourier integral operator U is given by the formula

$$(8)\qquad (Uv)(x) = \int e^{i\phi(x,\eta)} c(x,\eta)\hat{v}(\eta)d\eta(2\pi)^{-n},$$

where the real phase ϕ is (positively) homogeneous with degree 1 in η such that

$$\det \partial^2\phi/\partial x\partial\eta \neq 0,$$

and c is some amplitude behaving like a symbol. Some operators of this type were already introduced in 1957 in P. Lax's paper [54] as parametrices of hyperbolic operators. A fundamental theorem due to Y. V. Egorov [22] is that FIO are quantizing asymptotically canonical transformations in the sense that

$$(9)\qquad U^* a(x,D)U \equiv (a \circ \chi)(y,D) \mod \mathcal{P}^{m-1},$$

for any symbol a of order m, where χ is the canonical transformation naturally attached to the phase ϕ and $\mathcal{P}^{m-1}$ stands for pseudodifferential operators with order $m-1$.

Local Solvability

After Lewy's example (2) and Hörmander's work on local solvability, L. Nirenberg and F. Treves in 1970 [68], [69], [70], after a study of complex vector fields in [67] (see also the S. Mizohata paper [65]), introduced the so-called condition (Ψ) and provided strong arguments suggesting that this geometric condition should be equivalent to local solvability. The necessity of condition (Ψ) for local solvability of principal-type pseudodifferential equations was proved in two dimensions by R. Moyer in [66] and in general by Hörmander [44] in 1981. The sufficiency of condition (Ψ) for local solvability of differential equations was proved by R. Beals and C. Fefferman [8] in 1973. They created a new type of pseudodifferential calculus, based on a Calderón-Zygmund decomposition, and were able to remove the analyticity assumption required by L. Nirenberg and F. Treves. The sufficiency of that geometric condition was proven in 1988 in two dimensions by N. Lerner [55]. Later in 1994, Hörmander, in his survey article [47], went back to local solvability questions, giving a generalization of Lerner's article [56]. In 2006, N. Dencker [19] proved that condition (Ψ) implies local solvability with a loss of two derivatives.

More on Pseudodifferential Calculus

A most striking fact in R. Beals and C. Fefferman's proof was the essential use of a nonhomogeneous pseudodifferential calculus which allowed a finer localization than what could be given by conic microlocalization. The efficiency and refinement of the pseudodifferential machinery was such that the very structure of this tool attracted the attention of several mathematicians, among them R. Beals and Fefferman [7], Beals [6], and A. Unterberger [80]. Hörmander's 1979 paper [43], "The Weyl calculus of pseudodifferential operators," represents an excellent synthesis of the main requirements for a pseudodifferential calculus to satisfy; that article was used by many authors in multiple circumstances, and the combination of the symplectically invariant Weyl quantization along with the datum of a metric on the phase space was proven to be a very efficient approach.

The thirty-page presentation of the *Basic Calculus* in Chapter XVIII of [45] is concerned with pseudodifferential calculus and is an excellent introduction to the topic. R. Melrose's totally characteristic calculus [62] and L. Boutet de Monvel's transmission condition [12] are given a detailed treatment in this chapter. The last sections are devoted in part to results on new lower bounds by C. Fefferman and D. H. Phong [25]. Chapter XX in [45] is entitled "Boundary Problems for Elliptic Differential Operators." It reproduces at the beginning elements of Chapter X in [35] and takes into account the developments on the index problem for elliptic boundary problems given by L. Boutet de Monvel [12], [11] and G. Grubb [27]. Chapter XXIV in [45] is devoted to the mixed Dirichlet-Cauchy problem for second-order operators. Singularities of solutions of the Dirichlet problem arriving at the boundary on a transversal bicharacteristic will leave again on the reflected bicharacteristic. The study of tangential bicharacteristics required a new analysis and attracted the attention of many mathematicians. Among these works: the papers by R. Melrose [61], M. Taylor [78], G. Eskin [24], V. Ivriĭ [50], R. Melrose and J. Sjöstrand [63], [64], K. Andersson and R. Melrose [3], J. Ralston [74], and J. Sjöstrand [77].

Subelliptic Operators

A pseudodifferential operator of order m is said to be subelliptic with a loss of δ derivatives whenever

$$(10)\qquad Pu \in H^s_{loc} \Longrightarrow u \in H^{s+m-\delta}_{loc}.$$

The elliptic case corresponds to $\delta = 0$, whereas the cases $\delta \in (0, 1)$ are much more complicated to handle. The first complete proof for operators satisfying condition (P) was given by F. Treves in [79], using a coherent states method (see also Section 27.3 of Hörmander's [46]). Although it is far from an elementary proof, the simplifications allowed by condition (P) permit a rather compact exposition. The last three sections of Chapter XXVII in [46] are devoted to the much more involved case of subelliptic operators satisfying condition (Ψ), and one could say that the proof is extremely complicated. Let us cite Hörmander in [49]: *For the scalar case, Egorov* [21] *found necessary and sufficient conditions for subellipticity with loss of δ derivatives ($\delta \in [0, 1)$); the proof of sufficiency was completed in* [42]. *The results prove that the best δ is always of the form $k/(k+1)$ where k is a positive integer.…A slight modification of the presentation of* [42] *is given in Chapter 27 of* [46], *but it is still very complicated technically. Another approach which covers also systems operating on scalars has been given by Nourrigat* [71], [72] *(see also the book* [28] *by Helffer and Nourrigat), but it is also far from simple so the study of subelliptic operators may not yet be in a final form.*

Nonlinear Hyperbolic Equations

In 1996, Hörmander's book appeared [48]. The first subject which is treated is the problem of long-time existence of small solutions for nonlinear waves. Hörmander uses the original method of S. Klainerman [52]. It relies on a weighted L^∞ Sobolev estimate for a smooth function in terms of L^2 norms of $Z^I u$, where Z^I stands for an iterate of homogeneous vector fields tangent to the wave cone. The chapter closes with a proof of global existence in 3D when the nonlinearity satisfies the so-called "null condition," i.e., a compatibility relation between the nonlinear terms and the wave operator.

The last part of the book is concerned with the use of microlocal analysis in the study of nonlinear equations. Chapter 9 is devoted to the study of pseudodifferential operators lying in the "bad class" $S^0_{1,1}$. The results of Chapter 9 are applied in Chapter 10 to construct Bony's paradifferential calculus [9], [10]. One associates to a symbol $a(x, \xi)$, with limited regularity in x, a paradifferential operator and proves the basic theorems on symbolic calculus, as well as "Bony's paraproduct formula." Next, Bony's paralinearization theorem is discussed: it asserts that if F is a smooth function and u belongs to $C^\rho (\rho > 0)$, $F(u)$ may be written as $Pu + Ru$, where P is a paradifferential operator with symbol $F'(u)$ and R is a ρ-regularizing operator. This is used to prove microlocal elliptic regularity for solutions to nonlinear differential equations. The last chapter is devoted to propagation of microlocal singularities, where the author proves Bony's theorem on propagation of weak singularities for solutions to nonlinear equations.

Final Comments

After this not-so-short review of Hörmander's works on PDE, we see in the first place that he was instrumental in the mathematical setting of Fourier integral operators (achieved in part with J. Duistermaat) and also in the elaboration of a comprehensive theory of pseudodifferential operators. Fourier integral operators had a long heuristic tradition, linked to quantum mechanics, but their mathematical theory is indeed a major lasting contribution of Lars Hörmander. He was also the first to study what's now called *Hörmander's sum of squares* of vector fields and their hypoellipticity properties. These operators are important in probability theory and geometry but also gained a renewed interest in the recent studies of regularization properties for Boltzmann's equation and other nonlinear equations.

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Jean-Pierre Demailly

Lars Hörmander and the Theory of L^2 Estimates for the $\bar\partial$ Operator

I met Lars Hörmander for the first time in the early 1980s on the occasion of one of the "Komplexe Analysis" conferences held in Oberwolfach under the direction of Hans Grauert and Michael Schneider. My early mathematical education had already been greatly influenced by Hörmander's work on L^2 estimates for the $\bar\partial$-operator in several complex variables. The most basic statement is that one can solve an equation of the form $\bar\partial u = v$ for any given (n,q)-form v on a complex manifold X such that $\bar\partial v = 0$, along with a fundamental L^2 estimate of the form $\int_X |u|^2 e^{-\varphi} dV_\omega \le \int_X \gamma_q^{-1} |v|^2 e^{-\varphi} dV_\omega$. This holds true whenever φ is a plurisubharmonic function such that the right-hand side is finite and X satisfies suitable convexity assumptions, e.g., when X possesses a weakly plurisubharmonic exhaustion function. Here dV_ω is the volume form of some Kähler metric ω on X, and $\gamma_q(x)$, at any point $x \in X$, is the sum of the q smallest eigenvalues of $i\partial\bar\partial\varphi(x)$ with respect to $\omega(x)$. This was in fact the main subject of a PhD course delivered by Henri Skoda in Paris during the year 1976–77, and, to a great extent, the theory of L^2 estimates was my entry point into complex analysis of several variables. At the same time, I followed a graduate course of Serge Alinhac on PDE theory, and Lars Hörmander appeared again as one of the main heroes. I was therefore extremely impressed to meet him in person a few years later—his tall stature and physical appearance did make for an even stronger impression. I still remember that on the occasion of the Wednesday afternoon walk in the Black Forest, Hörmander was in a group of two or three that essentially left all the rest behind when hiking on the somewhat steep slopes leading to the Glaswaldsee, a dozen kilometers north of the Mathematisches Forschungsinstitut Oberwolfach.

It seems that Lars Hörmander himself, at least in the mid 1960s, did not consider his work on $\bar\partial$-estimates [13] to stand out in a particular way among his other achievements; after all, these estimates appeared to him to be only a special case of Carleman's technique, which also applies to more general classes of differential operators. In his own words, *Apart from the results involving precise bounds, this paper does not give any new existence theorems for functions of several complex variables. However, we believe that it is justified by the methods of proof.* In spite of this rather modest statement, the paper already permitted

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one to bypass the difficult question of boundary regularity involved in the Morrey-Kohn approach [22], [18], [19], [20]. For this, Hörmander observes that the Friedrichs regularization lemma applies to the particular situation he considers. Also, and perhaps more importantly, Hörmander's technique gives a new proof of the existence of solutions of $\overline{\partial}$-equations on pseudoconvex or q-convex domains, thus recovering the results of Andreotti-Grauert [1] by a more direct analytic approach. We should mention here that Andreotti-Vesentini [2] independently obtained similar results in the context of complete Hermitian manifolds through a refinement of the Bochner-Kodaira technique [3], [16], [17]. One year after the publication of [13], Lars Hörmander published his tremendously influential book *An Introduction to Complex Analysis in Several Variables* [14], which is now considered to be one of the foundational texts in complex analysis and geometry.

It took only a few years to realize that the very precise L^2 estimates obtained by Hörmander for solutions of $\overline{\partial}$-equations had terrific applications to other domains of mathematics. Chapter VII of [14] already derives a deep existence theorem for solutions of PDE equations with constant coefficients. More surprisingly, there are also striking applications in number theory. In 1970, Enrico Bombieri extended in this way an earlier result of Serge Lang concerning algebraic values of meromorphic maps of finite order: if a system of such functions has transcendence degree larger than the dimension and satisfies algebraic differential equations, then the set of points where they simultaneously take values in an algebraic number field is contained in a certain algebraic hypersurface of bounded degree. The proof combines use of Lelong's theory of positive currents with L^2 estimates for arbitrary singular plurisubharmonic weights; cf. [4]. It is crucial here to allow φ to have poles, e.g., logarithmic poles of the form $\log \sum |g_j|^2$ where the g_j's are holomorphic functions sharing common zeroes (the degree bounds were later refined by Henri Skoda [30] and [21]). Using Hörmander's techniques in a fundamental way, Henri Skoda gave further major applications to complex analysis and geometry. In [28] it is shown that analytic subsets of $\mathbb{C}^n$ possessing a given growth of the area at infinity can be defined by holomorphic functions with a precise control of the growth; such quantitative versions of Cartan's theorems A and B had also been anticipated in [14] (see e.g., section 7.6, "Cohomology with bounds"). The article [29], on the other hand, gives an almost optimal L^2 estimate for the solutions of Bézout equations $\sum g_j h_j = f$ in the ring of holomorphic functions (if f and the g_j's are given so that a certain quotient $|f|^2|g|^{-2(n+1+\varepsilon)}e^{-\varphi}$ is integrable, then h can be found such that $|h|^2|g|^{-2(n+\varepsilon)}e^{-\varphi}$ is integrable). This result, in its turn, yields profound results of local algebra, e.g., in the form of a fine control of the integral closure of ideals in the ring of germs of holomorphic functions [5]; see also [31] for more related geometric statements. It is remarkable that algebraic proofs of such results were found only after the discovery of the analytic proof. About the same time, Yum-Tong Siu [26] gave a proof of the analyticity of sublevel sets of Lelong numbers of closed positive currents; Siu's main argument again involves the Hörmander-Bombieri technique.

But the story does not stop there; Hörmander's L^2 estimates also have intimate connections with fundamental questions of algebraic geometry. In fact, Takeo Ohsawa and Kensho Takegoshi discovered in 1987 a very deep L^2 extension theorem: every L^2 holomorphic function defined on a subvariety Y of a complex manifold X can be extended as a holomorphic function F on X satisfying an L^2 bound $\int_X |F|^2 e^{-\varphi} dV_X \le C \int_Y |f|^2 e^{-\varphi} dV_Y$. This holds true provided φ is plurisubharmonic and suitable curvature assumptions are satisfied [24]. The initial proof used a complicated, twisted Bochner-Kodaira-Nakano formula, but it was recently discovered by Bo-Yong Chen [6], [7] that the Morrey-Kohn-Hörmander estimates were in fact sufficient to prove it, while improving the estimates along the way. The Ohsawa-Takegoshi L^2 extension theorem itself has quite remarkable consequences in the theory of analytic singularities, for instance a basic regularization theorem for closed positive currents [9] or a proof of the semicontinuity of complex singularity exponents [11]. Finally, [24] can be used to confirm the conjecture on the invariance of plurigenera in a deformation of projective algebraic varieties [27], [25]: this basic statement of algebraic geometry still has no algebraic proof as of this date! Another very strong link with algebraic geometry occurs through the concept of multiplier ideal sheaves: if φ is an arbitrary plurisubharmonic function, then the ideal sheaf of germs of holomorphic functions such that $|f|^2 e^{-\varphi}$ is integrable is a coherent analytic sheaf; the proof is essentially a straightforward consequence of the Hörmander-Bombieri technique. In general, if $(L, e^{-\varphi})$ is a singular hermitian line bundle on a compact Kähler manifold X, one has $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0$ as soon as the curvature of φ is positive definite (a generalization of the Kawamata-Viehweg vanishing theorem [15], [32]; cf. [8], [23], [10]). In case $i\partial\overline{\partial}\varphi$ is merely semipositive, one gets instead a surjective Lefschetz morphism $\omega^q \wedge \bullet :$ $H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(\varphi)) \to H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi))$ [12]. All the arguments use Hörmander's theory of L^2 estimates in one way or the other. Contrary to the exceedingly modest words of Hörmander, the L^2 existence theorem for solutions of $\overline{\partial}$-equations appears to be one of the most powerful theorems of contemporary mathematics!

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Michael Atiyah

I must have met Lars Hörmander in the early 1960s, probably on one of many trips to Lund, where I also collaborated with the other Lars (Gårding). At various stages Hörmander and I discussed elliptic differential equations and index theory. From these discussions and from his book, with its excellent treatment of distributions, I got my education in modern analysis.

Later we overlapped for a year at the Institute for Advanced Study in Princeton. By the time I arrived he had already decided to return permanently to Lund. But before he left he instructed me in the art of log-splitting with wedges, something which every Swedish boy picks up and which I found

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useful later when I acquired a Scandinavian log cabin in the Scottish Highlands.

I think the transition from Princeton to Lund was not easy for Lars. He was at home in the Swedish countryside and culture, but mathematically he was somewhat isolated, despite his fame and his international connections.

Lars had one of the most penetrating intellects I have ever come across. I remember when I was struggling with the linear algebra of Novikov additivity for the signature of manifolds, Lars just disposed of the problem in minutes.

Lars and I met at numerous international conferences. but I remember particularly a joint trip to Japan, where the tall blond Scandinavian towered over all the Japanese (and me), attracting universal attention, particularly from parties of Japanese schoolgirls.

With Atiyah and three Japanese ladies in Japan, 1969.

François Treves

This is much too short a space, for me, in which to condense a lifetime of complicated interaction with Lars Hörmander. We met for the first time in February 1958. I had finished the writing of my *thèse de Doctorat* and convinced a (nonmathematician) friend to drive us from Paris to Stockholm, where I was eager to meet Hörmander for a week of vacation in the northern snows. We arrived Saturday night and went directly to Drottninggatan (Queen Street, not pedestrian at the time), where the mathematics department was then located. I wanted to have a feel for the place, but what we saw was a not very wide, brightly lit street in downtown Stockholm with a surprising number of very drunk men ricocheting along the sidewalks, from the tightly parked cars to the walls. On Sunday morning Hörmander picked me up in his Volvo and drove us to the Mittag-Leffler Institute in Djursholm, where he was living. Unbeknownst to me I was the bearer of bad news from Paris: Lojasiewicz had just proved the divisibility of distributions by analytic functions (and much more), while Hörmander had just finished writing his paper on the division of distributions by polynomials (making use of the Seidenberg principle and implying the existence of a tempered fundamental solution for any PDE with constant coefficients).

At the time Hörmander was twenty-six, one year younger than I and already famous. Outside the circle of the participants in the Laurent Schwartz seminar at the Institut Henri Poincaré (a small circle indeed: Lions, Malgrange, Malliavin, Martineau, and a few others), nobody knew me—nobody, that is, except Hörmander. We had exchanged letters since 1956, and Schwartz had asked me to give a couple of lectures at the quarterly Bourbaki seminar on Hörmander's PhD thesis, recently published in *Acta Mathematica*. Those lectures (my first) had been a revelation to me, mainly about the standing of analysis in the mathematical firmament: they had been preceded by lectures in algebra and topology. With my arrival at the podium there was a mass exodus from the large Hermite auditorium. A few people stayed; I remember Claude Chevalley and Henri Cartan among my sparse public—out of pity, I guess, for the novice at the blackboard.

In 1959 Hörmander visited Berkeley (where I had my first job, an assistant professor position), bringing his very novel explanation of Hans Lewy's counterexample to the local solvability of a smooth, nowhere null, complex vector field L—in modern language, the nonvanishing of the Levi form $\frac{1}{2i}\left[L,\overline{L}\right]$ at a characteristic point. This breakthrough was destined to have a determinative impact on my future research, more than I could have imagined at the time. More importantly, from about that date on Hörmander's work was to have an overarching influence on the development of the modern theories of several complex variables and of linear PDE (with ramifications in probability theory). Prior sections of this article by other authors describe the scope of Hörmander's oeuvre, so I will limit myself here to a couple of very personal recollections.

My relations with Hörmander were always courteous and often cordial, even warm, I would say, in the last decades. There was one exception: at a conference in Nice in 1972 on the topic of Fourier integral operators and symplectic geometry, his attitude towards me seemed uncharacteristically cold, and I asked friends what could be the cause. Hörmander had recently submitted a paper to the *Annals* in which he proved a necessary and sufficient condition for a PDE with constant coefficients to have an analytic solution for every analytic right-hand side. That the heat equation in 3D did not have this property had been discovered earlier by E. De Giorgi, who, incidentally, openly

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refused to seek a necessary and sufficient condition, *so as not to discourage other mathematicians from getting interested in the problem.* Hörmander's paper had been returned to him for revision on the basis of some—rather mild I was told—objections by the referee. Apparently, Lars had convinced himself that I was the referee, which I was not, and he was showing to common friends a list of twenty points indicative of my personal style. I was never shown the list (nor the referee report!), but someone told me what point #1 was: that my correspondence was always typewritten, which was true. (I am, to this day, curious about the nineteen other "qualities" I shared with the culprit.) Nirenberg told Hörmander that *I had too much respect for him* to be the referee (a well-intentioned exaggeration, perhaps), while B. Malgrange put to me a sweet request: *May I tell Lars that the referee was not born in Europe?*. Bernard, I, and the others knew this to be true.

My last stay in Lund was part of a series of visits by mathematicians to celebrate the final settling, by N. Dencker, of the circle of solvability problems started fifty years earlier with the Lewy example (to which Nirenberg and I had made contributions, as well as R. Beals and Ch. Fefferman, R. Moyer, and, later, N. Lerner). During that visit, Lars took my wife and me on a day tour by car of Skåne: first to Ystad, which I had wished to visit because of Wallander's travails and which turned out to be an old, lovely small town, nothing like what I expected from reading Mankel's novels, and afterwards to Trelleborg, where my wife had worked in her early twenties. She left us to revisit old places. As in every one of my not-so-rare stays in Lund, the weather was splendid. We sat under the trees, two old professors indulging in a bit of nostalgia. That is when I learned that Hörmander had written a number of historical essays on the twentieth-century mathematical developments in which he had participated; they were in the safekeeping of Lund University for publication after his death. One of them, devoted to the $\bar{\partial}$ Neumann problem, had already circulated. Having read it, I was pretty confident that the unpublished ones would be as severely impartial and as valuable to interested historians and mathematicians.

Lars insisted on taking us to the train to Copenhagen the next morning. We both knew that this was the last time, for, though he appeared to be well, it was the tenth year of his illness and he had decided to terminate all medication.

During lunch in Ystad I had mentioned that in the diet to which celiac disease confined me, what I missed most was beer. A few weeks after our return home I received an email from his daughter with a list of gluten-free beers; I have been enjoying one of the brands ever since, for which I am very grateful to both him and his daughter. A minor, but still pleasurable, addition to my mathematical debt to Lars Hörmander.

Sigurdur Helgason

My contact with Hörmander started with my use of his 1963 book, which I used several times in a course on distributions. Already his work was having major impact on the large field of partial differential equations. At that time I was involved in the study of differential equations on a homogeneous space G/K invariant under the natural action of G. Constant coefficient differential operators are the first natural example. While invariant operators in the above sense are plentiful in nature, one's optimism is quickly dampened by Levy's example,

$$\delta_x + i\delta_y - 2i(x+iy)\delta_z,$$

of an operator which is not locally solvable yet closely related to an operator which is left invariant on the Heisenberg group.

At the Institute for Advanced Study in 1964 I had the opportunity of discussing these matters with Hörmander. The same afternoon he came up with the result that if G is a Lie group for which every first-order invariant operator is locally solvable, then G is either abelian or has a normal abelian subgroup of codimension one. This he derived quickly from his necessary condition of local solvability, specialized to first-order operators. The result was also proved by Cerezo and Rouvière with further explicit details.

This was typical of my experience with Hörmander. When asking him a mathematical question, one got back much more than just the answer. Another example of this is my questioning him about the property of a constant coefficient differential operator being a homeomorphism of the space of test functions onto its image. Ehrenpreis's treatment of this question did not convince me. Since I needed this result in my research, I consulted Hörmander. He was optimistic, but the full proof did not emerge until the second volume of his magnificent four-volume opus *The Analysis of Linear Partial Differential Operators*. Theorem 15.4.2 there gives an explicit intrinsic description of the topology of the Fourier transform space $\tilde{\mathcal{D}}$. Since Schwartz's topology of $\mathcal{D}$ is rather complicated, the topology of $\tilde{\mathcal{D}}$ is fairly complicated too. However, the result is powerful and the proof shows the touch of a master, also in the analysis of several complex variables. Since the proof can be reduced so as to need Cauchy's theorem in just one variable, I have found it ideally suited to a course on distribution theory. For example, it leads very quickly to the

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existence theorem $L\mathcal{D}' = \mathcal{D}'$ for any constant coefficient differential operator L.

Gerd Grubb

It was during my PhD studies at Stanford University (completed in 1966) when I first met Lars Hörmander. I did not follow his lectures then, but did so later during summer visits to Stanford in 1967 and 1971. The first time I asked him a question was on the Calderón projector put forward by Robert Seeley in [6], where he kindly explained to me the essentially equivalent construction in his *Annals* paper [3].

Back in Denmark around 1970 I organized an interuniversity study group with the aim of understanding the new results on hypoellipticity; it culminated with Lars's participation in a workshop we organized in Århus in spring 1972 (where some French colleagues also came: A. and J. Unterberger, C. Zuily, M. Derridj). That spring I was also occupied with an honorable task Lars had given me: to be the faculty opponent on the thesis [7] of Johannes Sjöstrand in Lund. (Here he kidded me with the fact that the part I found most strange was the one he had insisted on.)

Besides producing wonderful mathematics himself, Lars had a great influence on the work of all with whom he had contact. He inspired everyone around him to sharpen their efforts and formulations in research questions in analysis, to make simplifications by use of functional analysis such that one would reach the really hard questions that demanded original tricks or new theories. But being in his vicinity did not increase one's feeling of self-importance.

The Danish "hypoelliptic study group" was revived later when we were invited to read preliminary versions of his four-volume treatise [4]. I was allowed a small influence on the chapter on elliptic boundary problems. A particular effort was made by Niels J. Kokholm (who also received special thanks) at the same time as he was working on a thesis under Lars's guidance in Lund. Kokholm and I later carried a large project through [2], but he quit mathematics for more concrete IT jobs after this.

In 1985 we started the Danish-Swedish Analysis Seminar, where Lars and I, together with Anders Melin, organized one-day meetings, alternating between Lund and Copenhagen, once or twice per semester. This made it possible to have a sufficiently large audience for a lot of interesting guests, and it served the whole PDE community in Denmark.

In the 1980s it became possible to write manuscripts in TEX; Richard Melrose brought macros from MIT, and Lars passed them on to me, introducing me to this clever way to master mathematical formulation.

I have consulted Lars on many details in my ongoing research and refereeing jobs and have learned enormously from him; I have also given comments on some of his writings. We made one joint publication [1], on pseudodifferential operators satisfying the transmission condition (preserving smoothness up to a boundary) with symbols in $S^m_{\varrho,\delta}$-spaces, including results on Poisson operators of this type. The work was quite difficult; whenever I wrote something, a formulation or deduction by Lars would usually win over it, but after a lot of work back and forth it ended as a nice informative piece, I think.

A bit uncommonly, Lars detested being the center of a celebration. Therefore he often fled from his home in Lund when a special birthday was approaching. On the other hand, to organize a meeting gives one the chance to invite people in return for their hospitalities. In 1995, when Lars's retirement was approaching, he and I, jointly with Anders Melin and Johannes Sjöstrand, used the framework of the Danish-Swedish analysis seminar to arrange two 3-day meetings [5], where we fit in as many of his good international colleagues as possible, having feasts in Copenhagen and Lund each time. In 2006 (a birthday year) again many visitors came informally to Lund.

On the private side, we shared an interest in the wild flowers of Skåne and Blekinge, in particular rare Nordic orchids. This is a very sad occasion to think back on those years.

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Jean-Michel Bony

I first met Lars Hörmander at the International Congress of Mathematicians (Nice, 1970), but I was already quite familiar with his work. One of my first papers, devoted to sums of squares of vector fields, relied on his result of hypoellipticity. Above all, it is in his book *Linear Partial Differential Operators* that I learned the theory of PDE when I was a student in École Normale Supérieure. I could appreciate then the precision and the concision of his style: a complete exposition in eighteen pages of the theory of distributions, which was at that time the subject of a one-semester course in Paris.

Even though the words were not spoken, one can say that *microlocal analysis* was born at this ICM, with the lectures of Mikio Sato and Youri Egorov, and the plenary lecture of Lars Hörmander. For the first time, Hörmander defined the wave-front set, stated and proved his fundamental theorem on the propagation of singularities, and announced his monumental work (partly with J. J. Duistermaat) on Fourier integral operators.

A few months later I had the surprise of receiving a preprint of L. Hörmander, giving another proof, and an improvement, of my own contribution to the congress (an extension of Holmgren's theorem). I should not have been surprised. If L. Hörmander had been for forty years the foremost contributor to the theory of linear PDE, I think that it is due to three reasons: many outstanding theorems, of course, but also the fact that he gave us fundamental tools such as Fourier integral operators or his successive extensions of the pseudodifferential calculus. The third reason is that quite frequently he rewrote the results of other mathematicians, trying to extend their generality and above all to give proofs which link them to a small number of fundamental concepts and results. All this was converging towards a new treatise on the subject, which he had in mind as early as in the beginning of the 1970s.

I met L. Hörmander rather frequently after that, particularly during two long stays at the Mittag-Leffler Institute, where he organized two one-year thematic programs on PDE. During the first one, in 1974, he looked quite interested in hyperfunction theory, and I can find traces of our talks on this topic in Chapter 9 of his treatise. The manuscript was already quite advanced, carefully stored in three binders (the last volume was divided later), and I remember the clatter of his typewriter between the seminars. Ten years later the second program was mainly devoted to nonlinear PDE, the treatise was completed, and the typewriter was replaced by a noiseless computer.

My talks with Lars Hörmander were not limited to mathematics. He was fond of French literature, reading in French, for instance, novels by Stendhal and more recently *A la recherche du temps perdu*. He did not like to speak French, probably because his French, though quite good, was not perfect. However, he made an exception for his answer when he was made *Doctor Honoris Causa* of the University of Paris-Sud at Orsay.

The development of the theory of PDE has been and still is a wonderful journey. Without Lars Hörmander, this development would have been certainly much slower, and it could have produced a messy and tangled heap of results and specific methods. We are all indebted to him and will remain so for a long time.

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Christer O. Kiselman

Lars Hörmander was appointed professor at Stockholm University College effective January 1, 1957, not yet twenty-six years old. I started my studies there in the fall of 1957 and soon became aware of his existence. The student newspaper *Gaudeamus* published a report from the installation ceremony with the heading "Twenty-six-year-old mathematics machine solemnly installed."

During my first years I had no contact with Hörmander, but I had several teachers who were students of his: Benny Brodda, Vidar Thomée, Göran Björck, Stephan Schwarz, and Lars Nystedt. In October 1960 I talked with Olof Hanner about a possible continuation of my studies, and I mentioned that I would like to have Lars Hörmander as my advisor. Olof was not surprised; he just remarked that the reputation of his young colleague had spread efficiently.

Lars was always available for consultations. One knocked at his door—he could answer any question immediately. I never experienced any difficulty in talking mathematics with him. Most often he was typing articles or chapters of his book, the one which was to appear in 1963. The clattering was intense. He also typed lecture notes, which were mimeographed using the technology of the time. When I left his office the clattering resumed after zero seconds.

Lars lectured on partial differential equations during the academic year 1961–62. These lectures foreshadowed the Springer book that came out in 1963. We, his students, read the manuscript and commented on it. I wrote a licentiate thesis on a problem proposed by him concerning approximation of solutions to partial differential equations with constant coefficients.

During the academic year 1962–63, Lars gave a series of lectures on analysis in several complex

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variables. In this way he started the creation of his book that came out with Van Nostrand in 1966 and which is now one of the most quoted in the area. The organization of the lectures was exactly the one we now see in the book. Holomorphic functions were considered as solutions to differential equations, which was then a new approach for me and which gave powerful construction methods and made generalizations possible. It was a great experience to witness the birth of this book.

During the fall semester of 1963, Lars gave a series of seminars on convex and subharmonic functions. More than thirty years later, in 1994, he published his book *Notions of Convexity*, which takes up these topics with a unified treatment of convex, subharmonic, and plurisubharmonic functions.

Most important were three seminars on L^2-methods for the $\bar{\partial}$ operator, which he gave in the fall of 1963. The results then appeared in *Acta Mathematica* in 1965 in a groundbreaking article.

In 1964 Lars left Stockholm for Stanford and Princeton. He visited Stockholm in May 1965 and gave four lectures on pseudodifferential operators, an area that was then rather new.

Robert Oppenheimer offered me a membership in the Institute for Advanced Study in Princeton, NJ, for the academic year 1965–66. Lars was there then, and of course it was he who had arranged everything for me.

It became a most valuable year in every respect, both mathematically and culturally. I was there with my wife Astrid and our son Dan, who turned two during the fall. We arrived early on one of the first days in July while Lars and Viveka were in Sweden. They let us stay in their house during the summer. As a small service in return, we took care of their dog Shilly-Shally.

Lars gave a series of lectures at the institute with the title "Pseudo-differential operators and boundary problems." This was an elaboration and extension of the lectures he had given at Stockholm University in May 1965. Furthermore, Lars gave a seminar within the framework of the Current Literature Seminar on "The Lefschetz fixed point formula for elliptic complexes." I was in constant contact with Lars during that year and wrote a paper on the growth of entire functions and on analytic functionals. Another valuable contact was Miguel Herrera (1938–84), with whom I studied residue theory.

After that year I had many contacts with Lars concerning complex convexity and fundamental solutions as well as many other topics, especially after his return to Lund in 1968. I was the faculty opponent when his students Arne Enqvist and Ragnar Sigurdsson presented their PhD theses. On the last day of 2010, he sent me a new, strong theorem on the regularity of fundamental solutions, an excellent addition to the results in his four-volume book.

Photograph by Vidar Thomée.

With wife, Viveka, and Airedale terrier Shilly at their house in Princeton, 1965.

Nobody has been so important for my intellectual and scientific development as Lars.

Sofia Broström

My father was born on January 24, 1931, in a small fishing village in southern Sweden as the youngest of five children. His father was a school teacher—originally trained as a painter, he had paid his own way to a teaching degree at twenty-one—who had married a bright farmer's daughter who recalled her few years in school as a highlight of her life. Together they created an atmosphere characterized by a strong thirst for knowledge and education, and all five children graduated quickly from secondary school and then continued with higher education.

My father went through school even quicker than the others: he skipped two years of elementary school, with the result that he graduated with the usual interval of two years between the siblings, although in reality he was four years younger than his youngest sister. Thanks to an enthusiastic teacher in secondary school, Nils-Erik Fremberg, he had also finished the first semester of university mathematics when graduating at seventeen. Despite previous plans to become an engineer like his older brother, this made him decide to continue with more mathematics and physics, receiving his master's degree at nineteen

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With daughter Gisela in Princeton, 1960.

and his licentiate degree (PhD) a year and a half later in October 1951 at age twenty.

At the department Christmas party of that year he met my mother, Viveka. They married while my father was in military service and had their first daughter, Gisela, in October 1954. One year later my father received his doctorate through the dissertation that made him famous. Since my mother had only been able to stay at home to take care of Gisela for the first few months, much of the dissertation was actually written with my sister in his (pulled-out) desk drawer (which was just the right size for the carriage insert). He apparently didn't find this distracting, which was typical of my father—the only effect an interruption seemed to have on his work was the few minutes lost to the actual distraction, nothing more.

It is not for me, however, to give a full biography of my father's life, so let me skip to my own

Photograph by Eva Enderlein.

With daughter Sofia, 1971.

memories of him. I was born in 1965, almost eleven years after my sister, during the Princeton years. I adored my father as a child, and we spent countless hours together building with Legos, doing jigsaw puzzles, playing backgammon and Scrabble (for the latter my mother would also join), or doing carpentry. My father loved to work with his hands, especially with wood, having contemplated a future as a carpenter as a child.

Among other things he built a Barbie dollhouse for me with four rooms, each fitted with electricity and completely furnished—my sister also helping with some of the smaller details, like textiles or earthenware. And I loved the weeks before Christmas when he would spend long hours in the basement crafting presents for me—it was frustrating that I couldn't join, but at the same time exciting. One year I was deeply intrigued by screeching sounds coming from the basement. It turned out that he had built the kitchen for the doll house, complete with refrigerator, stove, sink, cupboards, drawers, even a broom closet, and that the strange sounds came from sanding down four coins to make them into stove plates.

I was very fortunate that my father always had a lot of spare time, so much so that my mother used to complain that she couldn't keep up with him, and this was even more so in the summers when he would take long periods away from work. On the island of Askerön off the west coast of Sweden my parents had bought a summer house by the sea, and there we spent much time outdoors together. Either in the forest—a highlight was an osprey nest that we would visit every day for years—or on the sea—my father loved sailing. When I was very young we had a SeaCat, which, to my great chagrin, was later traded in for a much smaller boat for day trips only. We also fished for plaice, with nets, together with a friend and his children.

When not actively engaged with the family or the house, my father would spend his spare time reading. He always read a lot, fiction as well as fact, mostly history or science. He had a very inquisitive mind and loved to learn new things up until the last days of his life. In all this he was much aided by his excellent memory; he seemed unable to forget anything he had learned and, unfortunately, unable to realize that other people did not have quite his powers of memory…he could be quite impatient with me. And I had to be careful when asking him a question: there was always the risk of a long and enthusiastic lecture, containing a lot more information than I wanted. For better or for worse, this seems to have been a heritable trait—my son complains about the same behavior in me.

My father set exceptionally high standards not only for himself but for everyone around him, including of course his children. Needless to say, this was difficult for us, especially until I was

With grandson Sander, 2003.

old enough to realize that he himself was the primary victim of his drive for perfection, not anyone else. He never felt that anything he did was quite good enough and was devastated if he made a mistake. Despite his demands for excellence, he did not, however, have career ambitions for us, just as he lacked such ambitions himself. His first goal when studying mathematics had been to become a teacher in secondary school (something he famously said to my mother on the evening they met), and the promotions and prizes he received in his career were never what he strived for—on the contrary, they weighed him down. He truly wanted us to do whatever we fancied, as long as we did what we did well and were able to support ourselves. He had chosen mathematics solely out of love for the subject and wanted us to do the same in our lives.

He was also a very loyal father. Many times I chose to do things he advised me not to do, but once my mind was made up he was always completely loyal with my decision. For example, my former husband and I bought a rundown house with a large garden despite his warnings. But once he realized that we were really going ahead, he quickly arranged for a generous loan and from day one was a faithful caretaker of the house and garden, effectively sheltering me from realizing what a crazy decision I had made. When he was no longer able to help, the house of course became impossible to keep and has now been sold.

In 1996 my son, Sander, was born, and Lars became a grandfather, something which delighted him no end. I think even more so since my sister had decided to end her life in 1978 at the age of twenty-three. When my son was born he could again see a continuation, a path into the future, and I think that this at least partially restored his peace of mind. And he was overjoyed that he was able to follow Sander all the way to sixteen, even to the publication of his first book (Sander is a photo artist). Before his death my father often said to me that he was content with his life, that it had been a good life. I do believe he died a happy man.