Logic's Lost Genius *and* Gentzen's Centenary

A Dual Review by Jeremy Avigad

Logic's Lost Genius: The Life of Gerhard Gentzen
Eckart Menzler-Trott
American Mathematical Society
and London Mathematical Society, 2007, US$97, xxii+440 pages
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Gentzen's Centenary: The Quest for Consistency
edited by Reinhard Kahle and Michael Rathjen
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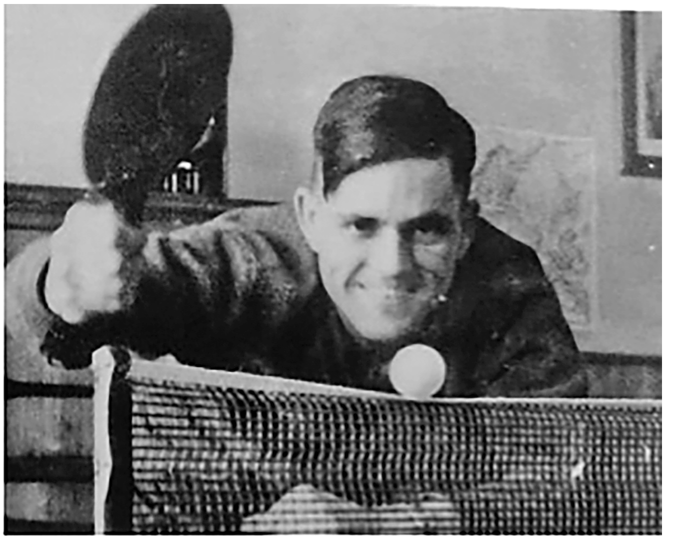
Among the mathematical disciplines, logic may have the dubious distinction of being the one that always seems to belong somewhere else. As a study of the fundamental principles of reasoning, the subject has a philosophical side and touches on psychology, cognitive science, and linguistics. Because understanding the principles of reasoning is a prerequisite to mechanizing them, logic is also fundamental to a number of branches of computer science, including artificial intelligence, automated reasoning, database theory, and formal verification. It has, in addition, given rise to subjects that are fields of mathematics in their own right, such as model theory, set theory, and computability theory. But even these are black sheep among the mathematical disciplines, with distinct subject matter and methods. The situation calls to mind the words of Georges Simenon, the prolific Belgian author and creator of the fictional detective Jules Maigret, who told *Life* magazine in 1958, "I am at home everywhere, and nowhere. I am never a stranger and I never quite belong." If we use his self-assessment to characterize mathematical logic instead, the description is apt.

One of the things that gives the field this peculiar character is its focus on language. Every subject has its basic objects of study, and those of logic include terms, expressions, formulas, axioms, and proofs. Logic was not even viewed as a part of mathematics until the middle of the nineteenth century, when George Boole's landmark 1854 treatise, *The Laws of Thought*, took the issue of viewing propositions as mathematical objects head on. Since the time of Aristotle, mathematics was commonly described as the science of quantity, encompassing both arithmetic, as the study of discrete quantities, and geometry, as the study of continuous ones. But Boole observed that propositions obey algebraic laws similar to those obeyed by number systems, thus making room for the study of "signs" and their laws within mathematics. Put simply, we can calculate with propositions just as we calculate with numbers.

Making sequences of symbols the subject of mathematical study was the cornerstone of *metamathematics*, the program by which David Hilbert hoped to secure the consistency of modern mathematical methods. By



George Gentzen, as pictured on the cover of *Logic's Lost Genius: The Life of Gerhard Gentzen*, **by Eckart Menzler-Trott.**

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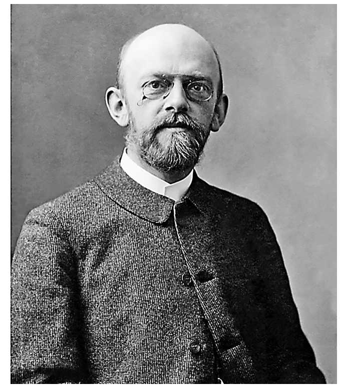
mathematizing things like formulas and proofs, Hilbert hoped to prove, using mathematical methods—in fact, using only a secure, "finitistic" body of mathematical methods—that no contradiction would arise from the new forms of reasoning. Hilbert gave a mature presentation of his *Beweistheorie*, or *proof theory*, in the early 1920s. Ironically, the representation of formulas and proofs as mathematical objects is an important component of Gödel's incompleteness theorems as well. Published in 1931, these showed that Hilbert's program could not succeed, in the following sense: No consistent system of mathematical reasoning that is strong enough to establish basic facts of arithmetic can prove its own consistency, let alone that of any larger system that includes it.

Nonetheless, proof theory is alive and well today. Its focus has expanded from Hilbert's program, narrowly construed, to a more general study of proofs and their properties. For Hilbert, as for Gödel, a proof was a sequence of formulas, each formula of which is either an axiom or follows from previous formulas by one of the stipulated rules of inference. Perhaps these qualify as mathematical objects, but they are the kinds of mathematical objects that only a logician could love. If there is one person who should be credited with developing a mathematical theory of proof worthy of the name, it is undoubtedly Gerhard Gentzen, whose work is now fundamental to those parts of mathematics and computer science that aim to study the notion of proof in rigorous mathematical terms.

Gentzen was born in Greifswald, in the northeastern part of Germany, in 1909. He began his studies at the University of Greifswald in 1928, but after two semesters he transferred to Göttingen, where he attended Hilbert's lectures on set theory and was introduced to foundational issues. He spent a semester visiting Munich and another visiting Berlin, and then returned to Göttingen in 1931. He learned about Hilbert's program and Gödel's results from Paul Bernays, whose collaboration with Hilbert ultimately culminated in the *Grundlagen der Mathematik*, the two volumes of which were published in 1934 and 1939.

Gentzen's work is fundamental to the study of proof in rigorous mathematical terms.

Hilbert and his students had studied classical first-order arithmetic as an important example of a classical theory whose consistency one would hope to prove by finitistic methods. In 1930 Arendt Heyting presented a formal axiomatization of intuitionistic first-order arithmetic in accordance with the principles of L. E. J. Brouwer. In 1932 Gentzen showed that the former could be interpreted in the latter, using an explicit translation that is now known as the *double-negation* translation. This showed, in particular, that the consistency of classical arithmetic can be derived from the consistency of intuitionistic arithmetic using finitistic means. Gentzen withdrew his submission, however, when he learned that Gödel himself had obtained the same result, by essentially the same method. The translation is often referred to as the Gödel-Gentzen translation, in honor of both.



David Hilbert hoped to establish a provably consistent foundation for mathematics.

The reduction of classical arithmetic to intuitionistic arithmetic seems to have encouraged Gentzen to think about the possibility of obtaining finitistic consistency proofs of either of these theories (and hence both). The second incompleteness theorem implies that such a proof could not be carried out within the theories themselves, so the challenge was to find suitably strong principles that could be argued to conform to the vague criterion of being finitistic. In order to do so, he needed a notion of deduction that was mathematically clean and robust enough to make it possible to study formal derivations in combinatorial terms. This was the subject of Gentzen's 1934 dissertation.

In fact, Gentzen developed two fundamentally different proof systems. The first, known as *natural deduction*, is designed to closely model the logical structure of an informal mathematical argument. Formulas in first-order logic are built up from basic components using logical connectives such "A and B," "A or B," "if A, then B," and "not A," as well as the quantifiers "for every x, A" and "there exists an x such that A." For each of these constructs, natural deduction provides one or more *introduction rules*, which allow one to establish assertions of that form. For example, to prove "A and B," you prove A, and you prove B. To prove "if A, then B," you assume A and, using that assumption, prove B. Natural deduction also provides *elimination rules*, which are the rules that enable one to make use of the corresponding assertions. For example, from "A and B," one can conclude A, and one can conclude B. The elimination rule for "A or B" is the familiar proof by cases: if you know that A or B holds, you can establish a consequence C by showing that it follows from each.

Gentzen also designed a system known as the *sequent calculus*, with a similar symmetric pairing of rules and, for some purposes, better metamathematical properties. In both cases, Gentzen considered *reductions*, steps that can be used to simplify proofs and avoid unnecessary detours. The *Hauptsatz* (main theorem) of his dissertation, now also known as the *cut elimination theorem*, is a seminal and powerful tool in proof theory. It shows that appropriate reductions in the sequent calculus can be used to transform any proof into one in a suitable normal form, in which every derived formula is justified, in a sense, from the bottom up.

While Gentzen completed this work, the Nazi party was on the rise, and the political and academic environment in Germany was deteriorating. Göttingen was particularly hard hit. Hermann Weyl, who had assumed Hilbert's chair in 1930, fled to the United States in 1933 with his wife, who was Jewish, and joined the Institute for Advanced Study. Emmy Noether fled similarly to Bryn Mawr. Bernays, who had become Gentzen's advisor, was summarily dismissed from his post in 1933 because of his Jewish ancestry.

In 1935 Gentzen submitted an article to the *Mathematische Annalen* describing a consistency proof for arithmetic. He began by extending the system of natural deduction to intuitionistic arithmetic, adding a rule to encapsulate proof by induction. The strategy was to show that proofs in the system can be reduced to ones in normal form, since from the description of the normal forms, it was then immediate that no such proof could conclude in a contradiction. A copy of the paper was sent to Weyl, who felt that in "the immediate future it should play the rôle of the standard work on the foundations of mathematics." The paper was also discussed by Gödel and Bernays on a boat to New York in 1935, though we do not know the content of their discussions. Apparently the paper met with criticism at the *Annalen*, and a letter that Gentzen wrote to Bernays suggests that the problem was that it was not sufficiently clear that the reduction procedure he described would always terminate. Gentzen rewrote the argument entirely, adapting it from natural deduction to a sequent calculus for classical arithmetic. This time he assigned to each proof an ordinal less than an ordinal known as ε_0 in such a way that with each reduction the associated ordinal decreases. Since, by definition, there is no infinite descending sequence of ordinals, every reduction sequence necessarily terminates.

The revised proof was published in the *Annalen* in 1936, but Gentzen's original proof is independently interesting, and one can show that the associated reduction procedure does in fact terminate. This version was published in English translation in 1969 and in the original German in 1974. Jan von Plato has provided a detailed history of these results, as discussed below.

It is perhaps a sign of the marginal role that syntax plays in most branches of mathematics that Gentzen's name is generally unfamiliar outside proof theory, but he is, today, considered to be a seminal figure in computer science. Natural deduction and the sequent calculus are fundamental to automated reasoning, where normal form theorems play a key role in reducing the space that algorithms have to search to find an axiomatic derivation. The connections between deduction and computation that Gentzen foreshadowed play an important role in the study of functional programming languages, where syntactic typing judgments are used to help ensure that programs meet their specifications. His ideas have been generalized to stronger logics, including second-order and higher-order logic, as well as modal logics and other special-purpose logics designed for reasoning about computation and computational processes. Although Gentzen's structural analyses were really byproducts of his work on the consistency problem, they are fundamental to contemporary proof theory. Sequents, rules, and normal forms are to that community of researchers what functions, operators, and derivatives are to the analyst, and now we can think of formal deduction only in those terms.

Kurt Gödel proved Hilbert's plan hopeless.

The first book under review, *Logic's Lost Genius*, is a biography of Gentzen that was originally published in German with the title *Gentzens Problem*. The work was then revised by the author, Eckart Menzler-Trott, and translated to English by Craig Smoryński and Edward Griffor. The appendices include a history of Hilbert's program by Smoryński, translations of three expository lectures delivered by Gentzen in 1935 and 1936, and a friendly and informative overview of Gentzen's work by von Plato.

Readers will enjoy following the backstage exchanges between Gentzen and many important figures in early mathematical logic.

Menzler-Trott does a good job of conveying the intellectual milieu of German foundational research. Readers will enjoy following the backstage exchanges between Gentzen and many important figures in early mathematical logic, including Weyl, Bernays, Gödel, Heyting, Alonzo Church, and Wilhelm Ackermann. But the greater and more moving drama is the utter collapse of the German intellectual environment as a result of the Nazi rise to power. It is painful to read first-hand reports of German science being purged of Jewish involvement by Nazi administrators and Nazi sympathizers within the academic community. And how did Gentzen respond? Disgracefully, in fact: in 1933 he voluntarily joined the paramilitary wing of the Nazi party militia, known as the *Sturmabteilung*, or *SA*. It is jarring to see Gentzen end letters to his old mentor in Greifswald, Martin Kneser, with a hearty "heil Hilter!" For those of us who admire Gentzen's work and contributions to logic, this raises knotty questions as to the extent to which we can admire someone's scientific contributions while isolating them from less commendable, and even deplorable, aspects of their lives.

But Menzler-Trott's portrayal of Gentzen is sympathetic. The sense we get from the stories of Gentzen's youth and from the letters he wrote to colleagues is that of someone bright, affable, creative, and polite, and above all devoted to his work. There is little to suggest that Gentzen had any interest in politics; joining the SA comes across as a pathetic attempt to keep himself in good graces with the establishment while maximizing his chances of securing financial support to continue with his research. In Menzler-Trott's assessment, Gentzen had "a certain passive, almost phlegmatic trait in all things which did not concern mathematics."

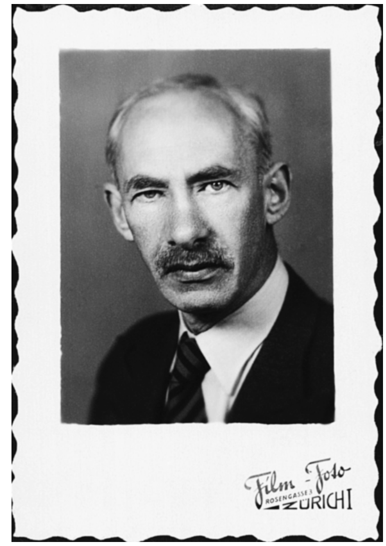
Indeed, he seems at times to have had no real sense of what was going on around him. In April of 1934 he wrote an upbeat letter to Bernays in which he complained of the difficulty of obtaining a teaching position and casually mentioned that he had joined the SA, "as has been urgently advised from various quarters." After relating his progress on consistency proofs, he asked, cheerfully, "Are you coming again to Göttingen for the summer semester?" It is almost as though he had forgotten that Bernays had been stripped of his license to teach and was trying to convince himself that nothing had really changed.

To be sure, being generally clueless and fixated on research is not an excuse for failing to take a stand against the Nazi atrocities and the indignities that were suffered by his colleagues. But I suspect that many readers of the *Notices* will find Gentzen's naiveté and preoccupation to be disarmingly familiar. We see these traits in ourselves and in our colleagues, and Gentzen's story challenges us to wonder whether we would have done better and, indeed, whether there are things we could be doing better in the present day.

In any case, Gentzen's attempts to withdraw into his work did not succeed, and he did not lead a happy or easy life. In 1939 he was called to military service as a radio operator inside Germany. In 1942 he was hospitalized in a state of nervous exhaustion and then released from military service. He was then assigned to teach mathematics at the German Charles University in Prague and chose to remain there after the Third Reich fell. As part of a general backlash against Germans in Prague, he was sent to a prison camp, where he died of malnutrition in August 1945. He was thirty-five years old.

Logic's Lost Genius is not easy reading. Its primary purpose is to serve as a historical record rather than an entertaining narrative, and Menzler-Trott accumulated a litany of names, places, dates, and events throughout the course of his prodigious research effort. Whenever possible, he lets source documents speak for themselves, and as a result many facts and opinions are conveyed through letters, scholarly reviews, firsthand narratives, government reports, and academic assessments. A long chapter details the Nazi transformation of logic and foundational research from 1940 to 1945, which had the goal of establishing racial purity and a proper "German" mathematics. Telling this story cannot have been an easy task for Menzler-Trott, who often lets his personal voice rise over the dry assemblage of facts in order to express his anger, frustration, and disgust. His admiration and respect for Gentzen is apparent throughout.

The second item under review, *Gentzen's Centenary*, is a very different work. It comprises a collection of essays that, taken together, provide a broad appraisal of Gentzen's mathematical legacy on the occasion of the 100-year anniversary of his birth. Although a few of the articles are targeted at logicians and proof theorists, most of the articles are written with a general mathematical audience in mind.



Paul Bernays, Gentzen's advisor, was summarily dismissed from his post in 1933 because of his Jewish ancestry.

The book is divided into four parts. The first, titled "Reflections," offers historical and philosophical views of Gentzen's work. Reinhard Kahle considers the meaning and importance of Gentzen's consistency proofs and explores modern variations on Hilbert's program. Michael Detlefsen argues that Gentzen's formalist position was less far-reaching than Hilbert's: whereas Hilbert felt that a finitistic consistency proof was sufficient to ground abstract mathematical methods, Gentzen held that a proper grounding of abstract mathematics would have to ascribe content to mathematical abstractions as well. Anton Setzer proposes a broader approach to Hilbert's program that combines non-mathematical and philosophical validation of basic principles with metamathematical study.

The second part, titled "Gentzen's Consistency Proofs," focuses on those. Wilfried Buchholz provides an analysis and presentation of Gentzen's original consistency proof using modern terminology and notation. Jan von Plato relies on archival work to describe the evolution of Gentzen's consistency proof from his original submission to the final result, and shows that the original version was equally prescient in introducing themes that were to become central to proof theory. Whereas Gentzen used a classical sequent calculus in the final version of the consistency proof, the one that used the notation for ε_0, Dag Prawitz presents a Gentzen-style normalization proof for a formulation of arithmetic in intuitionistic natural deduction, and Annika Siders presents a similar consistency proof for a system based on an intuitionistic sequent calculus. William Tait explains the constructive principles that can be used to ground Gentzen's original consistency proof, and Michael Rathjen gives a clean mathematical analysis of Goodstein's theorem, an interesting number theoretic result which, essentially as a result of Gentzen's analysis, can be shown to be independent of first-order arithmetic.

The third part, titled "Results," presents technical results that round out Gentzen's work. Sam Buss studies cut elimination procedures in terms of time and space complexity, Fernando Ferreira discusses a consistency proof for second-order arithmetic due to Clifford Spector, Herman Ruge Jervell explains how properties of ordinals are established in first-order arithmetic, and Wolfram Pohlers surveys some of the methods of contemporary ordinal analysis. Only the final part, "Developments," is aimed primarily at proof theorists. It includes substantial and interesting results by leading researchers in the area. The section includes a paper by Grigori Mints, a beloved and important figure in proof theory, who passed away just as the collection was about to go to press. The volume is movingly and appropriately dedicated to his memory.

Logic's Lost Genius and *Gentzen's Centenary* complement each other well, offering informative overviews of Gentzen's accomplishments and the intellectual and political environment in which they emerged. Contemporary textbooks provide sufficient evidence of the importance of Gentzen's work to modern logic. But the foundational background duly reminds us that the overarching goal of research in proof theory is a better understanding of the principles of reasoning and what it means to do mathematics. And the dark historical narrative reminds us that research doesn't take place in a vacuum; even the purest of mathematicians have to contend with the exigencies of their social, political, and institutional environments, and it is important to face them deliberately rather than accidentally.

Acknowledgments

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Jeremy Avigad's research interests include mathematical logic, interactive theorem proving, and the history and philosophy of mathematics.

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