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A NOTE ON NORMAL DILATIONS

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I. **Introduction.** Our purpose is to give certain sufficient conditions that a normal dilation of an operator be an extension from a reducing subspace. The first result of this kind, that we know of, is due to T. Andô [1] who considered compact normal dilations. In this note we use only assumptions about the nature of the spectrum; nevertheless, we are able to recover Andô's theorem.

Let A be an operator on a Hilbert space $\mathcal{K}$. Let P be the orthogonal projection of $\mathcal{K}$ onto a subspace $\mathcal{H}$. Let T denote the restriction of PAP to $\mathcal{H}$. The operator T is called a *compression* of A and A is called a *dilation* of T. If T^n is the compression of A^n $(n=0\ 1, 2, 3, \cdots)$ then T is called a *strong compression* and A a *strong dilation*. Let X be a compact subset of the plane containing $\sigma(A)$ and $\sigma(T)$, the spectra of A and T. The operator A is said to be an *X-dilation* of T if, for every rational function $r(\cdot)$ which is analytic on X, the operator $r(A)$ is a dilation of $r(T)$. These definitions were introduced by Halmos. Some other writers use "dilation" to mean what we call strong dilation. Sz-Nagy uses "projection" for compression. When T is a strong compression of A Andô calls $\mathcal{H}$ a "semi-invariant" subspace of A.

These notions are related to the more familiar concepts of invariant subspace and reducing subspace as follows. If $\mathcal{H}$ is an invariant sub-

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space (equivalently $PAP=AP$) and X contains $\sigma(A)$ and $\sigma(T)$ then A is an X-dilation of T. If $\mathcal{H}$ is reducing and X contains $\sigma(A)$ then A is an X-dilation of T. When $\mathcal{H}$ is invariant, we say A is an *extension* of T; when $\mathcal{H}$ is reducing, we say A is a *direct extension* of T and T is a *reduction* of A.

Summing up, we have five conditions:

(i) A is a dilation of T;

(ii) A is a strong dilation of T, $PA^nP=(PAP)^n$;

(iii) A is an X-dilation of T, $Pr(A)P=r(PAP)$;

(iv) A is an extension of T, $PAP=AP$; and

(v) A is a direct extension of T, $AP=PA$.

These are also expressed as: T is a (i) compression, (ii) strong compression, (iii) X-compression, (iv) restriction, and (v) reduction of A. One sees at once

$$\text{(v)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}.$$

There are examples which show that no arrow can be reversed unless more assumptions are made.

It is known [2, p. 60] that if both T and A are normal, then (iv) implies (v). T. Andô [1] has proved that if A is compact and normal, then (v) follows from (ii). It will be shown here that we need only assume that A is normal and $\sigma(A)$ is nowhere dense and does not separate the plane. This result depends on the density of the set of polynomials in the space, $C(\sigma(A))$, of complex continuous functions on $\sigma(A)$. If the set of rational functions is dense in $C(X)$, then (iii) implies (v).

II. Statement and proof of results.

REMARK (ANDÔ). Let E and P be projections. The operator $Q=PEP$ is a projection if and only if P and E commute.

PROOF. If Q is a projection we have that $\|EPx-Qx\|^2=0$, by expansion to a sum of inner products. Now note

$$EP = Q = Q^* = (EP)^* = P^*E^* = PE.$$

The converse is trivial.

LEMMA. *Let T be a normal operator with a normal dilation A, and X be a compact set containing $\sigma(A)$ and $\sigma(T)$.*

If the set

$$D = \{f \in C(X) \mid f(A) \text{ is a dilation of } f(T)\}$$

is dense in $C(X)$ then T is a reduction of A.

PROOF. For every pair of vectors x in $\mathcal{H}$ and y in $\mathcal{K}$, if $f \in D$ we have

(1) $(f(A)x, Py) = (f(A)Px, P^2y) = (Pf(A)Px, Py) = (f(T)x, Py).$

Denoting the spectral measures of A and T by $E(\cdot)$ and $F(\cdot)$, respectively, (1) may be written as

$$\int f(\lambda)d(E(\lambda)x, Py) = \int f(\lambda)d(F(\lambda)x, Py). \tag{2}$$

Since D is dense in $C(X)$ and the measures $(E(\cdot)x, Py)$ and $(F(\cdot)x, Py)$ are regular, it follows that they are identical. That is for each Borel set δ

$$(E(\delta)x, Py) = (F(\delta)x, Py)$$

and therefore

$$(PE(\delta)x, y) = (F(\delta)x, y).$$

Since y is arbitrary $PE(\delta)x = F(\delta)x$ for all x in $\mathcal{H}$. This means that $PE(\delta)P$ acts as a projection in $\mathcal{H}$ and since $PE(\delta)P$ acts as 0 in $\mathcal{H}^\perp$ it is evident that $PE(\delta)P$ is a projection. It follows from the Remark that P commutes with $E(\delta)$ and from the spectral theorem that P commutes with $\int \lambda dE(\lambda) = A$.

THEOREM 1. *Let A be a normal strong dilation of T. If $\sigma(A)$ is nowhere dense and does not separate the plane, then T is a reduction of A.*

PROOF. Since T is a strong dilation

$$(PAP)^n = PA^nP; \qquad n = 1, 2, 3, \cdots,$$

for every polynomial $p(\cdot)$, PAP commutes with $Pp(A)P$. By Lavrentieff's theorem [**4**] the polynomials are dense in $C(\sigma(A))$ and thus the functional calculus for normal operators implies that A^* is in the uniform closure of $\{p(A)\}$, and so PA^*P is in the uniform closure of $\{Pp(A)P\}$. Hence PAP commutes with PA^*P and is therefore normal. The operator T is normal because it is the reduction of the normal operator PAP.

Now with D being the set of polynomials and $X = \sigma(A)$ it follows from the Lemma that T is a reduction of A.

COROLLARY (ANDÔ). *Every strong compression of a compact normal operator is a reduction.*

The method for proving Theorem 1 can be used to obtain other similar results. We shall refrain from maximum generality and will be satisfied to state one theorem on X-dilations. If the rational functions with poles not in X are dense in $C(X)$ we call X an $\mathcal{R}$-*set*. Bishop [**3**] has given a characterization of $\mathfrak{R}$-sets which includes the Hartogs-Rosenthal result that every set of zero planar Lebesgue measure is an $\mathfrak{R}$-set. Mergelyan [4] has obtained other sufficient conditions X be an $\mathfrak{R}$-set.

THEOREM 2. *If X is an $\mathfrak{R}$-set and T has a normal X-dilation A, then T is a reduction of A.*

The proof follows the outlines of that of Theorem 1.

As one more variation on this theme we state:

THEOREM 3. *If A^n is a normal dilation of T^n for $n=1, 2, 3, \cdots, k$ and $\sigma(A)$ is a finite set of not more than k points, then T is a reduction of A.*

Note that if $k=2$ and A is a projection, Theorem 3 reduces to the Remark at the beginning of this section.

Donald Sarason has informed the author (personal communication) that he has obtained Theorems 1 and 2 by different methods.

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